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A Methodological Approach to Chaos: Are Economists Missing the Point?

"A very slight cause which escapes our notice determines a considerable effect which we cannot fail to see, and then we say that this effect is due to chance."

—Poincaré

THERE IS INCREASING interest among economists in a new field of study that may offer an alternative explanation for the seemingly random behavior of many economic variables. This research, which originated in the physical and biological sciences, concerns a phenomenon called deterministic chaos.[1]

Contrary to the common usage of the word, chaos in this context describes the behavior of a variable over time which appears to follow no apparent pattern but in fact is completely deterministic, that is, each value of the variable over time can be predicted exactly. In fact, one "chaologist" describes chaos as ". . . lawless behavior governed entirely by law."[2]

To demonstrate the difficulty in determining whether a variable is random or chaotic, figures 1a and 1b show two time series of a variable; one series is a random variable, whose actual value cannot be known with certainty, and the other is a chaotic variable, whose value can be predicted with certainty. Even the most practiced observer, however, would have difficulty determining which of these series, if any, is not random. As a result, most economists would model or estimate both time series as random processes. The chaotic series is described by a very simple deterministic equation and identified later in this paper.

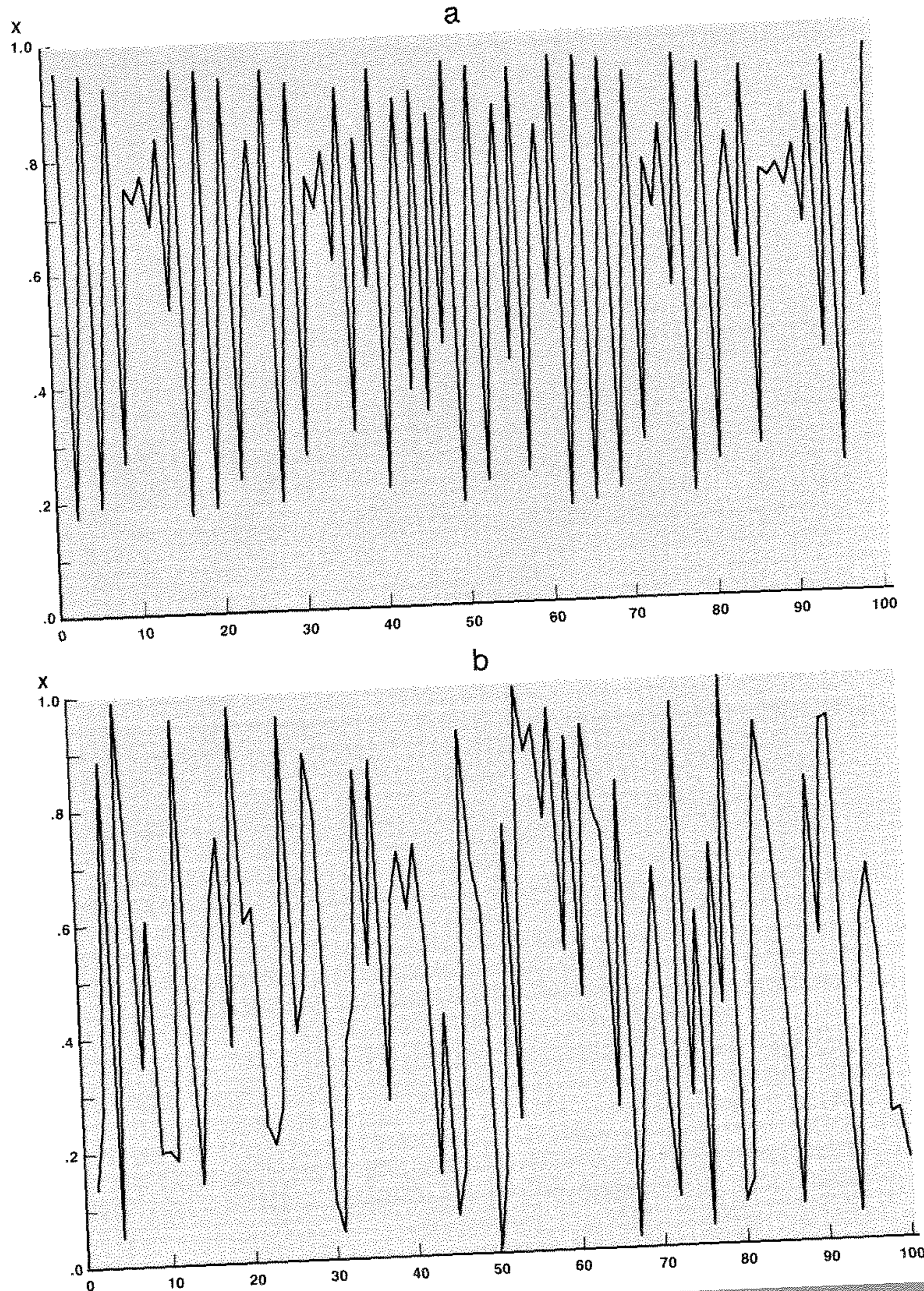
Often, behavior that cannot be explained by standard theories and modeling techniques is attributed to random forces, even when there is no theoretical reason to do so. This paper argues that economists are perhaps not using the appropriate types of models and empirical techniques to explain the behavior of some economic variables and that the choice of methodology needs to be more closely examined.

The study of chaos is a recent phenomenon in the biological and physical sciences and is just

[1]The terms "deterministic chaos" and "chaos" are used interchangeably here, although deterministic chaos is the more precise description.

[2]Stewart (1989), page 17.

Figure 1
Random or Chaotic?



now beginning to be applied to economics. Unfortunately, many of the empirical tests for chaos are imprecise and, because of mathematical constraints, the theoretical models used to generate chaos are generally limited to systems with only one or two explanatory variables. Both of these factors restrict the usefulness of applying chaos to economic systems. Nevertheless, the theory of deterministic chaos has attracted a great deal of attention, both in the popular press and in academic circles. The discussion that follows attempts to clarify some of the issues and suggests some ways to incorporate chaos into economics.

