Douglas Fisher and Adrian Fleissig

Douglas Fisher is professor of economics at North Carolina State University. Adrian Fleissig is assistant professor of economics at the University of Texas-Arlington. *The authors wish to acknowledge the assistance of Ron Gallant, Douglas Pearce, Walter Thurman and Michael Belongia, who made important suggestions for the revision of this paper.*

Money Demand in a Flexible Dynamic Fourier Expenditure System

IN WELL-KNOWN SURVEYS of the growing literature on expenditure systems, Deaton and Muellbauer (1980) and Pollak and Wales (1992) describe many of the shortcomings of the existing work in this genre. Among the problems they list that inhibit the acceptance of these methods, the ones that seem most critical to us are (1) the failure to link theory to application, (2) improper aggregation techniques, (3) imprecise estimation of partial derivatives, (4) the failure of locally integrable models at some data points and (5) the misspecification of the dynamics. We can address several of these problems by extending the Fourier Flexible Form of Gallant (1981). Most notably, his technique provides global flexibility and arbitrarily accurate estimates of partial derivatives. In fact, the technique is capable of approximating the unknown function (an aggregator function, for example) to any desired degree of accuracy. The version of the Fourier model in current use, however, is static in nature, which inhibits its application to time-series data; in particular, studies by Gallant (1981), Ewis and Fisher (1985), and Fisher (1989, 1992), all employ the static model and all produce residuals that are not white noise for each share; see also Barnett, Fisher, and Serletis (1992). This may be due to inadequately modeled dynamics; in fact, there are no examples of a dynamic Fourier in the literature. The task of this paper is to produce and evaluate two dynamic alternatives in the context of the Fourier model.

In the traditional literature on consumer choice, the indirect utility function is approximated by a specific functional form in order to obtain expenditure shares and estimates of the important own- and cross-elasticities. One might attempt to estimate a parametric model, of course, but the results of such exercises have not been satisfactory. The chief problem has been model failure, partly related to the choice of specific (nonflexible) functional forms. To finesse this problem, a flexible functional form can be employed in order to estimate the unknown indirect utility function. Diewert (1974) defines a flexible functional form as a second-order approximation to an arbitrary twice continuously differentiable function $f(x)$ at any given point x^*; the popular translog is an example. The difficulty, however, is that this definition, and the resulting approximation, fails to impose precision on the partial derivatives of the function. Indeed, it is well-known that away from the point of approximation, the translog can perform quite poorly in its task of tracking the unknown function. The result is imprecise estimation of the expenditure shares.

Gallant (1981) developed the Fourier flexible form in order to approximate the unknown indirect utility function and its first derivatives arbitrarily accurately within a Sobolov norm. The first derivatives are important since the expenditure shares are derived by differentiation. The Fourier model, with its global properties, can then provide integrability over a finite region for the estimated model, assuming convergence. In particular, since integrability normally implies a convex closure over a finite region, one can presume desirable separability properties for data examined under the Sobolov norm. This contrasts, as noted, with the possible lack of closure on procedures that provide an approximation only at a single point in the data space; in particular, it contrasts with locally integrable models (such as the Translog).

In this paper, we produce two versions of the dynamic Fourier expenditure system; these are then compared with the static model in various ways. In section two we briefly discuss the static model before going into considerable detail over what we will be calling the "time-series approach" to making the Fourier model dynamic. This basically follows the lead of Anderson and Blundell (1982, 1983), whose results are both well-known and have been applied in the literature on flexible functional forms (see Serletis, 1991). In section three, we continue with a second version of the dynamics, this time involving the construction of the dynamic Fourier utility function. We term this the "dynamic utility function approach." In section four, we present examples of the two dynamic models in order to clarify the ideas and explain the notation. It is here possible to establish clear distinctions between the models in the context of the Fourier. In section five, we go over the procedures used to prepare the data, and in section six, finally, we discuss estimates of the two dynamic models that utilize the U.S. data previously described. We also discuss how the two models perform in comparison with their static equivalents. Our conclusions follow.

THE TIME SERIES APPROACH

Following Gallant (1981), the static Fourier flexible form approximation of an indirect utility function h($\mathbf{v}$) may be written as

$$(1)\ h_k(v,\theta) = a_0 + b'v + \tfrac{1}{2}v'Cv + \sum_{\alpha=1}^{A}\sum_{j=-J}^{J} a_{j\alpha}e^{ijk'_\alpha v},$$

where

$$C = -\sum_{\alpha=1}^{A} a_{o\alpha}k_\alpha k'_\alpha \text{ and } a_{j\alpha} = \bar{a}_{-j\alpha},$$

a_o, $a_{o\alpha}$ and b are real-valued, and $\mathbf{v}$ is a vector of the expenditure-normalized user costs of the particular assets involved in the exercise (Gallant, 1981). In this expression the overbar denotes complex conjugation and i is the imaginary number. A multi-index k_α, is an n-vector with integer components and is used to denote partial differentiation of the utility function (see the example in section four). The elements of a multi-index can be considered to be the weights (when multiplied by v) of the normalized price indexes.

In an empirical investigation, it is actually more convenient to work with a sine/cosine formulation rather than the exponential just written and so the following form is generally employed:

$$(2)\ h_k(v,\theta) = u_o + b'v + \frac{1}{2}v'Cv + \sum_{\alpha=1}^{A}\left(u_{o\alpha} + 2\sum_{j=1}^{J}[u_{j\alpha}\cos(jk'_\alpha v) - w_{j\alpha}\sin(jk'_\alpha v)]\right),$$

in which

$$C = -\sum_{\alpha=1}^{A} u_{o\alpha}k_\alpha k'_\alpha.$$

After differentiating equation 2 and applying Roy's identity, Gallant arrives at the following set of equations:

$$(3)\ y_i(v,\theta) = \frac{v_i b_i - \sum_{\alpha=1}^{A}\left(u_{o\alpha}v'k_\alpha + 2\sum_{j=1}^{J} j[u_{j\alpha}\sin(jk'_\alpha v) + w_{j\alpha}\cos(jk'_\alpha v)]\right)k_{i\alpha}v_i}{b'v - \sum_{\alpha=1}^{A}\left(u_{o\alpha}v'k_\alpha + 2\sum_{j=1}^{J} j[u_{j\alpha}\sin(jk'_\alpha v) + w_{j\alpha}\cos(jk'_\alpha v)]\right)k'_\alpha v},$$

for $i = 1, \ldots, n$ expenditure shares. This system is what is estimated with a vector of error terms appended. Equation (3) can be more compactly expressed as:

$$(4)\ y_{it} = f(v_t, \theta).$$

Note that we have attached a time subscript in order to emphasize the static nature of the equations. This completes the discussion of the static Fourier Flexible model.

