Optimal Monetary Policy under Negative Interest Rate

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Abstract

In responding to the extremely weak global economy after the financial crisis in 2008, many industrial nations have been considering or have already implemented negative nominal interest rate policy. This situation raises two important questions for monetary theories: (i) Given the widely held doctrine of the zero lower bound on nominal interest rate, how is a negative interest rate (NIR) policy possible? (ii) Will NIR be effective in stimulating aggregate demand? (iii) Are there any new theoretical issues emerging under NIR policies? This article builds a model to show that (i) money injections can remain effective even when the nominal bank lending rate has reached zero or become negative; (ii) it is a good policy to keep the nominal interest rate as low as possible by purchasing government bonds with money; and (iii) the conventional wisdom on the notion of the liquidity trap and the Fisherian decomposition between the nominal and real interest rate can be invalid.

Keywords: Monetary Policy, Quantitative Easing, Liquidity Preference, Liquidity Trap, Banking, Money Demand.

JEL codes: E12, E13, E31, E32, E41, E43, E51.

1 Introduction

In responding to the prolonged weak aggregate demand (especially in investment) during the post-financial-crisis and global secular stagnation period, many industrial nations have considered or actually implemented negative nominal interest rate policies. For example, countries that have already adopted negative interest rate policies include Denmark, Hungary, Japan, Sweden, and Switzerland (Figure 1). In fact, the nominal interest rate in the entire euro area has been negative since 2014, with Denmark and Switzerland having the lowest level at 750 basis points below zero (-0.75%).
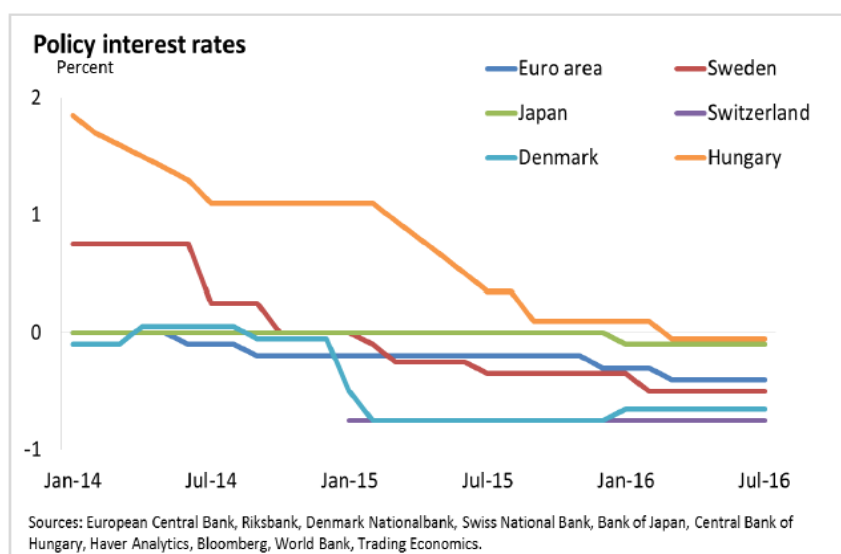
Figure 1. Negative Interest Rates Across Some Industrial Countries.

This situation raises two important questions for monetary theories: (i) Given the widely held doctrine of the zero lower bound on nominal interest rate, how is a negative interest rate (NIR) policy possible? (ii) If NIR is possible, will it be effective in stimulating aggregate demand? (iii) Is it desirable to keep the nominal interest rate so low for so long?

This article uses a theoretical model with an explicit micro-founded money market to shed light on these issues. It shows that the answers to the above questions are soundly "Yes." Namely, it is possible for the nominal interest rate to go below its "zero lower bound (ZLB)"; it is possible for the conventional money policies to remain effective in stimulating aggregate output even when

the nominal interest rate has become zero or even negative; and it is desirable to keep the nominal interest rate as low as possible by taking the government bonds out of the money market and replenish them with plenty of cash.

These results are counter to a large body of the existing theoretical literature. For example, a growing literature argues that the correct policy response to the post financial crisis is to issue (supply) plenty of government bonds to meet the liquidity demand of firms and households. Yet the governments of most industrial nations have tried to keep the nominal interest rate as low as possible by actively buying back their public debts. The rationale behind the theory is that (i) government bond is as liquid as cash in serving as a store of value but better than cash in serving as collateral for relaxing borrowing constraints, because bond pays a positive nominal interest rate. Therefore, a high interest rate, rather than low interest rate, is optimal and desirable to improve the efficiency of credit allocations, especially during a recession.

We disagree. The intuition behind our results are as follows. First, interest rate on loans imposes a big cost on borrowers, especially during recessions. Second, the largest bond holders in any economy are not firms or households, but the commercial banks; thus, bond yields directly affect banks' lending rates. Therefore, policy makers' reactions around the world to the great recession and stagnation by lowering the nominal interest rate is the right thing to do.

But how low can the nominal interest rate be? We show that if there are costs of carrying money by the private sector, then the nominal interest rate of lending institutions such as commercial banks can be negative. Furthermore, such a negative interest rate policy can be effective in stimulating aggregate output, and it is desirable.