This article first reviews how economic variables typically are modeled by describing and evaluating several techniques of economic modeling using a simple model of output and population growth.[3] Next, chaos is defined and its properties demonstrated. The advantages and pitfalls of applying the theories of chaos to economics are then discussed and illustrated.

ECONOMIC MODELING

There are many different ways to build economic models. Four such possibilities are examined here for the case in which all variables are completely deterministic.[4] The types of models examined here are static linear, static nonlinear, dynamic linear and dynamic nonlinear. A further distinction, which proves to be significant, is also drawn between discrete and continuous time dynamic models. A simple model of output, where labor is the only input, is used to illustrate each approach to modeling as well as the restrictiveness of many common modeling techniques. In addition, focusing on economic modeling allows us to show that chaotic dynamics can only arise in certain types of models that have often been excluded, *a priori*, by economists.

Static Models

The simplest type of economic model is a static linear model, in which variables do not change over time and are related in a proportionate manner. Consider, for example, the following simple production function, which has only labor as an input:

(1) $Y = AN \quad A > 0,$

where Y is output, which is completely consumed by workers (there is no saving or investment), N is labor employed and A is the productivity parameter. This equation states that output is positively related to the amount of labor employed. Given the value of A and the labor supply, the exact value of output can be determined.

This type of model is highly restrictive; any change in labor changes output by a constant percentage. Hence, the production function exhibits constant returns to scale.

Allowing the model to be nonlinear (that is, not necessarily proportionate) provides a more general model in which equation 1 is a special case. An example of a nonlinear production function is given by:

(2) $Y = AN^{\alpha} \quad A > 0, \alpha > 0.$

If $\alpha = 1$, this model is identical to the one shown in equation 1. By not restricting α to equal one, however, this model can be used to examine the case in which output can vary disproportionately with respect to changes in labor. This is illustrated in figure 2, which shows the relationship between output and labor for different values of α (for simplicity, $A = 1$). Notice that if α is between zero and one, the production function exhibits decreasing returns to scale (that is, output increases less than proportionately with respect to a change in labor); if α is greater than one, production is characterized by increasing returns to scale (output increases more than proportionately with respect to an increase in labor). Empirical tests of actual production relationships can be performed to determine if α is actually different from or equal to one.

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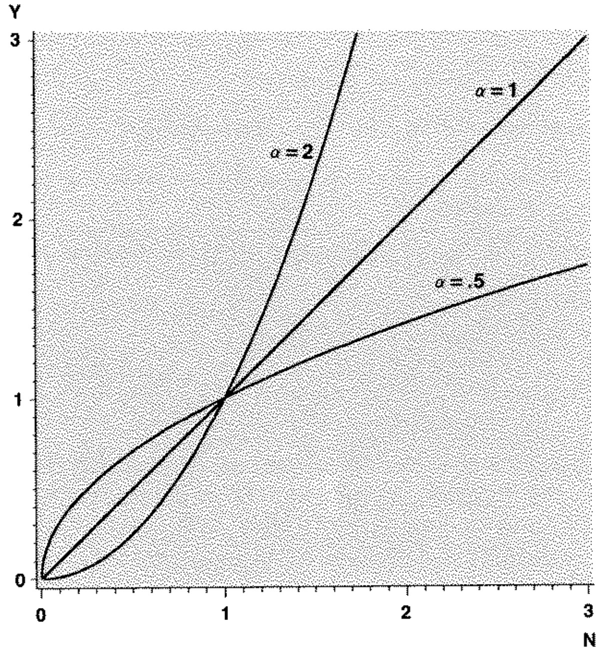
[3]There is a growing body of theoretical literature incorporating chaos into many different types of economic models. These models include Benhabib and Day (1981), Deneckere and Pelikan (1986), Grandmont (1985), De Grauwe and Vansanten (1990), Kelsey (1988), Day and Shafer (1985) and Stutzer (1980). For surveys of the theoretical literature, see Kelsey (1988) and Baumol and Benhabib (1989).

[4]There is also a burgeoning field in stochastic (random) modeling, which incorporates the assumption of randomness used in econometric models into the theoretical literature. Recent papers also look at the properties of chaos in the presence of a random component [see, for example, Kelsey (1988)]. For simplicity, this paper focuses only on purely deterministic systems.

Figure 2
Linear vs. Nonlinear Production Functions
$Y = AN^{\alpha}$



Linear Dynamic Models

One disadvantage of these static models is that they can be used to describe the relationship between output and employment only if the labor force or population remains constant over time.[5] Suppose instead that we want to examine the behavior of output over time as it is related to a continuously changing labor force. A standard equation borrowed from Haavelmo (1954), used to describe the growth of the labor force where that growth is dependent on the level of output, is given by:

(3) $\dot{N}(t)/N(t) = C - DN(t)/Y(t) \quad C > 0, D > 0,$

where C and D are constants, and a dot over a variable means the change in the variable with respect to a very small change in time. This type of equation is called a differential equation.

Equation 3 states that the percentage rate of change of the labor force $[\dot{N}(t)/N(t)]$, where time is continuously changing, equals the difference between the rate of birth, C, and the rate of death, given by $DN(t)/Y(t)$, where $N(t)/Y(t)$ is the number of individuals who have to subsist on each good at time t.

Using the linear production function given in equation 1 and substituting it into equation 3 provides a linear specification of the percentage change in the population:

(4) $\dot{N}(t)/N(t) = C - D/A.$

Notice that when the production function is linear the rate of death, D/A, is constant.