Consumption, monetary and production theories use past variables—in the utility function, in the constraints, or by time-series methods—to model habit persistence, adjustment costs and/or expectations. In a demand systems approach, incorporating dynamics in any of these ways complicates the calculation of the restrictions, which still must hold. In the following exercises we present results for the time-series function and, in section three, for the utility function. We present the models first, including with each a discussion of the restrictions, before presenting examples of both.

For the time series model, applying an ARMA (p,q) directly to equation (4) is one approach toward modeling the dynamic behavior of the consumer. This approach is taken by Anderson (1980) for the special case when $f(v_t, \theta)$ is linear in the expenditure-normalized prices v_t and the parameters θ. He shows that adding up, as the direct result of adopting the ARMA approach, implies four additional restrictions. Anderson and Blundell (1982, 1983) extend the results for the case in which $f(v_t, \theta)$ may be nonlinear in the parameters but linear in the normalized prices V, i.e., $f(v_t, \theta) = \pi(\theta)v_t$. When applying an ARMA$(p,q)$ to equation (4), they can extract a term, $y_{t-p} - \pi(\theta)v_{t-q}$, the gap between the shares lagged p periods and normalized prices lagged q periods, representing the long-run structure for a system of simultaneous equations. This approach is not applicable when the matrix $\pi(\theta)$ cannot be extracted, as is the case with the Fourier flexible functional form; as a consequence, we use an alternative approach for analyzing the long-run structure. First, an ARMA(p,q) is applied to equation (4). The result is:

$$(5)\ A(L)y_t = B(L)f(v_t, \theta).$$

Here, where L is the lag operator, the terms $A(L)$ and $B(L)$ represent the following distributed lags

$$A(L) = I + A_1L + A_2L^2 + \ldots + A_pL^p$$

$$B(L) = I + B_1L + B_2L^2 + \ldots + B_qL^q.$$

Consider the following ARMA(1,1):

$$(6)\ y_t = A_1^* y_{t-1} + f(v_t, \theta) + B_1^* f(v_{t-1}, \theta) + e_t.$$

As in Anderson and Blundell (1982, 1983), the addingup restrictions require a transformation A_1^* of A_1 where the columns of A_1^* must sum to zero, and $a_{ij}^* = a_{ij} - a_{in}$ for $i=1,\ldots,n$ and $j=1,\ldots,n\text{-}1$. Similar restrictions for the matrix B_1^* apply. In sum, then, the dynamics appear as lagged shares y_{t-1} and lagged normalized prices v_{t-1}.

THE DYNAMIC UTILITY FUNCTION APPROACH

Individuals are unlikely, generally, to be able to adjust their consumption plans instantaneously. Rather than apply an arbitrary lag to the shares derived from a static optimization exercise, an attractive alternative is to allow past behavior to affect current decisions directly through the utility function. We can define the set of past decisions on a commodity to be an $nx1$ vector of shares ($\boldsymbol{s}$) that are functions of all past values of v:

$$(7)\quad s = f(v_{t-r}) \quad for\ r=1,\ldots,n\text{-}1.$$

Here, each share depends on its own lagged normalized price and the lagged normalized prices of the remaining n-1 shares. In this case, the representative consumer's dynamic indirect utility function can be expressed as

$$(8)\quad U = U(v,s),$$

where $v = P/M$ and s represents the dynamics. M is total "expenditures" on this class of assets. This is, in effect, a structural approach for obtaining dynamic shares since the dynamics are embedded in the decision process rather than appearing as dynamic extensions of the static shares (as in the time-series model). It produces a new version of the Fourier model, accordingly. To begin with, we will let $s = x_{t-1}$, so that each share depends on its own lagged value as well as on lags from the remaining n-1 shares.

The dynamic Fourier Flexible Form is defined as

$$(9)\quad g_k^d(z,\theta) = u_o + b'z + \tfrac{1}{2}z'Cz + \sum_{\alpha=1}^{A}\sum_{j=-J}^{J} a_{j\alpha}e^{ijk'_\alpha z}$$

and

$$C = -\sum_{\alpha=1}^{A} u_{o\alpha}K_\alpha K'_\alpha \qquad z = \begin{pmatrix} v'_t \\ v'_{t-1} \end{pmatrix}.$$

Parallel to equation 2, we may express the model as

$$(10)\quad g_k^d(z,\theta) = u_o + b'z + \tfrac{1}{2}z'Cz$$

$$+\sum_{\alpha=1}^{A}\left(u_{o\alpha} + 2\sum_{j=1}^{J}[u_{j\alpha}\cos(jk'_\alpha z) - w_{j\alpha}\sin(jk'_\alpha z)]\right),$$

in which

$$C = -\sum_{\alpha=1}^{A} u_{o\alpha}K_\alpha K'_\alpha$$

In this formulation, a multi-index is now a 1 by $(r+1)$ (n) vector with integer components; in the static case, it was 1 by (n). Here, r is the number of lags. The dynamic shares for this problem are obtained by applying Roy's identity to equation 10:

$$(11)\quad y_i = \frac{v_{it}b_i - \sum_{\alpha=1}^{A}\left(u_{o\alpha}z'k_\alpha + 2\sum_{j=1}^{J} j[u_{j\alpha}\sin(jk'_\alpha z) + w_{j\alpha}\cos(jk'_\alpha z)]\right)k_{i\alpha}z_i}{\sum_{i=1}^{n} b_i v_{it} - \sum_{\alpha=1}^{A}\left(u_{o\alpha}z'k_\alpha + 2\sum_{j=1}^{J} j[u_{j\alpha}\sin(jk'_\alpha z) + w_{j\alpha}\cos(jk'_\alpha z)]\right)k'_\alpha z},$$

where $i = 1,\ldots,n$. This can be more compactly expressed as

$$(12)\quad y_t = f(v_t, v_{t-1}, \theta).$$

In this model, adding up is guaranteed, and no additional restrictions need to be applied at the estimation stage.