However, most existing theories argue that zero is the lower bound on nominal interest rate and that this zero lower bound can be reached only if the economy is in a liquidity trap at which further monetary injections would have no effect because real money demand is infinite under the Friedman rule (see Krugman, 1998).

This argument can be false. Under borrowing constraints and with financial intermediation, the demand for money is not necessarily infinite at or below the zero lower bound (ZLB) of the nominal interest rate, in sharp contrast to the conventional wisdom. This happens because agents can always insure themselves against idiosyncratic risk by borrowing in the credit market (through financial intermediation) without the need to carry infinite amount of cash when nominal interest rate on credit is zero (or negative). In addition, since loan payment is still a burden despite low interest rate, agents opt to bear only a finite amount of nominal debt even with zero or negative interest rate on the loan. This means that there is still room for monetary authority to provide liquidity to further relax borrowing constraints and stimulate consumption and output even in a "liquidity trap" where the nominal interest rate has reached its zero lower bound or even become negative.

We demonstrate these ideas in a micro-founded general-equilibrium model of Bewley (1980) and Lucas (1980). We show that the conventional understanding on the notion of "liquidity trap" is incomplete. The conventional view of the liquidity trap is a situation at the Friedman rule where the real rate of return to money equals the time discount rate. In this case, the conventional wisdom holds that agents opt to hoard infinite amount of nominal balances so that standard borrowing constraints (such as the cash-in-advance constraints) no longer bind. In such a "liquidity trap" equilibrium, further monetary injections would have zero effect in stimulating consumption or investment (despite sticky prices) since agents are already satiated with money (liquidity).

We show, however, that this conventional wisdom is not valid in general. In our model, not only the Friedman rule is not a necessary condition for a liquidity trap, but also the demand for money is not necessarily infinite at the Friedman rule and borrowing constraints can still be binding for the liquidity constrained agents. Consequently, the aggregate economy remains responsive to lowering the nominal interest rate further below the ZLB. How much can the nominal interest rate go below the ZLB depends on the costs of holding money by the private sector.

We also show that the conventional Fisherian decomposition between the nominal and real interest rates is not necessarily correct or economically meaningful. The conventional Fisherian relationship holds that the nominal interest rate and the inflation rate form a log-linear relationship with the time discount factor ρ, namely, $1 = e^{-\rho}\frac{1+i}{1+\pi}$, or approximately $i = \rho + \pi$. Thus, the ratio $\frac{1+i}{1+\pi}$ or the difference $(i - \pi)$ would measure (or "define") the real rate of return to money (or to any nominal financial assets). We show that this notion of the real interest rate is a model-dependent object that does not hold in general and not even make economic sense in certain situations. For example, in our theoretical model the analogous Euler equation ("Fisherian relationship") is given by $1 = e^{-\rho}h(i, \pi)$ where the ratio $\frac{1+i}{1+\pi}$ is still an increasing function of the inflation rate. That is, keeping ρ constant, the the nominal rate i increases more than one-for-one with the inflation rate π. In this case, defining $\frac{1+i}{1+\pi}$ as the "real" interest rate becomes misleading.

This conventional view about the Fisherian decomposition is often rationalized by the intuitive argument that when a competitive lender issues a one-dollar loan to a borrower, she opts to add an inflation component to the nominal interest rate so that she can break even in purchasing power (real return) when the loan is repaid one period later. In this competitive setting without time discounting, the real rate of return to a dollar is ensured to be one: $1 = \frac{1+i}{1+\pi}$. Thus, the competitive nominal interest rate is related to the inflation rate one-for-one and the real interest rate should be defied as $\frac{1+i}{1+\pi}$, or approximately $(i - \pi)$.

However, suppose the financial market arrangement is such that the competitive market interest rate i is a twice increasing function of the inflation rate, i.e., $\frac{\partial^2 i}{\partial \pi^2} > 0$, such that when the inflation

rate rises by one percentage point the competitive market interest rate rises by more than one percentage point. In this case, the proper definition of the real interest rate should no longer be $\frac{1+i}{1+\pi}$ or $(i - \pi)$ but something else. And in this situation if the lender continues to define the real lending rate as in the traditional way, she would make negative profits.

Most importantly, we find that in the absence of any risk taking behaviors, the optimal monetary policy is always to keep the market interest rate for loans as low as possible. This is in stark contrast with a large body of the existing literature, which shows that with heterogeneous agents the optimal monetary/fiscal policy is to keep the interest rate as high as possible so as to achieve constrained efficiency in credit allocations (see, e.g., Williamson, 2012; Wen, 2014). The rationale behind this result is that under borrowing constraints and constant returns to scale production technologies, the productive agents are unable to borrow at full capacity. As a result, the equilibrium market interest rate is too low such that some less productive agents also opt to borrow — because they are unable to benefit from lending out their credit resources under the low interest rate. Thus the government should issue enough bonds to push up the market interest rate to enable only the most productive agent to borrow and the rest to lend, thus correcting the misallocations of credit resources (market failures) under borrowing constraints.