Solving equation 4 yields the following solution for the population:

(5) $N(t) = Ke^{(C-D/A)t},$

where K is an arbitrary constant.[6]

This solution has the property that, unless the rate of birth (C) is *exactly* equal to the rate of death (D/A)—in which case the population will equal K—the population will either rise exponentially or fall to zero. This result is highly restrictive, however, because the likelihood of either the two rates being identical or the population increasing infinitely is, in reality, very small. In other words if $C \neq D/A$, the system is unstable.[7] Unfortunately, in models of other types of economic variables, results that greatly restrict the possible values of the parameters of the models are not uncommon. In addition, because of the complexity of many economic models, the implications of restricting the value of the parameters to determine the solution or to ensure a

[5]For expositional ease, the terms "population" and "labor force" are used interchangeably.

[6]Equation 4 is solved by the variable separable method of solving differential equations found in most calculus books. K is the constant of integration, which can be determined by choosing an initial condition.

[7]In fact, stability is an important issue which is frequently ignored or abstracted from in economics. Stability is important because, for example, an unstable equilibrium is not a sustainable equilibrium. Stability is also important in the choice between linear and nonlinear models. Linear models have three possible cases: stable converging dynamics (such as when C = D/A in the model above), unstable dynamics (when $C \neq D/A$) and cyclical dynamics, which is the least common of the three. In nonlinear models, however, cyclical dynamics are far more common, and exploding dynamics may not occur. Thus, it is also important to consider the desirable and realistic stability properties when choosing a model. Obviously the nonlinear case is more general and the most realistic for variables that exhibit cyclical variation. For the purpose of this paper, however, the issue of stability is ignored.

stable solution are not always as obvious as in the population growth model. Because linear differential equations are far simpler to solve than nonlinear differential equations, and because their solutions are more often stable and easier to interpret, however, they are used in economic models more often than may be appropriate.

Nonlinear Dynamic Models

Combining the nonlinear production function given by equation 2 with the description of population growth given in equation 3 provides a less restrictive model of population growth:

(6) $\dot{N}(t)/N(t) = C - DN(t)^{1-\alpha}/A.$

Unlike equation 4, equation 6 allows the labor force to vary more or less as the current labor force changes. Unfortunately, the price of the generality provided by such nonlinear differential equations is that most either cannot be solved or have solutions so complex the results cannot be interpreted. Not surprisingly, economists often avoid these types of models.

The model used here, however, was chosen for its tractability and can be solved for the value of labor at any time t.[8] All that is necessary for a stable solution is that the production function exhibits decreasing returns to scale ($0 < \alpha < 1$). Regardless of the value of the other parameters, if α is between zero and one, the population will reach a stable equilibrium level. Hence, in contrast to the dynamic linear model discussed previously, the results of this model are more realistic and provide a more general description of population and output growth.

Discrete Models

One problem with using continuous time models in economics is that data are available only in distinct intervals (daily, weekly, monthly, etc.). One approach typically taken by economists, therefore, is to convert these continuous time models into discrete time models. Discrete dynamic models are called *difference* equations; they measure time in distinct intervals rather than the *differential* equations used above, which measure time continuously. Equation 6 can be transformed into a difference equation by letting the rate of change of N (previously given by $\dot{N}$) equal the difference between the value of N at time t and t + 1. Thus, equation 6 becomes

(7) $(N_{t+1} - N_t)/N_t = C - D(N_t/Y_t),$

where $Y_t = AN_t^{\alpha}$.

Combining these equations and simplifying the result yields:

(8) $N_{t+1} = N_t\,[(1+C) - DN_t^{1-\alpha}/A]\,,$

which, following Stutzer (1980), can be rewritten using a change of variables as

(9) $X_{t+1} = k\,X_t(1-X_t^{1-\alpha}),$

where $k = 1 + C$.[9]

The models shown in equations 8 and 9 describe the most general specification of population and output growth given the assumptions made above. Behavior is not restricted to being linear, nor is population or output restricted to remaining constant over time. On the other hand, as noted earlier, these more general models often cannot be solved or have solutions without any economic interpretation. Nevertheless, unless there are theoretical reasons for assuming relationships are static or linear, dynamic nonlinear models, which provide the most general specification of behavior, should at least be considered in economic analysis. Although generality for its own sake is not a desirable goal, using a more general model would be appropriate when simpler models have solutions that are highly unrealistic or when parameters have to be restricted beyond reason (as in the model presented here). In addition, if economists want to test their models and results empirically, then these variables should be modeled in the form in which they are estimated—discrete form.[10] As it turns out, these types of nonlinear dynamic models can exhibit chaos.

[8]For the solution and discussion of this model, see Haavelmo (1954), pp. 24-29.

[9]Allowing $N_t = [A(1+C)/D]^{1/(1-\alpha)}\,X_t$ only changes the scale of the population and has no effect on the general characteristics of the solution. For further discussion of this procedure and the solution, see Stutzer (1980).

[10]Although the model given by equation 6 is stable in continuous time, it is not necessarily stable in discrete time since in discrete time this model can generate chaotic dynamics for certain parameter values. Differential equations can also exhibit chaos, although only in more complicated models. This is discussed in greater detail later.

AN INTRODUCTION TO THE THEORY OF CHAOS

The possibility that chaos exists in economic variables has strong implications for the way in which economics is modeled. For example, some variables that appear to be random processes, like one of the variables shown in figure 1, might in fact not be random at all; instead it might be completely explained using the appropriate deterministic model. This section demonstrates the properties of chaos, using a simple model.

In the most general sense, the term chaos is used to describe the behavior of a variable over time that appears random but, in fact, is deterministic; more precisely, given the initial value of the variable, all future values of the variable can be calculated with exact precision.[11] In contrast, the value of a random variable can never be predicted with certainty.