EXAMPLES OF THE TWO MODELS

In the two models just presented, the dynamics are captured in quite different ways. For the time-series approach, the dynamics enter in the form of lagged shares and lagged expenditure-normalized prices. In the dynamic utility function model, the dynamics enter only as lagged normalized prices in each of the share equations. The dynamic models can be more clearly compared with an example, which is what we now present. Note that we use what are termed "multi-indices" in the process of estimating the Fourier model. This is a notational convenience, as we have explained, for expressing the partial differentiation of the indirect utility function and can be considered as weights (linear combinations $k\alpha'v$) of normalized prices.

In this example we will be looking at four share equations, with $A=4$ and $J=1$ in the Fourier model. The multi-indices used for the time-series approach, assuming an ARMA(1,0), are:

$$k_\alpha = \begin{pmatrix} k_{1\alpha} \\ k_{2\alpha} \\ k_{3\alpha} \\ k_{4\alpha} \end{pmatrix} \text{ where } k_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ k_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$k_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},\ k_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ with } V = \begin{pmatrix} V_{1t} \\ V_{2t} \\ V_{3t} \\ V_{4t} \end{pmatrix}.$$

Note that V defines the four expenditure-normalized prices. The multi-indices are set up in the same way as in Gallant (1981) and one must be careful, when taking partial derivatives, to ensure that the corresponding $k_{i\alpha}$ is used. In this example, the first element of each of the multi-indices, zero or one, corresponds to the first element in V; this is the normalized price, V_{1t}. Since the dynamics are modeled by adding lagged expenditure shares, the dimension of the multi-indices, which only appears in $f(v_t,\theta)$ in equation 5, stays the same when one moves from the static to the dynamic time series model.

On the other hand, in the dynamic utility approach, the inclusion of lagged normalized prices increases the length of each multi-index [see $f(v_t, v_{t-1}, \theta)$ in equation 12]; we use the following eight indices, accordingly:

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, k_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, k_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, k_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$k_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, k_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, k_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, k_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this case the vector of normalized prices is

$$z' = [v_{1t}, v_{2t}, v_{3t}, v_{4t}, v_{1,t-1}, v_{2,t-1}, v_{3,t-1}, v_{4,t-1}].$$

The first four elements of each k_α correspond to the static part of the vector z and the last four elements of each k_α to the dynamic elements of z. This separation of multi-indices enables one to test the static against the dynamic utility function because each multi-index has an associated parameter.

THE CONSTRUCTION OF DIVISIA MONETARY AGGREGATES

Most of the studies of money demand in the literature employ monetary aggregates that are simple sums of their components (for example, *M1 = Currency plus deposits*) and are constructed essentially without benefit of index-number theory. While simple-sum aggregation might serve policy makers well when interest rate fluctuations are relatively mild, it is at a disadvantage when the relative interest rates on the monetary components fluctuate significantly. A Divisia index is an alternative approach for aggregating data that is based directly on economic theory. The Divisia index, indeed, is designed to internalize the substitution effects (at constant utility) that arise from relative price changes. In fact, the simple-sum index cannot produce this result unless the components of the proposed aggregation are perfect substitutes. We have reason to believe this is not the case for the monetary aggregates in common use.

Having a satisfactory procedure such as the Divisia does not, however, tell us exactly what set of assets to consider or how to group the subsets of the data for efficient estimation. A procedure that is available is the linear NONPAR program of Varian (1982, 1983), which is based directly on the Generalized Axiom of Revealed Preference (GARP). Satisfaction of GARP on a set of data implies that there exists a non-satiated, concave, monotonic utility function across that particular set. Such a set of data, if it exists, can be examined for logical groupings, again using the program NONPAR. If such groupings can be established—that is, if weak separability holds—then, according to the Leontief-Sono definition of separability, the marginal rates of substitution between any two commodities in the monetary index are independent of changes in relative prices outside the monetary group. This group is then available for (Divisia) aggregation.

On the quarterly U.S. data from 1970:1 to 1985:2, Swofford and Whitney (1987) have constructed a set of real per capita measures of monetary quantities and a set of related nominal user costs to represent the prices of these quantities. With M1 denoting narrow money (excluding the deposits of businesses); OCD, other checkable deposits; SD, savings deposits in financial institutions; and STD, small time deposits in financial institutions, they find that the following arrangement passes the necessary and sufficient conditions for the General Axiom of Revealed Preference:

U[V(DUR, NONDUR, SERV, LEIS), M1, OCD, SD, STD].

Here, the first three items in the equation refer to components of total consumption, while LEIS refers to leisure (evaluated at the wage rate). Note that SD and STD describe vectors of the liabilities of the various financial institutions *(for example, SD = small time deposits in com-*

mercial banks, S&Ls, and so on).[1] Also, notice that in the arrangement just listed, the consumption and leisure activities are separable from the financial assets but not the converse. This implies the existence of an aggregate utility function defined across these monetary entities (for this time and place).

Because of the failure to establish a subgrouping of the monetary assets, it proves necessary to work with the following four aggregate commodities:

M1, OCD	*A1*
SDCB, SDSL, SDSB, SDCU	*A2*
STDCB, STDTH, STDCU	*A3*
DUR, NONDUR, SERV, LEIS.	*A4*

Here, *SDCB* and so on are savings deposits at commercial banks, S&Ls, mutual savings banks and credit unions, while STDCB and so on are small time deposits at commercial banks, thrifts and credit unions. To attempt to preserve the economic characteristics of this set of data up to a third-order remainder term, Divisia index numbers are constructed from the individual quantities and their associated user costs; these are designated as A1, ..., A4. Note that M1 and OCD are summed for convenience; this can be justified by further noting that the correlation coefficient between the user costs of these two items is .994.