However, in our model lenders (depositors) do not benefit directly from a higher lending rate, while credit borrowers are penalized by high lending rates. Therefore, the optimal policy should be to reduce the nominal interest rate to stimulate public borrowing by increasing money (or decreasing bond) supply. The low interest rate policy will achieve efficient risk sharing between lenders and borrowers.

These findings are demonstrated in this paper in an analytically tractable infinite-horizon model of money and banking. In what follows, Section 2 presents the benchmark model to show how the nominal interest rate is determined in the money market equilibrium, Section 3 discusses the issue of negative interest rate, Section 4 studies the model's implications for monetary policy, Section 5 shows the effectiveness of conventional monetary policy under negative interest rate, and Section 6 concludes.

2   The Benchmark Model

The model is based on Dong and Wen (2017) and Wen (2015), which is a stochastic general-equilibrium version of Bewley (1980, 1983) and Lucas (1980).[1] The key friction in the Bewley-Lucas model is no-short-sale constraint on nominal balances, so agents cannot completely smooth

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[1] Both models feature incomplete heterogeneous agents and financial markets. The only difference between the two models is the specific form of borrowing constraint in that Bewley imposes the non-negativity constraint $m_t \geq 0$ while Lucas imposes the cash-in-advance constraint $m_t \geq p_t c_t$. As shown by Wen (2010), these two models are equivalent in many of their implications.

idiosyncratic shocks by engaging in mutually beneficial lending and borrowing directly among themselves.

However, with this setup, there is an ex post inefficiency since some agents are holding idle balances while others face a binding liquidity constraint. This creates needs for risk sharing, as suggested by Lucas (1980). But without necessary information- and record-keeping technologies, households are unable to share risks through lending and borrowing among themselves.

We assume that a community bank emerges to resolve the risk-sharing problem by developing the required information technologies. The function of the bank is to accept nominal deposits from households and make nominal loans to those in need. To start with, we assume that deposits pay zero interest rate and all households voluntarily deposit their idle cash into the bank as a safety net. The benefit of making deposits is that bank members are qualified for loans when needed, as is typical for a "credit union" existed in many communities. This provides enough incentives for agents in the community to pull together their cash resources.

To simplify the banking sector, assume all deposits are withdrawn at the end of each period (100-percent reserve banking), and all loans are one-period loans that charge the competitive nominal interest rate $1 + \tilde{\imath}_t$, which is determined by demand and supply of loans in the community. Any profits earned by the bank are redistributed back to community members as lump-sum transfers.

Similar banking arrangements have been studied by Holmström and Tirole (1998) in their classic paper of public liquidity provision and recently by Berentsen, Camera, and Waller (2006) and Wen (2015). But this segment of the literature does not analyze the issues of the liquidity trap and negative nominal interest rate.

To make the results comparable to the existing literature, we assume zero deposit rate in the benchmark model. Since deposits do not earn interests, the lower bound on the nominal interest rate in the money (credit) market is zero; namely, $\tilde{\imath}_t \geq 0$. We will show how to relax the model to generate negative nominal interest rate in the next section.

There is a continuum of ex ante identical households indexed by $i \in [0, 1]$. As in Lucas (1980), each household is subject to an idiosyncratic preference shock to the marginal utility of consumption, $\theta(i)$, which has the distribution $F(\theta) \equiv \Pr[\theta(i) \leq \theta]$ with support $[\theta_l, \theta_h]$. Leisure enters the utility function linearly as in Lagos and Wright (2005).[2] A household chooses consumption $c(i)$, labor supply $n(i)$, nominal balance $m(i)$ and credit borrowing $b(i)$ to maximize lifetime utility. We assume that aggregate money stock grows at rate $\mu \gtreqqless 0$.

Following Wen (2009, 2015), we assume that in each period the decisions for labor supply and investment on interest-bearing assets (such as capital) must be made before observing the

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[2] The linearity assumption simplifies the model by making the distribution of wealth degenerate. However, unlike Lagos and Wright (2005), the distribution of money holdings in our model is not degenerate but well-defined.

idiosyncratic preference shock $\theta(i)$. Thus, if there is an urge to consume in period t, money stock is the key asset that can be adjusted most quickly to buffer the random preference shock compared with the capital stock. In addition, borrowing constrained households can take loans from the community bank to relax their borrowing constraints by paying the nominal interest rate $\tilde{\imath}$. These assumptions imply that households may find it optimal to carry money as a store of value (inventories) to cope with demand uncertainty, even though money is not essential for exchange and pays zero (or even negative) nominal interest rate. As in the standard literature, any aggregate uncertainty is resolved at the beginning of each period and is orthogonal to idiosyncratic uncertainty.

The time line of events is as follows: In the beginning of each period, aggregate shocks are realized, each household then makes decisions on labor supply, taking as given the initial wealth from last period. After that, idiosyncratic preference shocks are realized, and each household chooses consumption, the amount of nominal balances to be carried over to the next period, and whether or not to borrow from or lend to the community bank. Given such an environment, it is clear that agents with idle cash will not take a loan in that period and that agents who take loans must be cash constrained. It is also possible for a cash-constrained agent not to take any loans if the urge to consume is not high enough to justify the interest rate payment on a loan. Hence, in terms of cash balances, there may exist three types of households ex post in each period: depositors, borrowers, and agents who neither deposit nor borrow.