More formally, a function is chaotic if, for certain parameter values, the following two conditions hold: First, the function never reaches the same point twice under any defined interval of time. In this case, the function is said to exhibit *aperiodic* behavior. Second, the time path is sensitive to changes in the initial condition, so that a small change in the value of the initial condition will greatly alter the time path of the function.[12]

Chaos only arises in certain types of nonlinear dynamic systems, although not all nonlinear dynamic equations are chaotic. Moreover, equations that can be characterized as chaotic need not exhibit chaos for all parameter values. Rather, functions that can exhibit chaos will do so only for certain parameter values. This is explained by example below.

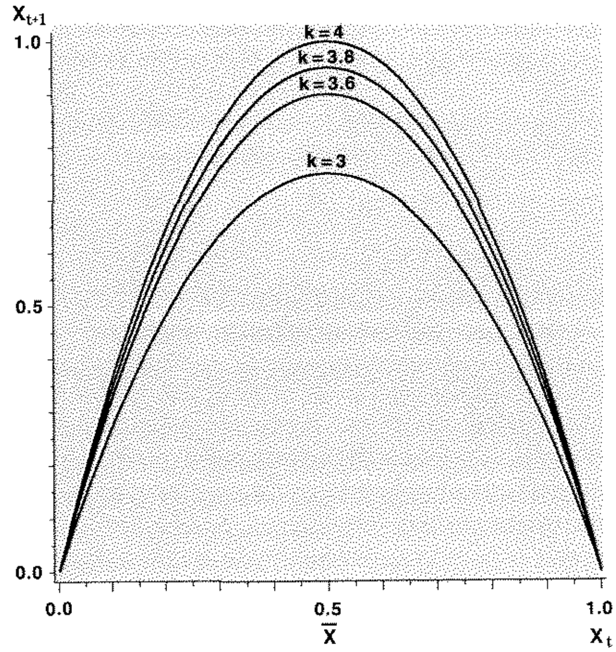
And Now for Something Completely Different . . .

The properties of chaos can be demonstrated using a simple mathematical equation, called the logistic growth equation. While this model has no particular economic interpretation, it is the simplest model that exhibits chaos and provides reasonably interpretable graphical results. This equation is given by:

$$(10)\quad X_{t+1} = k\,X_t(1-X_t),\ 0 < X_t < 1,\ 0 < k < 4.$$

Equation 10 describes the time path of a variable, X (which for expositional purposes is called a population), that is a function of its previous value and a parameter k. To demonstrate chaotic behavior in this simple framework, the value of X can only take on values between zero and one. The value of k, the only parameter in the equation, is called the "tuning" parameter; it determines the steepness of the function. Figure 3 shows the function given in equation 10 for various values of k. Increases in the population below $\bar{X}$ increase future values of X more than proportionately. Past this point, the population begins to decrease.[13] For larger values of k,

Figure 3
The Logistic Growth Curve For Various Values of k
$X_{t+1} = kX_t(1 - X_t)$

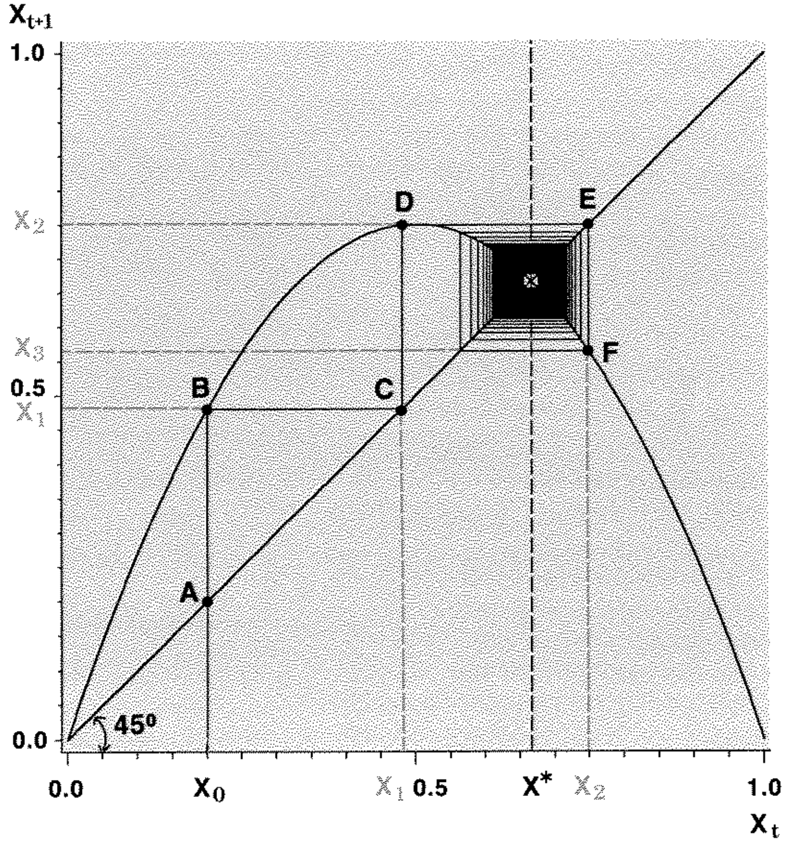


[11]For simplicity, only single-variable equations are discussed. Although chaos exists in multivariate economic systems, tests for chaos in these systems are just beginning to be developed, and the mathematics of such systems are extremely complex.

[12]There are many different characterizations of deterministic chaos, but they all include the one used here. For more rigorous definitions and discussion of the different definitions, see Li and Yorke (1975), Brock and Dechert (1988) and Melese and Transue (1986). For a good mathematical description of chaos and the mathematical tools used in the theory of chaos, see Devaney (1989).

[13]This behavior is similar to that of a total product curve where, once the marginal product becomes negative, further increases in an input decreases output.

Figure 4
A Stable Time Path for a Logistic Growth Curve
$X_{t+1} = 3\,X_t(1 - X_t)$
$X_0 = .20 \quad t = 1 \text{ to } 500$



the absolute value of the rate of change of X is larger.