Putting all the pieces together, then, we have monetary data (and user costs) that satisfy an empirical test for revealed preference, we have aggregated the data in a way that is designed to preserve their economic characteristics in the face of changes in relative prices and, finally, we propose to estimate the elasticities using a model which can come arbitrarily close to the elasticities implied by the true (but unknown) aggregate indirect utility function known to be defined (by the GARP test) over these entities. Note, especially, that satisfaction of GARP implies that there is a firm link between the indirect utility that is actually estimated and the underlying utility function that actually generates these data.

EMPIRICAL RESULTS

In our empirical work, we compare the results of the estimation of the three systems: the static, the time series dynamic and the utility function dynamic. Because the static theory is nested in each of the two dynamic theories, we present the results in that form. The comparisons are in terms of the significance of the coefficients, the characteristics of the residuals and the relevance of the dynamic formulations using the results of the Gallant-Jorgenson (1979) chi-square test. Unfortunately, the two dynamic approaches are not nested, so that we cannot compute a Gallant-Jorgenson test statistic. We do, however, offer a comparison utilizing the other statistics just mentioned. As it turns out, neither model has a clear advantage, although we do prefer the dynamic utility model in view of its economic properties and adequate performance. We also offer some comparisons with earlier work that utilized the static Fourier model over the same data space (Fisher, 1992). Here, there are dramatic differences in the estimated elasticities of substitution; we believe the dynamic results (utilizing the estimates from the dynamic utility approach) are considerably more reasonable than the earlier static results.

The share equations, with the across-equations restrictions, were estimated in the SAS system using PROC MODEL with nonlinear seemingly unrelated regression. The results for the dynamic time-series model appear in Table 1.

In this table, the *Bs* correspond to the quadratic terms in the Fourier Flexible Form, the *Us* and *Vs* to the Fourier series expansion, and the *As* to the lagged shares y_{t-1}.

These results describe reasonable fits, with 10 of the 12 adjustment parameters (A_{ij}) having *t*-

[1]The original variables were supplied by the Federal Reserve and appear in several publications by Farr and Johnson (1985a, 1985b). In this study, the monetary data are employed in per capita real form (where the latter is achieved by deflation with the CPI). SD represents savings deposits in commercial banks, S&Ls, mutual savings banks and credit unions, while STD represents the small time deposits of the same institutions. OCD is other checkable deposits and includes NOW accounts. See Swofford and Whitney's two papers for more details on the construction of the data.

As discussed in Swofford and Whitney (1987, 1988), the data were prepared as follows. Each monetary asset is deflated by the consumer price index for urban areas. OCD includes super NOW accounts. The user cost is the concept defined by Barnett (1978). For leisure, the quantity is 98 hours less average weekly hours worked during the quarter (times 52). The wage rate measures the opportunity cost of time. The consumption figures are taken from Department of Commerce data that also provides the implicit deflator for each category. A 10 percent depreciation rate is used in calculating the one-period holding cost of a durable good.

Table 1
Time Series Model: Dynamic Fourier Flexible Functional Form

Nonlinear SUR summary of residual errors

Eqn.	DF model	DF error	SSE	MSE	Root MSE	R-square	Adjust R-square
SM1	9	53	0.00438	0.0000827	0.00909	0.915	0.902
SM2	9	53	0.01650	0.0003113	0.01764	0.848	0.825
SM3	9	53	0.02445	0.0004613	0.02148	0.923	0.912

Nonlinear SUR parameter estimates

Parameter	Estimate	Approximate standard error	"T" Ratio	Approximate prob>|T|
B1	0.175462	0.08184	2.14	0.0367
B2	0.007578	0.25542	0.03	0.9764
B3	−0.448512	0.15285	2.93	0.0049
U01	−0.007955	0.02828	0.28	0.7796
U11	−0.009762	0.00710	1.37	0.1752
W11	0.023864	0.02545	0.94	0.3526
A11	0.419255	0.07229	5.80	0.0001
A12	0.259118	0.03873	6.69	0.0001
A13	0.197632	0.02071	9.54	0.0001
A14	0.190888	0.04239	4.50	0.0001
U02	−0.014187	0.01355	1.05	0.3000
U12	0.011554	0.01015	1.14	0.2599
W12	−0.019444	0.00809	2.40	0.0197
A21	−1.000371	0.13602	7.35	0.0001
A22	0.958742	0.06774	14.15	0.0001
A23	−0.034480	0.03080	1.12	0.2680
A24	0.766254	0.08967	8.55	0.0001
U03	0.020581	0.00692	2.97	0.0044
U13	−0.002235	0.00216	1.03	0.3062
W13	0.002259	0.00146	1.55	0.1272
A31	0.572904	0.11232	5.10	0.0001
A32	−0.381781	0.07622	5.01	0.0001
A33	0.276142	0.04249	6.50	0.0001
A34	−0.119175	0.08974	1.33	0.1899
U04	−0.009973	0.00554	1.80	0.0778
U14	−0.000714	0.00236	0.30	0.7638
W14	0.000171	0.00287	0.06	0.9527

N = 62
Objective = 2.0164
Objective*N = 125.0495
The Aij represent the dynamics.

values in excess of 2. Note that it is the surfaces of $(\partial/\partial x)g^*(x)$ and $(\partial^2/\partial x\partial x')\, g^*(x)$ that one aims to estimate accurately; it is not required that all parameters be significant. The coefficients capturing the dynamics tend to be the most significant parameters. We also calculated the autocorrelation and partial autocorrelations for each of the three share equations; the residuals here were white noise. In order to compare the dynamic and static results, we apply the Gallant-Jorgenson chi-square test to provide a comparison with the static equivalent of the time series model. The test statistic uses the value "objective*N" in the table. For the static model (the estimates are not shown here), the value of this statistic is 527.9597; for the dynamic it is 125.0495 as shown in the table. The value of the Gallant-Jorgenson statistic is then 527.9597-125.0495 = 402.9102 with degrees of freedom equal to the difference in the number of parameters, of 27-15 = 12. This calculation decisively rejects the static model.