2.1 Household Problem

Household i takes the bank's real profit income (Π) as given and chooses consumption, capital investment, labor supply, money demand, and credit borrowing to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta_t(i) \log c_t(i) - n_t(i) \right\},$$

subject to

$$c_t(i) + \frac{m_t(i)}{P_t} - \frac{b_t(i)}{P_t} \leq \frac{m_{t-1}(i) + \tau_t}{P_t} - (1 + \tilde{\imath}_{t-1}) \frac{b_{t-1}(i)}{P_t} + w_t n_t(i) + \Pi_t, \tag{1}$$

$$m_t(i) \geq 0, \tag{2}$$

$$b_t(i) \geq 0, \tag{3}$$

where $\tilde{\imath}_t$ denotes the nominal interest rate on bank loans, and τ_t denotes lump-sum money injections (evenly distributed across households). Notice that we have assumed that the deposit rate is zero. We defer our analysis of negative deposit rate to a later section. The non-negativity constraints

on nominal balances (m_t) and loans (b_t) capture the idea that households cannot borrow or lend outside the banking system.[3]

2.2 Financial Intermediation, Credit Supply and Demand

In the money (credit) market, the aggregate supply of nominal credit (total deposits) is $M_t = \int m_t(i)\,dF(\theta)$, and the aggregate demand for credit is $B(\tilde{\imath}_t) = \int b_t(i)\,dF(\theta)$. Note that credit demand cannot exceed supply $(M_t \geq B_t)$ because the loan rate $(\tilde{\imath}_t)$ will always rise to clear the money market, and in the benchmark model the nominal loan rate cannot be negative (or below the deposit rate) because people have the option not to deposit (we defer the analysis of negative interest rate). Hence, the credit market-clearing conditions are characterized by the following complementarity slackness conditions:

$$(M_t - B_t)\,\tilde{\imath}_t = 0; \quad M_t \geq B_t, \tilde{\imath}_t \geq 0. \tag{4}$$

That is, the nominal loan rate will reach its zero lower bound if liquidity (credit) supply exceeds its demand: $M_t > B_t$. However, a zero nominal interest rate does not necessarily imply that $M_t > B_t$, since it is also possible that $M_t = B_t$ and $\tilde{\imath}_t = 0$. On the other hand, if credit demand B_t exceeds supply M_t, the nominal interest rate will rise to clear the market, so that $M_t = B_t$ whenever $\tilde{\imath}_t > 0$.

The bank's balance sheet in period t is given by

$$\underbrace{M_t}_{\text{deposit}} + \underbrace{(1+\tilde{\imath}_{t-1})B_{t-1}}_{\text{loan payment}} \implies \underbrace{M_{t-1} + \tau_t}_{\text{withdraw}} + \underbrace{B_t}_{\text{new loan}} + \underbrace{\Pi_t}_{\text{profit income}}, \tag{5}$$

where the left-hand side is total inflow of bank liquidity in period t and the right-hand side is total outflow of bank liquidity in period t. Specifically, the LHS indicates that in the beginning of period t the bank receives aggregate loan payment $(1 + \tilde{\imath}_{t-1})\,B_{t-1}$ from households with interest, and accepts total deposit M_t. The RHS indicates that the bank makes new loans in the total amount B_t to the households, and at the end of period t it faces withdrawal of $M_{t-1} + \tau_t$. It also redistributes total profit income $\Pi_t = (1 + \tilde{\imath}_{t-1})\,B_{t-1} - B_t$ back to the households in lump sum. Note that aggregate money withdraw includes money injections τ_t, which ensures that households' initial money holdings are consistent with aggregate household budget constraint. An interpretation of this is that lump-sum money injection is distributed to households through the banking system.

Aggregate money supply follows the law of motion:

$$\tilde{M}_t = \tilde{M}_{t-1} + \tau_t. \tag{6}$$

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[3]The consumption-utility function can be more general without losing analytical tractability. For example, the model can be solved as easily with the CRRA utility function $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

If the nominal stock of money grows over time, then the nominal stock of loans (credit) also grows at the same rate except in a liquidity trap. It can be shown that the aggregate household budget constraint is always satisfied by setting $M_t = \tilde{M}_t$.