For certain values of the tuning parameter ($k \leq 3$), the system is stable; this means the population will reach some sustainable steady-state value which differs from $\overline{X}$.

Figure 4 illustrates how the time path for X_{t+1} is solved graphically. The parabola represents equation 10 when k is equal to three; all values of X_t and X_{t+1} must lie on this curve. The 45-degree line depicts the points where $X_{t+1} = X_t$, which is required for a steady-state equilibrium. In this example the initial value (when $t = 0$) is .20. To determine the value of X_1, draw a line between the initial value (X_0) and the parabola (line segment X_0B). To find the value of X_2, set $X_t = X_1$ by drawing a line from point B to the 45-degree line (point C). Then draw a straight line from point C to the parabola. This is the value of X_2 (point D). This process, called iteration, can be used to determine as many subsequent values of X as is desired, once the initial value is determined.[14] As we can see in figure 4,

[14]Notice that X_{t+1}, which must always lie on the parabola, can be either above the 45-degree line (as in point D) or below it (as in point F). For precision, the equation is solved numerically and then graphed.

the population appears to be converging to a steady-state equilibrium value at 2/3 (X^*).

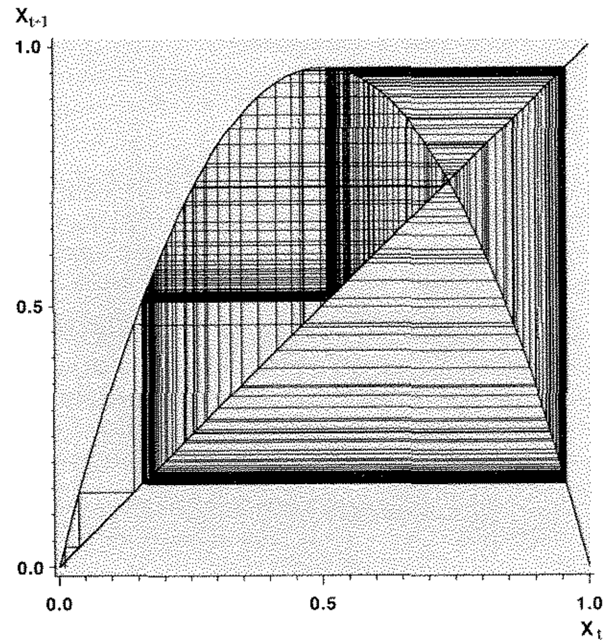
If the value of k increases past three, however, the equilibrium point becomes unstable and the time path exhibits a two-period cycle, where the variable alternates between two values. Further increases in k produce a four-period cycle (that is, the time path repeats the same sequence of numbers every fifth iteration), then an eight-period cycle, and so on, with the periodicity increasing by 2^n (n = 1,2,3, . . .). If k increases past a certain point called the "point of accumulation" (for this function, it occurs at k = 3.5700), the time path enters into a region in which the function can exhibit chaos.[15] In the chaotic region ($3.57 < k < 4$ for this function), there can be both an infinite number of periodic cycles and an infinite number of initial conditions that produce an aperiodic time path.[16] Using this simple example, we can demonstrate some of the properties of chaos in graphical form.

Properties of Chaos

An example of aperiodic behavior is seen in figure 5. The first 500 iterations are shown in this figure (that is, t = 1, 2, . . . 500), and *no single point is ever reached twice.*[17] In fact, no matter how many times this equation is iterated, X_t never has the same value twice.[18] If the data are plotted as a time series, it would look similar to figure 1a, the chaotic series in figure 1, and one might conclude that the data are generated by a random process, such as figure 1b, because they follow no obvious pattern. This is not the case here; the data in figure 1a and figure 5 were generated from models without a random component and therefore are completely deterministic.

The other characteristic of a chaotic function is that its time path is sensitive to the choice of initial values. An example of how changing the initial condition can affect the time path is shown in figure 6. In this figure, the values of X_t are plotted against time, as in figure 1. This diagram demonstrates how changing the initial value, X_0, at the fourth decimal place (from .0101 to .0100) causes the time paths generated by equation 10 to deviate substantially from each other.[19] Although not all sections of the time path differ as dramatically as the one shown here, figure 5 graphically demonstrates that the choice of an initial condition or, for forecasting purposes, the choice of a time interval (that is, determining where to start the sample), can greatly alter the results. In fact, despite

Figure 5
A Logistic Growth Curve Exhibiting Chaos
$X_{t+1} = 3.82840\, X_t(1 - X_t)$
$X_0 = .0101 \quad t = 1 \text{ to } 500$



[15]This process of increasingly complex periodicity is called bifurcation and is discussed in most papers on chaos. For a nontechnical discussion of bifurcation, see Gleick (1987) and Stewart (1989). For a more analytical treatment of bifurcation, see May (1976) and Baumol and Benhabib (1989). A more rigorous discussion of the relationship between bifurcation and chaos is given in Li and Yorke (1975).

[16]Notice that not every initial condition gives rise to an aperiodic time path.

[17]This property is unlikely to be found in actual data, however, because of rounding. Although theoretically, aperiodicity is required for chaos, tests for chaos in actual data take a different approach, thus avoiding the problem.