Table 2

Utility Function Model: Dynamic Fourier Flexible Functional Form

Nonlinear SUR summary of residual errors

Eqn.	DF Model	DF Error	SSE	MSE	Root MSE	*R*-Square	Adjust *R*-square
SM1	9	53	0.01057	0.0001995	0.01412	0.796	0.765
SM2	9	53	0.01885	0.0003556	0.01886	0.826	0.800
SM3	9	53	0.03803	0.0007175	0.02679	0.880	0.862

Nonlinear SUR parameter estimates

Parameter	Estimate	Approximate standard error	"T" ratio	Approximate prob>|T|
B1	−0.163039	0.38557	0.42	0.6741
B2	−1.187580	0.32865	3.61	0.0007
B3	−1.028066	0.35163	2.92	0.0051
U01	0.016757	0.03275	0.51	0.6110
U11	−0.002319	0.01498	0.15	0.8776
W11	0.027816	0.02235	1.24	0.2187
U05	0.002500	0.02955	0.08	0.9329
U15	0.009921	0.01450	0.68	0.4968
W15	−0.064331	0.03009	2.14	0.0372
U02	−0.111290	0.03773	2.95	0.0047
U12	0.061322	0.01864	3.29	0.0018
W12	0.022699	0.01756	1.29	0.2023
U06	−0.008559	0.01160	0.74	0.4637
U16	0.006524	0.00869	0.75	0.4561
W16	0.001616	0.00993	0.16	0.8713
U03	−0.006461	0.01692	0.38	0.7041
U13	0.006416	0.01338	0.48	0.6335
W13	−0.113939	0.00800	1.42	0.1604
U07	−0.013340	0.01127	1.18	0.2419
U17	0.000898	0.00873	0.10	0.9185
W17	−0.016214	0.01118	1.45	0.1530
U04	−0.070695	0.01156	6.12	0.0001
U14	−0.010984	0.01223	0.90	0.3730
W14	−0.070924	0.00896	7.91	0.0001
U08	0.123303	0.00850	1.45	0.1530
U18	0.024230	0.01013	2.39	0.0204
W18	0.020177	0.01199	1.68	0.0982

N = 62
Objective = 2.3732
Objective*N = 147.1362

The dynamic utility model features interaction among the asset choices over time. This characteristic distinguishes the dynamic utility system from the time series approach. For this model the results are not quite as satisfactory as those just given. They follow in Table 2. Here, the *R*-squares are slightly lower, the objective*N statistic is higher, and there are fewer significant parameters. The static Fourier is nested within the dynamic utility function in terms of the multi-indexes (see section four). Consequently, we analyze the reduction in the residuals due to the dynamic specification (see Gallant, 1981). The residual sum of squares from the dynamic model is less than half the size of those obtained from the static model.

Quite often, the methods discussed to this point would be applied to systems of demand equations, as they are here. While the estimated structural equations themselves might be of interest, and for the dynamic versions presented here they could be used to generate forecasts, a typical concern is the elasticity of substitution among the assets. What the Fourier provides in this connection is precise estimates of a set of own- and cross-elasticities of substitution (and income) at each data point. This can reveal the nature of the substitutability or complementarity among the assets and the time-series behavior of this concept.

While we do not wish to explore the fine points of the data set just examined, a further illustration, because it reveals an important characteristic of the dynamic models, is in order. For the more interactive dynamic utility function model, Table 3 presents the estimates of the Allen partial elasticities of substitution among the four commodity bundles studied here. In the table, *Eij* is the elasticity of substitution between *Ai* and *Aj*. The Fourier Flexible Form provides a global approximation and hence we can calculate the asymptotic standard errors for each elasticity *(Eij)* at each point in time and then evaluate the significance of the estimate. The *Tij* in the table are the corresponding *t*-statistics for *Eij*.

Here, we show a complete set of substitution elasticities along with their associated *t*-values. Note that a positive value for the elasticity indicates substitution, while a negative indicates a complementary relation.

Several things stand out in Table 3. Most importantly, the elasticity of substitution between cash and savings assets (*E12* in the table) and between cash and time deposits (*E13*) are very precisely estimated at all data points. This was not the case for static estimates published elsewhere (Fisher, 1992). While we cannot say a priori what value of the elasticity of substitution is high, an elasticity over unity, as most are in the first column of the table, could be termed "elastic." Note that the result here is that cash and savings accounts are substitutes, as many would expect on the basis of intuition.

More provocatively, however, cash and time deposits appear to be "elastic" complements. This spells trouble for a simple-sum M3 definition of money, if these results are correct, since the simple-sum approach to aggregation treats all components as (perfect) substitutes. Clearly, we are not in a position to doubt our results. We have adopted a rigorous aggregation-theoretic approach and tied the empirical work to that as closely as our data would permit. In fact, the very theory we are using can be invoked in our defense: Economic theory does not say whether commodities will be substitutes or complements *in practice*. That is, in practice, economic agents decide what assets are substitutes and what are complements. Our results indicate that they use cash and time deposits as if they are complements, at least over this data sample. We also should point out that this is not an unusual finding in this literature (see the survey in Barnett, Fisher and Serletis, 1992).

Another interesting finding, and one that demonstrates the power of the dynamic approach, is that the elasticities shown in Table 3 are much more stable than those obtained from the static model. For this comparison, we refer to the elasticities produced in the static Fourier from the same data set, as published in Fisher (1992). In Figure 1 we show the results for the substitution relation between cash and savings deposits. Note especially that the two series are scaled differently, an adjustment necessary because the static estimates fluctuate so wildly. While both series are generally positive (indicating that they are substitutes), the static estimates are occasionally negative (although they were not significantly less than zero). This sort of result is not ruled out by economic theory, but is still hard to explain in terms of the known characteristics of these assets.

In Figure 2 we present a comparison between the results for the static Fourier and the dynamic utility model where the former results are, again, drawn from the earlier study. In this case we compare cash *(A1)* and small time deposits *(A3)*, a relation that is consistently that of complementarity in Table 3.