2.3 Household Decision Rules

The equilibrium decision rules follow a two-cutoff strategy with two cutoff values $\underline{\theta}_t < \bar{\theta}_t$, which fully characterize the distribution of the allocations in the economy, as the following proposition shows:

Proposition 1 *Defining* $x_t \equiv \frac{m_{t-1}(i)}{P_t} + w_t n_t(i) + \Pi_t - (1+\tilde{\imath}_t)\frac{b_{t-1}}{P_t}$ *as household net worth, the optimal decision rules for consumption, money demand, borrowing, and net worth are given, respectively, by*

$$
c_t(i) = \begin{cases} \frac{\theta_t(i)}{\underline{\theta}_t} x_t & \text{if } \theta_t(i) < \underline{\theta}_t \\ x_t & \text{if } \underline{\theta}_t \le \theta(i) \le \bar{\theta}_t \\ \frac{\theta(i)}{\bar{\theta}_t} x_t & \text{if } \theta(i) > \bar{\theta}_t \end{cases} , \tag{7}
$$

$$
\frac{m_t(i)}{P_t} = \begin{cases} \left[1 - \frac{\theta_t(i)}{\underline{\theta}_t}\right] x_t & \text{if } \theta(i) < \underline{\theta}_t \\ 0 & \text{if } \theta(i) \ge \underline{\theta}_t \end{cases} , \tag{8}
$$

$$
\frac{b_t(i)}{P_t} = \begin{cases} 0 & \text{if } \theta(i) \le \bar{\theta}_t \\ \left[\frac{\theta(i)}{\bar{\theta}_t} - 1\right] x_t & \text{if } \theta(i) > \bar{\theta}_t \end{cases} , \tag{9}
$$

$$
x_t(i) = \underline{\theta}_t w_t R_t, \tag{10}
$$

where the cutoffs $\{\bar{\theta}_t, \underline{\theta}_t\}$ *depend only on aggregate state, not on individual's history.*

Proof. See Dong and Wen (2017) and Wen (2015). ■

Note that there are three possible regimes (cases) for money and credit demand: (i) If $m_t(i) > 0$, then $b_t(i) = 0$; namely, a household has no incentive to take a loan if it has idle cash in hand. (ii) If $b_t(i) > 0$, then $m_t(i) = 0$; namely, a household will take a loan only if it runs out of cash. (iii) It is possible that a household has no cash in hand but does not want to borrow money from the bank because the interest rate is too high; namely, $m_t(i) = b_t(i) = 0$. Which of the three situations prevails in each period depends on the urge to consume or the realized value of the preference shock $\theta_t(i)$. It will be proven that household decision rules follow a trigger (cutoff) strategy and there

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exist two cutoff values, $\underline{\theta}$ and $\bar{\theta}$ with $\underline{\theta} < \bar{\theta}$. If $\theta(i) < \underline{\theta}$, since the urge to consume is low, then $m_t(i) > 0$; if $\theta(i) > \bar{\theta}$, since the demand for liquidity is high, then $b_t(i) > 0$; if $\underline{\theta} \leq \theta \leq \bar{\theta}$, then $m_t(i) = b_t(i) = 0$.

Proposition 2 *The two cutoff variables are jointly determined by the following two equations: the no-arbitrage condition,*

$$\frac{\bar{\theta}_t}{\underline{\theta}_t} = (1 + \tilde{\imath}_t), \tag{11}$$

and the Euler equation that determines the liquidity premium (R_t) of money,

$$1 = \beta E \frac{P_t w_t}{P_{t+1} w_{t+1}} R(\underline{\theta}_t, \bar{\theta}_t), \tag{12}$$

where the liquidity premium R_t is given by

$$R_t \equiv \int_{\theta(i)<\underline{\theta}} dF(\theta) + \int_{\underline{\theta} \leq \theta(i) \leq \bar{\theta}} [\theta(i)/\underline{\theta}_t] \, dF(\theta) + \int_{\theta(i)>\bar{\theta}} [\bar{\theta}_t/\underline{\theta}_t] \, dF(\theta). \tag{13}$$

Notice that $R_t \geq 1$ and $\bar{\theta}_t \geq \underline{\theta}_t$. Furthermore, $R_t \geq 1$ if and only if $\bar{\theta}_t \geq \underline{\theta}_t$. Also, $R = \frac{1+\pi}{\beta}$ holds in the steady state, where the real wage is constant and $\pi = \frac{P_{t+1}}{P_t}$ is the inflation rate.

2.4 Aggregation

Aggregating the decision rules in Proposition 1 across households gives aggregate consumption (C), aggregate real money demand $(\frac{M}{P})$, aggregate real credit demand $(\frac{B}{P})$, and aggregate household net worth, respectively:

$$C_t = D(\underline{\theta}_t, \bar{\theta}_t) x_t, \tag{14}$$

$$\frac{M_t}{P_t} = H\left(\underline{\theta}_t, \bar{\theta}_t\right) x_t, \tag{15}$$

$$\frac{B_t}{P_t} = G\left(\underline{\theta}_t, \bar{\theta}_t\right) x_t, \tag{16}$$

$$\frac{M_{t-1} + \tau_t}{P_t} + w_t N_t + \Pi_t - (1 + \tilde{\imath}_{t-1}) \frac{B_{t-1}}{P_t} = x_t = w_t R_t \underline{\theta}_t, \tag{17}$$