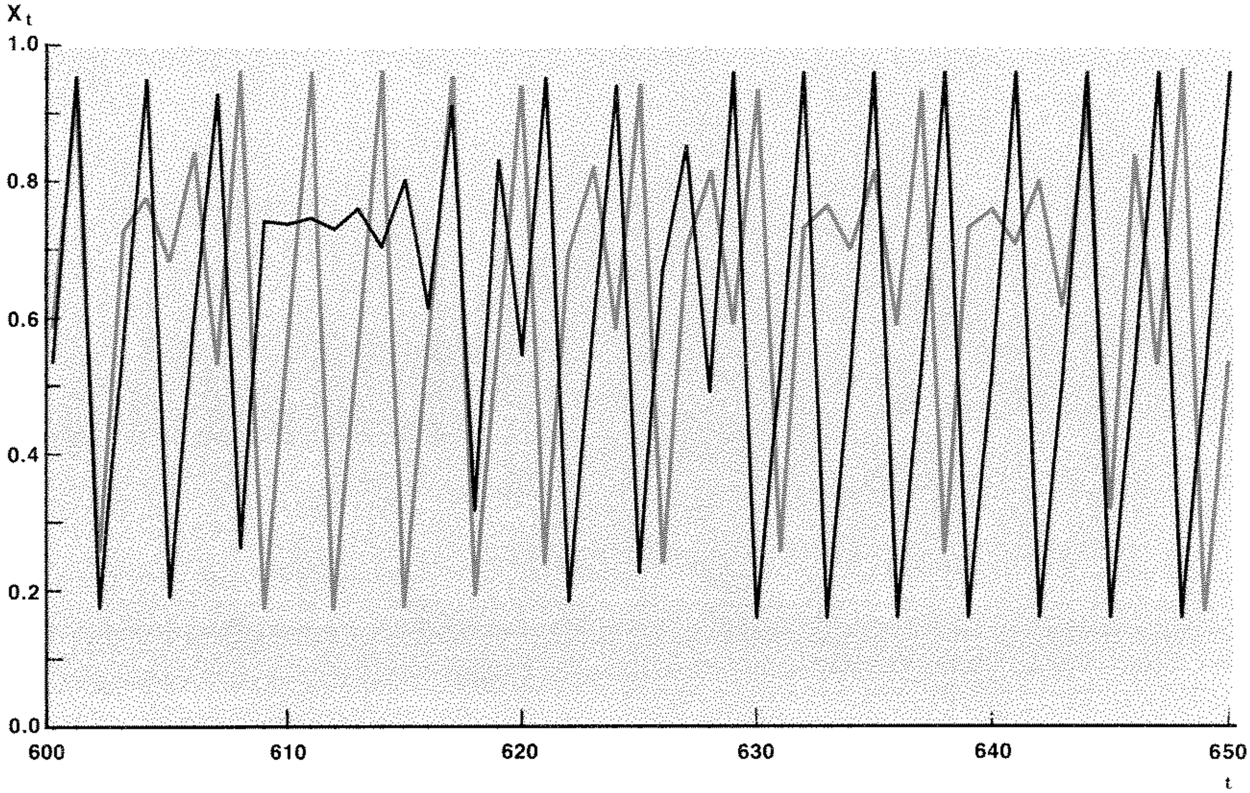
[18]The time path can avoid having repeat values because the number of possible points between zero and one is infinite.

[19]Although it sometimes looks like the function is periodic, this appearance is a result of the lack of precision of the printer and the scale of the graph. In fact, there are no periodic points in this function.

Figure 6
A Segment of the Time Trend Showing Sensitivity to Initial Conditions

$X_{t+1} = 3.82840\ X_t(1 - X_t)$

$X_0 = .0101$ $X_0 = .0100$



seemingly trivial differences in the initial conditions, the time path produced by one initial value will not necessarily be similar to the time path generated by a marginally different initial value. In general, the two time paths that arise from the different initial values will have periods during which they are arbitrarily close together and periods during which they deviate substantially.

Chaotic functions also exhibit sensitivity to very small changes in the parameter values. A third- or fourth-order change in the value of a parameter can alter the time path from stable to chaotic or vice versa.

Sensitivity to changes in the parameter values is illustrated in figure 7. Here, a fifth-order change in the value of the tuning parameter (from 3.82840 to 3.82844) produces not only a substantially different time path from the one in figure 5, but also one that exhibits periodic rather than chaotic behavior.[20]

Are Attractors Strange?

Another feature often found in chaos, although neither necessary or sufficient for chaos, is a strange attractor.[21] The properties of attractors and strange attractors are best illus-

[20]Recall that when a function is in a chaotic region (that is, when the parameters are such that the function can exhibit chaos), there can be both periodic and aperiodic time paths.

[21]The only examples of chaos without the presence of a strange attractor are found in certain types of dissipative systems.

Figure 7
A Logistic Growth Curve With Periodic Points
$X_{t+1} = 3.82844\ X_t(1 - X_t)$
$X_0 = .0101 \quad t = 1 \text{ to } 500$

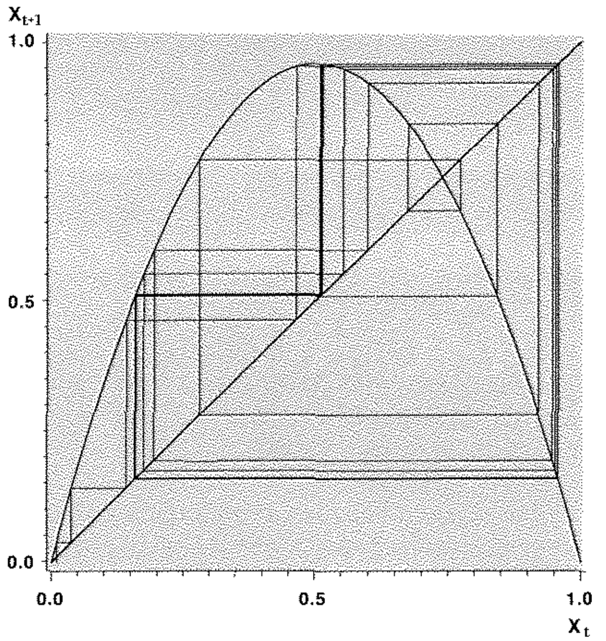
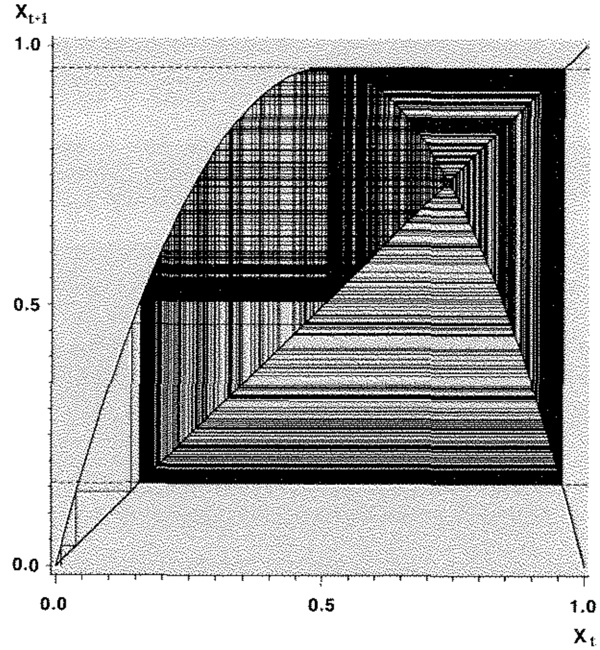