Once again the dynamic elasticities are relatively constant. In addition, the static elasticities are sometimes positive and sometimes negative (and statistically so, in both cases, at some dates). Clearly, then, the complementary relationship between cash and small time deposits is clearly established in the dynamic utility function results. We note that such results are quite common in this literature (see Barnett, Fisher and Serletis, 1992).

CONCLUSIONS

In the introduction to this paper, we listed five areas in which existing studies of expenditure systems frequently fall short, in Diewert's opinion. Obviously, the innovation of this paper is to convert a static system into a dynamic one; this deals with one of his concerns. Diewert is also concerned that existing studies do not link the theory to the application firmly enough. This we have attempted to address both by setting out

Table 3

Substitution Elasticities: Dynamic Utility Model

	E12	T12	E13	T13	E14	T14	E23	T23	E24	T24	E34	T34
70:2	1.291	2.983	−0.250	2.367	0.509	1.394	−0.614	1.106	0.146	1.153	−1.286	14.957
70:3	1.276	3.163	−0.259	2.532	0.445	1.329	−0.601	1.164	0.102	0.857	−1.291	14.973
70:4	1.255	3.389	−0.270	2.751	0.371	1.218	−0.592	1.241	0.049	0.438	−1.301	14.906
71:1	1.167	3.669	−0.272	3.026	0.290	1.107	−0.610	1.462	−0.047	0.491	−1.269	14.076
71:2	1.163	3.744	−0.270	3.071	0.273	1.057	−0.610	1.490	−0.068	0.718	−1.280	14.219
71:3	1.166	3.896	−0.278	3.216	0.233	0.932	−0.595	1.491	−0.084	0.890	−1.307	14.273
71:4	1.115	4.050	−0.279	3.368	0.195	0.832	−0.602	1.609	−0.129	1.480	−1.292	13.771
72:1	1.101	3.896	−0.265	3.202	0.236	0.987	−0.625	1.643	−0.124	1.436	−1.256	13.693
72:2	1.077	4.083	−0.266	3.382	0.188	0.825	−0.619	1.712	−0.165	1.979	−1.271	13.523
72:3	1.052	4.269	−0.273	3.596	0.134	0.623	−0.605	1.763	−0.198	2.450	−1.282	13.165
72:4	1.010	4.451	−0.282	3.873	0.065	0.328	−0.582	1.820	−0.236	3.052	−1.282	12.547
73:1	1.033	4.524	−0.297	4.058	0.046	0.230	−0.568	1.755	−0.226	2.766	−1.314	12.436
73:2	0.995	4.672	−0.335	4.543	−0.047	0.261	−0.527	1.738	−0.238	3.071	−1.320	11.475
73:3	1.041	4.739	−0.446	5.808	−0.150	0.875	−0.421	1.359	−0.125	1.304	−1.383	9.225
73:4	1.008	4.726	−0.465	5.944	−0.194	1.167	−0.412	1.383	−0.141	1.538	−1.375	8.904
74:1	0.920	4.791	−0.507	5.954	−0.302	2.083	−0.381	1.464	−0.176	2.098	−1.323	8.123
74:2	1.152	4.725	−0.426	5.233	−0.074	0.386	−0.419	1.228	−0.067	0.666	−1.434	10.784
74:3	1.284	4.358	−0.364	4.063	0.090	0.375	−0.444	1.091	0.010	0.083	−1.460	13.096
74:4	1.184	4.615	−0.392	4.663	−0.010	0.049	−0.460	1.292	−0.079	0.811	−1.438	12.212
75:1	1.231	4.094	−0.306	3.543	0.169	0.669	−0.548	1.353	−0.074	0.702	−1.397	14.327
75:2	1.216	3.977	−0.281	3.352	0.204	0.790	−0.579	1.416	−0.094	0.898	−1.374	14.565
75:3	1.199	4.134	−0.294	3.577	0.162	0.659	−0.567	1.442	−0.111	1.085	−1.381	14.093
75:4	1.190	4.117	−0.287	3.539	0.168	0.685	−0.576	1.471	−0.121	1.195	−1.373	14.101
76:1	1.169	4.137	−0.282	3.551	0.165	0.683	−0.587	1.526	−0.139	1.413	−1.361	13.984
76:2	1.137	4.249	−0.286	3.713	0.136	0.588	−0.588	1.596	−0.166	1.761	−1.352	13.559
76:3	1.107	4.347	−0.291	3.880	0.107	0.482	−0.586	1.654	−0.189	2.083	−1.346	13.145
76:4	1.055	4.499	−0.303	4.207	0.049	0.240	−0.576	1.738	−0.224	2.632	−1.333	12.371
77:1	1.041	4.487	−0.291	4.157	0.056	0.273	−0.586	1.785	−0.239	2.827	−1.321	12.366
77:2	1.030	4.515	−0.295	4.235	0.041	0.206	−0.581	1.797	−0.245	2.935	−1.319	12.199
77:3	0.997	4.611	−0.318	4.586	−0.020	0.109	−0.555	1.805	−0.257	3.201	−1.316	11.491
77:4	0.977	4.651	−0.330	4.813	−0.054	0.299	−0.539	1.801	−0.262	3.253	−1.313	10.997
78:1	1.003	4.625	−0.323	4.654	−0.022	0.118	−0.549	1.772	−0.248	2.987	−1.322	11.359
78:2	1.031	4.612	−0.328	4.605	−0.008	0.044	−0.546	1.715	−0.227	2.663	−1.338	11.573
78:3	0.982	4.718	−0.420	5.550	−0.159	0.944	−0.463	1.578	−0.202	2.430	−1.349	9.671
78:4	0.994	4.692	−0.527	6.350	−0.274	1.722	−0.357	1.232	−0.104	1.061	−1.388	7.894
79:1	0.992	4.653	−0.569	6.378	−0.329	2.097	−0.319	1.119	−0.079	0.788	−1.401	7.546
79:2	1.000	4.632	−0.576	6.336	−0.335	2.119	−0.313	1.090	−0.071	0.716	−1.407	7.581
79:3	1.067	4.658	−0.545	6.507	−0.260	1.529	−0.335	1.083	−0.051	0.492	−1.437	8.127
79:4	1.139	4.448	−0.618	6.911	−0.319	1.806	−0.237	0.718	−0.062	0.464	−1.528	7.332
80:1	1.194	4.103	−0.634	6.714	−0.289	1.456	−0.212	0.569	0.113	0.782	−1.596	7.133
80:2	1.299	4.479	−0.503	5.621	−0.074	0.332	−0.301	0.751	0.128	0.907	−1.545	9.679
80:3	1.267	4.593	−0.472	5.429	−0.067	0.313	−0.353	0.927	0.052	0.427	−1.510	10.522
80:4	1.265	3.983	−0.590	6.200	−0.164	0.707	−0.224	0.524	0.202	1.035	−1.626	7.546
81:1	1.296	3.675	−0.586	5.700	−0.110	0.409	−0.208	0.427	0.268	1.128	−1.679	7.396
81:2	1.323	3.688	−0.552	5.332	−0.040	0.140	−0.223	0.440	0.289	1.212	−1.677	7.778
81:3	1.382	3.333	−0.581	4.869	−0.024	0.071	−0.135	0.223	0.422	1.295	−1.783	7.238
81:4	1.329	3.513	−0.573	5.281	−0.052	0.172	−0.193	0.364	0.328	1.244	−1.707	7.567
82:1	1.319	3.653	−0.562	5.401	−0.054	0.191	−0.215	0.427	0.292	1.221	−1.680	7.760
82:2	1.338	3.774	−0.532	5.211	−0.011	0.038	−0.238	0.478	0.283	1.286	−1.659	8.378
82:3	1.395	4.130	−0.435	4.379	0.102	0.362	−0.330	0.688	0.207	1.252	−1.577	11.357
82:4	1.349	4.221	−0.380	4.034	0.117	0.443	−0.416	0.940	0.081	0.599	−1.505	13.092
83:1	1.460	3.505	−0.323	2.887	0.358	0.951	−0.444	0.772	0.237	1.454	−1.558	15.850
83:2	1.479	3.353	−0.328	2.730	0.410	0.996	−0.414	0.671	0.324	1.823	−1.593	15.791
83:3	1.485	3.344	−0.341	2.768	0.401	0.953	−0.383	0.610	0.359	1.881	−1.632	15.192
83:4	1.483	3.396	−0.342	2.826	0.385	0.940	−0.388	0.629	0.340	1.832	−1.622	15.251
84:1	1.483	3.421	−0.347	2.874	0.373	0.917	−0.382	0.622	0.339	1.814	−1.628	15.054
84:2	1.493	3.241	−0.326	2.581	0.438	0.978	−0.381	0.580	0.382	1.855	−1.654	15.085
84:3	1.465	3.392	−0.388	3.119	0.318	0.775	−0.317	0.507	0.388	1.744	−1.695	12.801
84:4	1.470	3.640	−0.376	3.254	0.287	0.782	−0.367	0.643	0.296	1.676	−1.623	14.393
85:1	1.471	3.447	−0.329	2.814	0.379	0.971	−0.413	0.696	0.294	1.714	−1.579	15.821
85:2	1.463	3.319	−0.306	2.615	0.429	1.067	−0.444	0.733	0.291	1.733	−1.547	16.458