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where the three functions $\{D, H, G\}$ are defined as

$$D \equiv \int_{\theta(i)<\underline{\theta}} \frac{\theta(i)}{\underline{\theta}} dF(\theta) + \int_{\underline{\theta}\leq\theta(i)\leq\bar{\theta}} dF(\theta) + \int_{\theta(i)>\bar{\theta}} \frac{\theta(i)}{\bar{\theta}} dF(\theta) \geq 0, \tag{18}$$

$$H \equiv \int_{\theta(i)<\underline{\theta}} \left[1 - \frac{\theta(i)}{\underline{\theta}}\right] dF(\theta) \geq 0, \tag{19}$$

$$G \equiv \int_{\theta(i)>\bar{\theta}} \left[\frac{\theta(i)}{\bar{\theta}} - 1\right] dF(\theta) \geq 0, \tag{20}$$

and they satisfy the relationship:

$$D_t + H_t - G_t = 1, \tag{21}$$

where D is the aggregate propensity to consume, H the aggregate propensity to save, and G the aggregate propensity to borrow. Notice that the aggregate (consumption) velocity of money is given by

$$V_t \equiv \frac{P_t C_t}{M_t} = \frac{D(\underline{\theta}_t, \bar{\theta}_t)}{H\left(\underline{\theta}_t, \bar{\theta}_t\right)}, \tag{22}$$

which is time varying and bounded in the interval $\left[\frac{E(\theta)/\theta_h}{1-E(\theta)/\theta_h}, \infty\right).$[4]

2.5 The Liquidity Trap

Suppose the steady-state inflation rate is π and the real wage is w. In a steady state, we have the following system of equations:

$$R(\underline{\theta}, \bar{\theta}) = \frac{1+\pi}{\beta}, \tag{23}$$

$$1 + \tilde{i} = \bar{\theta}/\underline{\theta}, \tag{24}$$

$$C = D(\underline{\theta}, \bar{\theta})x, \tag{25}$$

$$\frac{M}{P} = H(\underline{\theta})x, \tag{26}$$

$$\frac{B}{P} = G(\underline{\theta}, \bar{\theta})x, \tag{27}$$

$$x = \underline{\theta}w\frac{1+\pi}{\beta}, \tag{28}$$

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[4]Wen (2010) shows that in the heterogeneous-agent cash-in-advance model of Lucas (1980), the velocity of money is also time varying and bounded by an interval similar to this one.

$$(M - B)\tilde{\imath} = 0, \quad M - B \geq 0, \quad \tilde{\imath} \geq 0, \tag{29}$$

where the functions $\{R, D, H, G\}$ are defined in equations (12), (18), (19), and (20).

Proposition 3 *If $1 + \pi > \beta$, the above equation system uniquely solves for $\{\underline{\theta}, \overline{\theta}\}$ with the property $\overline{\theta} > \underline{\theta}$ and $\tilde{\imath} > 0$. At the Friedman rule (when $1 + \pi = \beta$), we have $\tilde{\imath} = 0$ and $\underline{\theta} = \overline{\theta} = \theta^*$. However, θ^* is not unique. We have two possible solutions: either $\theta^* = E(\theta)$, or $\theta^* = \theta_h$.*

Proof. First, by (13), it is clear that $R(\underline{\theta}, \overline{\theta}) = 1$ if and only if $\overline{\theta} = \underline{\theta}$; and $R > 1$ if and only if $\overline{\theta} > \underline{\theta}$. Hence, by (23) we have $1 + \pi \geq \beta$ if and only if $\overline{\theta} \geq \underline{\theta}$. Second, since $\overline{\theta} \geq \underline{\theta}$, we have $\tilde{\imath} \geq 0$ and $M = B$. But $M = B$ is equivalent to $H(\underline{\theta}) = G(\underline{\theta}, \overline{\theta})$ or

$$\int_{\theta(i) < \underline{\theta}} [\underline{\theta} - \theta(i)] \, dF(\theta) = \int_{\theta(i) > \overline{\theta}} [\theta(i) - \overline{\theta}] \, (\underline{\theta}/\overline{\theta}) \, dF(\theta),$$

which implies

$$\int_{\theta < \underline{\theta}} \frac{\underline{\theta} - \theta}{\underline{\theta}} dF(\theta) = \int_{\theta > \overline{\theta}} \frac{\theta - \overline{\theta}}{\overline{\theta}} dF(\theta). \tag{30}$$

When $1 + \pi > \beta$, $R > 1$, the above equation implies that for any value of $\underline{\theta}$ close enough to the lower bound θ_l, there exists a unique value of $\overline{\theta}$ close enough to the upper bound θ_h such that the area measured by the left-hand-side of (30) equals the area measured by the right-hand-side of (30) for any non-degenerate distribution function $F(\theta)$. Plugging this relationship implied by (30), $\overline{\theta}(\underline{\theta})$, into the continuous and single-valued relation $R(\underline{\theta}, \overline{\theta}) = \frac{1 + \pi}{\beta}$ uniquely determines the value of $\underline{\theta}$. Given $\underline{\theta}$, $\overline{\theta}$ can then be uniquely determined by (30). Equation (13) also implies $R > 1$ if and only if $\overline{\theta} > \underline{\theta}$. Equation (24) implies $\tilde{\imath} > 0$.