Figure 8
The Strange Attractor for a Chaotic Function
$X_{t+1} = 3.82840\ X_t(1 - X_t)$
$X_0 = .0101 \quad t = 1 \text{ to } 1500$



trated by example. In a stable system, the time path converges to an equilibrium point (for example, X^* in figure 4). The equilibrium point is also called the attractor, because the time path is "attracted" to the equilibrium point. Another possibility is that the time path has two attractors, and the system oscillates between them, never remaining at one equilibrium point. This is found in predator/prey population models, where the population grows until it is so large it begins to die off and then shrinks to a level so small it begins to grow again.

A "strange attractor" is the name given to the case where there is a region, rather than a finite set of points, that attracts the time path of the variable. That is, after some number of iterations, which varies depending on the function, the time path of the variable is completely contained in this region (the strange attractor). Thus, even though the path is aperiodic and therefore never reaches an equilibrium in the standard sense, it also never leaves the strange attractor and therefore is not unstable (for example, never goes to positive or negative infinity). An example of this is shown in figure 8, which takes the same numerical example as in figure 5, but iterates it 1500 rather than 500 times. In this picture, the values of X are still contained in the same area as in figure 5, but the distribution of points is becoming denser. The bounded region (shown by the dotted line in figure 8) is the strange attractor for this function. If the function is iterated further, the area within the bounded region would appear to be a solid block, although the function would never have the same value twice. In fact, the existence of a strange attractor is an important way to distinguish between a random and chaotic time path.[22]

[22]Another definition of a strange attractor is an attractor with fractal dimension. In fact, if a strange attractor exists, the variable has fractal dimension. Random variables have infinite dimension, however. As a result, tests for dimension are one of the main ways data are tested to determine if they are chaotic. For the purpose of this paper, however, the issue of fractals and fractal dimension will be ignored. For a discussion of these topics, see Mandelbrot (1983) and Gleick (1987).

LESSONS FROM CHAOS

Although economists are beginning to incorporate chaos into their economic and econometric models, there has been little discussion of the ways in which chaotic dynamics are useful or realistic for economic models. Clearly, chaos holds considerable appeal for economists who are looking for a deterministic explanation of the apparent randomness in economic variables. Economists frequently assume randomness when they are unable to explain the behavior of an economic variable empirically. The presence of an alternative explanation, chaos, will require them to consider more carefully the rationale behind their assumptions.

One problem with incorporating chaos into economics is that, while economists can either postulate an equation and test it for the presence of chaos or, alternatively, see if the data themselves are chaotic, it is especially difficult to identify the correct functional form that characterizes the data. The choice of a functional form is always a problem in economics, but, as previously discussed, it is particularly difficult to model nonlinear dynamics. This problem is exacerbated because, as a result of the mathematics required, the study of nonlinear dynamics has, until recently, been relatively limited in general and largely ignored in economics.[23]

Even when it is possible to estimate nonlinear dynamic equations, the models themselves often cannot be solved analytically. Without explicit solutions to these models, their usefulness is extremely limited. Obviously, the difficulty of determining the "true" underlying model from a data series is a problem whether or not chaos exists. The "discovery" of chaos, however, has focused much more attention on this problem, especially if the data are nonlinear.

Economic Modeling and Chaos

The study of chaos emphasizes the importance of rigorously modeling the dynamics of a system rather than merely taking a static model (like equations 1 and 2) and adding time subscripts and an error term. Although these simpler models may be more likely to have solutions with explicit results that can be tested empirically, the dynamics that arise may not capture the behavior of the variable of interest. The richness of a model may be found in explaining the behavior of a variable over time as much as in the direct, time-independent (or time-constant) relationship between the variables.

In addition, the study of deterministic chaos illustrates some of the pitfalls of first differencing a dynamic model to convert it to discrete time, as was done in the model of population growth presented above. This practice is common in economics because data are only available in discrete intervals.

As is shown in Stutzer (1980) and demonstrated here, there are first-order differential equation models (such as equation 6) which converge to a steady-state equilibrium that are chaotic when expressed in discrete time (equation 9). Thus, the dynamic properties of the discrete analog of a differential equation cannot be assumed to be the same. In fact, it has been shown that, although chaos can arise in first-order *difference* equations, it can only arise in third-order or higher *differential* equations.[24] As a result, an economist must be careful about either converting a continuous time dynamic model into a discrete model (such as converting equation 6 into equation 7), or taking a static model and simply adding a time subscript, rather than postulating a model that is dynamic (in either discrete or continuous time) and estimating or solving it in that form. The choice of the appropriate type of model should depend on the economic variables being described rather than analytical convenience. This issue is particularly important if a continuous-time dynamic model is estimated in discrete time using the steady-state equilibrium properties of the continuous-time solution. The discrete-time equation that is being estimated may not reach a steady state at all, or the solution could differ qualitatively from that found in the continuous-time version of the model.