Figure 1
Substitution Elasticities Between Cash (A1) and Savings Deposits (A2), 1970-1985

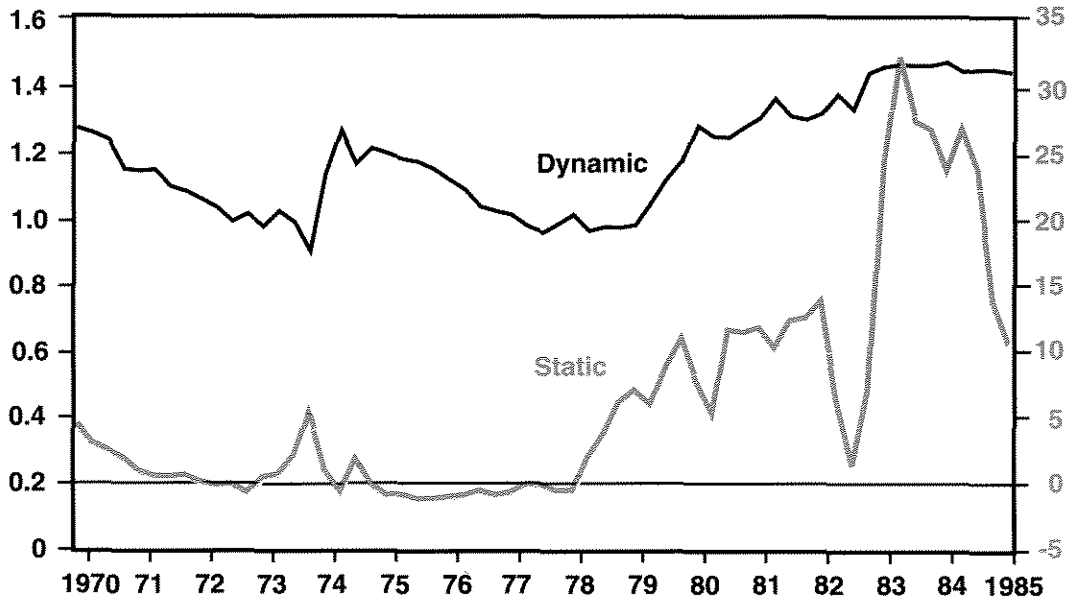
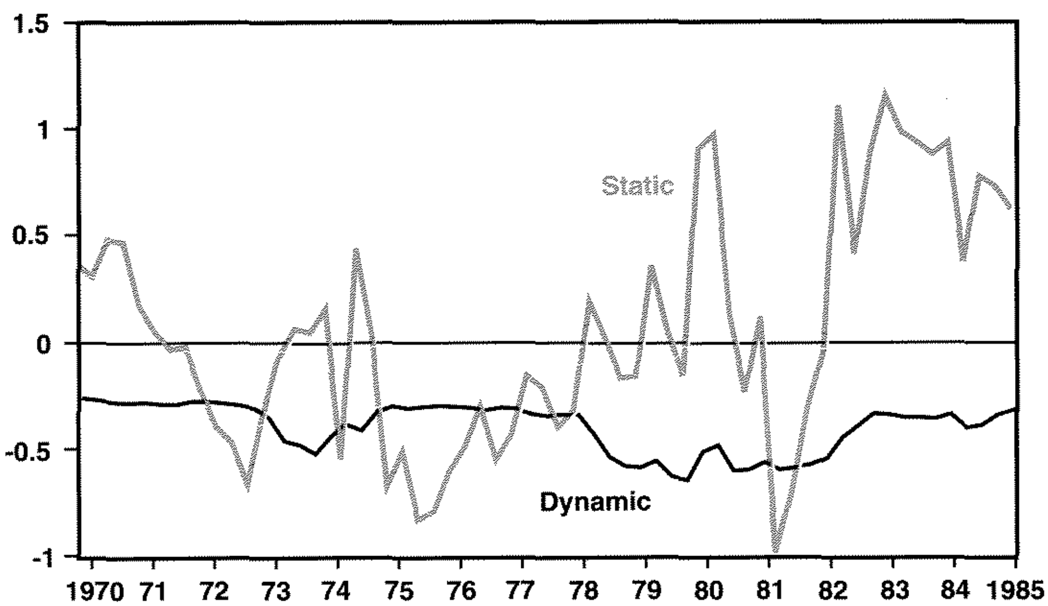