When $1 + \pi = \beta$, by equation (13) we must have $R = 1$ and $\overline{\theta} = \underline{\theta} = \theta^*$, and $\tilde{\imath} = 0$. In this case, if $M_t = B_t$, then equation (30) implies the solution

$$F(\theta^*) - \frac{\int_{\theta < \theta^*} \theta dF(\theta)}{\theta^*} = \frac{\int_{\theta > \theta^*} \theta dF(\theta)}{\theta^*} - [1 - F(\theta^*)],$$

which implies

$$\theta^* = \int \theta dF(\theta) \equiv E(\theta).$$

However, it is also possible that $M_t > B_t$ when $\tilde{\imath} = 0$, then the LHS of equation (30) is greater than its RHS; since $\overline{\theta} = \underline{\theta} = \theta^*$, it must be true that $\theta^* = \theta_h$. In this case, we also have $G_t = 0$ and $0 < H_t \equiv \int_{\theta(i) < \theta_h} \left[1 - \frac{\theta(i)}{\theta_h}\right] dF(\theta) < \infty$, i.e., money demand is finite and credit demand is zero

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(net worth is $x = \theta_h w$). Note that money demand becomes infinite if and only if the upper bound $\theta_h = \infty$. But this case is ruled out by assumption. ∎

If we define the situation with $M_t > B_t$ as a liquidity trap, this Proposition suggests that Friedman rule does not necessarily imply a liquidity trap, but a liquidity trap necessarily implies the Friedman rule. In addition, real money demand is finite even in the liquidity trap. This result is in contrast to the argument of Grandmont and Laroque (1976) and Bewley (1980) who argue that real money demand becomes infinity at the Friedman rule.

In a liquidity trap, money is such an attractive asset to hold that any additional money injection will be hoarded by the private sector (albeit the amount is finite), thus increasing liquidity supply (deposits) in the banking system. On the other hand, since agents have enough cash to meet their urge to consume (the highest shock is $\theta_t = \theta_h$), the need for borrowing cease to exist. In this case, the nominal interest rate on loans cannot decrease further below zero, and the demand for loans will not be further stimulated to absorb the excess supply of liquidity. Hence, conventional monetary policy will cease to be effective in stimulating credit demand and aggregate spending through the credit lending channel of the banking system.

2.6   The Essentiality of Money

Proposition 4 *There exists a finite upper limit for the inflation rate, $\pi_{\max} = \beta \frac{E(\theta)}{\underline{\theta}} - 1 > 0$, such that if $\pi \geq \pi_{\max}$, then the optimal demand for real balances $\frac{M}{P} = 0$; namely, no household is willing to hold cash if inflation is at or above $\pi_{\max}$. In this case, every household holds zero liquidity (money and credit) and we have $\frac{B}{P} = 0, R = \frac{E(\theta)}{\underline{\theta}}, D = 1, H = G = 0$, and $1 + \tilde{i} = \frac{\theta_h}{\theta_l}$.*

Proof. By (13), we have $\frac{\partial R}{\partial \underline{\theta}} < 0$ and $\frac{\partial R}{\partial \bar{\theta}} > 0$. Hence, given the support of θ, the maxim value of R ($R_{\max}$) is reached either when $\underline{\theta} = \theta_l$ or $\bar{\theta} = \theta_h$ or both. By Proposition 1 and (30), $\underline{\theta} = \theta_l$ if and only if $\bar{\theta} = \theta_h$. Hence, there exists π_h such that R is at its maximum value $R_{\max} = \frac{1 + \pi_h}{\beta}$ if and only if $\underline{\theta} = \theta_l$. Then by equations (13), (18), (19) and (20), we have $R = \frac{E(\theta)}{\theta_l}$, $D = 1$, $H = 0$ and $G = 0$, so by (24), (26) and (27), we have $\tilde{i} = \frac{\theta_h}{\theta_l} - 1$, $\frac{M}{P} = \frac{B}{P} = 0$. Since there is no credit supply in the banking system, we must also have $\frac{B}{P} = 0$ in equilibrium regardless of the nominal loan rate. Notice that the high nominal interest rate is caused by high inflation rate, not vise versa. ∎

This Proposition suggests that money is essential in the economy, in the sense that agents opt to hold it only when the benefit of holding it exceeds the cost of holding it. The welfare cost of inflation in this model is an order of magnitude larger than that in representative models with cash-in-advance (CIA) constraints. However, the welfare cost of inflation is equally large in

a heterogeneous-agent model with CIA constraints as in Lucas (1980). Hence, in contrast to the claims of Lagos and Wright (2005), the large welfare cost in their model is not due to bargaining per se, but rather to heterogeneity (see Wen 2010 and Wen 2015).

3 Monetary Policy under Negative Interest Rate

This section consider two issues: (i) negative interest rate and (ii) real effects of conventional monetary policies under negative interest rate.

Suppose that agents always incur a cost of storing money at home (holding a large sum of cash can be costly). Without loss of generality, assume that the cost per period is proportional to money holdings: namely, cost $= i_t^c m_t(i) \geq 0$. The banking system, however, can minimize such costs through economy of scale. Given this, then the community bank can opt to charge the depositors a negative nominal deposit rate $i_t^d < 0$ such that depositors still find it beneficial to put money in the bank as long as $i_t^d \geq -i_t^c$.