[23]For recent work in nonlinear dynamics, see Grandmont (1987).

[24]The "order" of an equation refers, for a differential equation, to the highest power attained by the derivative and, for a difference equation, the highest degree of differencing. For a more complete discussion, see Chiang (1984).

Empirical Applications of Chaos in Economics

There are generally two approaches used in the empirical literature to test for the presence of deterministic chaos in economic and financial data. The first approach tests for the presence of nonlinearities in the data.[1] Since chaos only arises in nonlinear systems, finding nonlinearities in the data suggests that testing directly for the presence of chaos is appropriate. In addition, the presence of nonlinearities in the data provides information to theorists modeling these types of economic systems. Because testing for nonlinearities in the data is much simpler (and less controversial) than testing for chaos, these tests are often performed first.

Many macroeconomic time series have been found to behave in a nonlinear manner. Brock and Sayers (1988) find such evidence in data for quarterly employment (1950-83), quarterly unemployment (1949-82), monthly post-war industrial production and pig-iron production (1877-1937). Nonlinearities have also been found in the Divisia M1 monetary aggregates.[2] Other studies have found nonlinearities in financial data as well. For example, Hinich and Patterson (1985a, 1985b) find strong evidence of nonlinearity in daily stock returns.

The second approach is to test directly for the presence of chaos.[3] There are many problems with testing directly for chaos using economic data, however. The most obvious, and perhaps the most important, is the sensitivity of chaotic systems to small changes in the parameter values and initial conditions. For these tests to be accurate, the data need to be especially exact. This degree of precision presents a particular problem for economics, where controlled experiments are essentially impossible, especially on the macro level. Data collection is far from perfect, and the quality of the data declines as the degree of aggregation increases, introducing measurement error in the data. In addition, because of rounding, the data are not as precise as they should be. For this reason, tests for chaos are not simply tests for aperiodicity.

The quantity of high-quality data is also extremely important. Even if the results show aperiodicity for a sample of 100 observations, the system need not be aperiodic. The existing empirical tests for the presence of chaos require an extremely large number of highly accurate data. Rarely are both of these available to econometricians. As a result, any evidence from tests for chaos should be viewed with caution.

Given these caveats, some statistical tests, originating in the physical sciences, do look for the presence of chaos in economic data. These tests are run on variables that have long time series available, are not aggregate variables, and are thus more likely to provide accurate results. Tests have found evidence consistent with chaos in exchange rates (Ellis, 1990), daily gold and silver prices on the London market (Frank and Stengos, 1988) and in the Divisia monetary aggregates (Barnett and Chen, 1988).[4]

[1]For a discussion of the tests used, see Brock and Sayers (1988), as well as the other papers cited above.

[2]For a definition and discussion of the Divisia monetary aggregates, see Barnett and Spindt (1982).

[3]A description of the actual empirical techniques used to test for chaos is beyond the scope of this paper. For a description of the tests available for chaos and their drawbacks, see Barnett and Hinich (forthcoming), Brock (1986) and Ramsey (1989).

[4]This is by no means a comprehensive survey of the empirical literature applying chaos to economic and financial data. For a more comprehensive discussion of the empirical work on chaos, see Barnett and Hinich (forthcoming) and Ramsey (1989).

Econometrics and Chaos

The study of deterministic chaos also offers several lessons for econometricians. If forecasting is a goal of economic modeling, inappropriate modeling techniques in the presence of chaos become more costly. If the data are chaotic, forecasting is close to impossible since a small error in the value of the initial condition can lead to highly inaccurate predictions (see, for example, figure 6). Similarly, an error in any parameter value can also produce incorrect forecasts (see figures 5 and 7). Thus, it is important to realize the limitations of economic forecasts in the presence of chaotic variables.

Chaos does not have to be present in the data to find the sort of fluctuating behavior (although without any clearly defined periodicity) that is often found in economic data. Nonlinear nonchaotic models often can generate time paths that appear random, and testing for nonlinearities is the likely next step for future research in this area. In fact, empirical economists are beginning to test for both nonlinearities and chaos in economic data (see insert on page 47). As a result, more work needs to be done in understanding nonlinear estimation so that economic models can describe a greater variety of behavior and be more accurate as well. In addition, the existence of chaos suggests that economists might want to try nonlinear specifications of a variable before resorting to modeling it as a random variable. This in turn will help to improve the quality of economic forecasts in the presence of nonlinear variables.

In addition, the use of a random component in estimation does not necessarily imply that the variable itself is random, but rather that other relevant variables might be excluded from the regression. Although each of these other variables could have a small influence on the system by itself, the total effect of these excluded variables could be substantial. Given both the difficulty in detecting what these missing variables might be and data limitations, such a complex system might best be approximated by a random variable, even if there is no true randomness in the variable being estimated. In fact, some argue (see, for example, Kelsey, 1988) that, since economic models do not include such (chaotic) phenomena as weather and other biological factors which can influence economic variables, it "seems inevitable that we will have random terms in our equations."[25]

CONCLUSION

The study of deterministic chaos and its subsequent application to economics has opened a new realm of possibilities for economists trying to explain cyclical or erratic behavior in economic variables. As discussed above, chaos has implications for both theoretical modeling and empirical applications in economics. By illustrating explicitly how restrictive the assumption of linearity can be, the study of chaos emphasizes the importance of allowing for the possibility of nonlinear behavior. The use of chaos in economics also has offered new explanations for behavior that, until recently, has been able to be explained only by random forces.

The techniques that have arisen from the study of chaos in the physical and biological sciences are in their infancy. As these techniques become more refined, and economists become better trained in working with these types of models, their ability to explain the behavior of variables such as exchange rates, business cycles and stock prices is likely to improve. That possibility alone is sufficient reason for economists to take a closer look at deterministic chaos in particular and nonlinear dynamics in general.

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[25]Kelsey (1988), p. 12.