Figure 2
Substitution Elasticities Between Cash (A1) and Time Deposits (A3), 1970-1985



the theory and by dealing with two of his further concerns: aggregating in a consistent fashion and employing a system that provides arbitrarily accurate estimates of the partial derivatives of the system. What we did not do, which is in his list of concerns, is examine the model at every data point.

In our results, the dynamic models derived and estimated appear clearly superior to the (nested) static models. We are not able to compare the two dynamic formulations directly, because they are not nested, but we find the statistical performance of the time series approach to be superior, while the dynamic-utility approach seems better able to capture the economic interactions among the assets studied. Furthermore, most of the estimated share equations produced white noise residuals, and this is a characteristic that is not shared by the static estimates, whether of the nested form in this paper or in the earlier (static) Fourier results that we have been using as a benchmark.

For the dynamic utility model, we have produced a set of elasticities of substitution and charted those between cash (M1 + OCD) and savings deposits and between cash and small time deposits. The former are shown to be substitutes in the dynamic system, and, more important, to be much more stable than static estimates produced in an earlier study. The latter are actually complements, although the negative elasticity of substitution is generally less than minus one, a finding that is not without foundation theoretically. We anticipate that further study of the model and/or the U.S. data will provide further observations on this phenomenon.

REFERENCES

Anderson, G. J. "The Structure of Simultaneous Equations Estimation: A Comment," *Journal of Econometrics* (October 1980), pp. 271–76.

________, and R. W. Blundell. "Estimation and Hypothesis Testing in Dynamic Singular Equation Systems," *Econometrica* (November 1982), pp. 1559–72.

________. "Testing Restrictions in a Flexible Demand System: An Application to Consumers' Expenditure in Canada," *Review of Economic Studies* (July 1983), pp. 397–410.

Barnett, William A. "The User Cost of Money," *Economics Letters* (vol. 1, no. 2, 1978), pp. 145–49.

________, Douglas Fisher, and Apostolos Serletis. "Consumer Theory and the Demand for Money," *Journal of Economic Literature* (December 1992), pp. 2086–119.

Deaton, Angus, and John Muellbauer. *Economics and Consumer Behaviour.* Cambridge University Press, 1980.

Diewert, W. E. "Applications of Duality Theory," in M.D. Intriligator and D.A. Hendricks, *Frontiers of Quantitative Economics.* North-Holland, 1974, pp. 106–206.

Ewis, Nabil A., and Douglas Fisher. "Toward a Consistent Estimate of the Substitutability between Money and Near Monies," *Journal of Macroeconomics* (spring 1985), pp. 151–74.

Farr, Helen, and Deborah Johnson. "Revisions in the Monetary Services (Divisia) Indexes of Monetary Aggregates," *Special Studies Paper* no. 189. Board of Governors of the Federal Reserve System, 1985a.

________, and ________. "Revisions in the Monetary Services (Divisia) Indexes of the Monetary Aggregates," *Staff Study* no. 147. Board of Governors of the Federal Reserve System, 1985b.

Fisher, Douglas. *Money Demand and Monetary Policy.* University of Michigan Press, 1989.

________. "Money-Demand Variability: A Demand-Systems Approach," *Journal of Business & Economic Statistics* (April 1992), pp. 143–51.

Gallant, A. Ronald. "On the Bias in Flexible Functional Forms and an Essentially Unbiased Form: The Fourier Flexible Form," *Journal of Econometrics* (February 1981), pp. 211–45.

________, and Dale W. Jorgenson. "Statistical Inference for a System of Simultaneous, Nonlinear, Implicit Equations in the Context of Instrumental Variable Estimation," *Journal of Econometrics* (October/December 1979), pp. 275–302.

Pollak, Robert A., and Terence J. Wales. *Demand System Specification and Estimation.* Oxford University Press, 1992.

Serletis, Apostolos. "The Demand for Divisia Money in the United States: A Dynamic Flexible Demand System," *Journal of Money, Credit and Banking* (February 1991), pp. 35–52.

Swofford, James L., and Gerald A. Whitney. "Nonparametric Tests of Utility Maximization and Weak Separability for Consumption, Leisure, and Money," *Review of Economics and Statistics* (August 1987), pp. 458–64.

________, and ________. "A Comparison of Nonparametric Tests of Weak Separability for Annual and Quarterly Data on Consumption, Leisure, and Money," *Journal of Business and Economics Statistics* (April 1988), pp. 241–46.

Varian, Hal R. "The Nonparametric Approach to Demand Analysis," *Econometrica* (July 1982), pp. 945–73.

________. "Nonparametric Tests of Consumer Behaviour," *Review of Economic Studies* (January 1983), pp. 99–110.