In this setting, the lower bound of the nominal lending interest rate is i_t^d, and it is determined by the following complementarity slackness conditions analogous to equation (4):

$$(M_t - B_t)\left(\tilde{\imath}_t - i_t^d\right) = 0; \quad M_t \geq B_t, \tilde{\imath}_t \geq i_t^d. \tag{31}$$

When deposits earn interests (positive or negative), the household budget constraint (1) becomes

$$c_t(i) + \frac{m_t(i)}{P_t} - \frac{b_t(i)}{P_t} \leq \frac{\left(1 + i_{t-1}^d\right)m_{t-1}(i) + \tau_t}{P_t} - (1 + \tilde{\imath}_{t-1})\frac{b_{t-1}(i)}{P_t} + w_t n_t(i) + \Pi_t, \tag{32}$$

equation (11) becomes

$$\frac{\overline{\theta}_t}{\underline{\theta}_t} = \frac{1 + \tilde{\imath}_t}{1 + i_t^d}, \tag{33}$$

and bank's period-t balance sheet (5) becomes

$$\underbrace{M_t}_{\text{deposit}} + \underbrace{(1 + \tilde{\imath}_{t-1})B_{t-1}}_{\text{loan payment}} \Longrightarrow \underbrace{\left(1 + i_{t-1}^d\right)M_{t-1} + \tau_t}_{\text{withdraw}} + \underbrace{B_t}_{\text{new loan}} + \underbrace{\Pi_t}_{\text{profit income}}, \tag{34}$$

and bank profits become $\Pi_t = (1 + \tilde{\imath}_{t-1})B_{t-1} - B_t - i_{t-1}^d M_{t-1} > (1 + \tilde{\imath}_{t-1})B_{t-1} - B_t$.

Lemma 5 *At the Friedman-rule steady state with $\frac{1+\pi}{\beta} = 1$, we have $R = 1$, $\tilde{\imath} = i^d$ and $\overline{\theta} = \underline{\theta} \equiv \theta^*$.*

Lemma 6 *When $R = 1$ and $\tilde{\imath} = i^d$, the cutoff θ^* can take two possible equilibrium values: an interior solution $\theta^* = E(\theta) \in (\theta_l, \theta_h)$ or a corner solution $\theta^* = \theta_h$.*

At the Friedman rule inflation rate, aggregate money holdings (and hence aggregate deposits) are large enough (albeit finite) to drive the money-market interest rate down towards its lower bound $i^d < 0$. In this case the economy may enter a liquidity trap. However, analogous to the benchmark model with ZLB, only the corner solution $\theta^* = \theta_h$ is consistent with the traditional notion of "liquidity trap" where money (credit) supply in the banking sector exceeds credit demand and the borrowing constraint ($\frac{m_t(i)}{P} \geq 0$) no longer binds across states for all households. However, it is important to note that even in this case the demand for real balances is not infinite despite negative nominal interest rate on loans, in sharp contrast to the conventional wisdom.

In other words, there are two possible regimes at the Friedman rule: (i) Aggregate money demand (M_t) and aggregate credit demand (B_t) are both finite with the property $M_t = B_t$. (ii) Aggregate money demand is finite but exceeds aggregate credit demand, $M_t > B_t = 0$. The first regime happens when θ^* is interior and the second regimes happens if θ^* is a corner solution at θ_h. The level of aggregate money demand and credit demand can be solved by equations (19), (20) and (28), which yield

$$\frac{M}{P} = \begin{cases} \left(\int_{\theta(i)<E(\theta)} \left[1 - \frac{\theta(i)}{E(\theta)}\right] dF(\theta)\right) wE(\theta) & \text{if } \theta^* = E(\theta) \\ \left(1 - \frac{E(\theta)}{\theta_h}\right) w\theta_h & \text{if } \theta^* = \theta_h \end{cases}, \tag{35}$$

$$\frac{B}{P} = \begin{cases} \left(\int_{\theta(i)>E(\theta)} \left[\frac{\theta(i)}{E(\theta)} - 1\right] dF(\theta)\right) wE(\theta) & \text{if } \theta^* = E(\theta) \\ 0 & \text{if } \theta^* = \theta_h \end{cases}. \tag{36}$$

At the corner solution $\theta^* = \theta_h$, the borrowing constraint never binds for any household in any circumstances because households opt to hoard enough real balances (albeit finite amount) such that they no longer have the need to borrow despite negative interest rate on loans. It is important to note that the negative interest rate $\tilde{i}$ is an equilibrium phenomenon due to the lack of credit demand rather than infinite credit supply (or money demand). This is in sharp contrast with the argument of the existing literature. The implications for monetary policies are discussed below.

4   Implications for Optimal Monetary Policies

In many existing monetary models with micro foundations (e.g., Andolfatto and Williamson, 2015), the government should supply as much bond as possible to make the market interest rate as high as possible. This implication does not hold here. Instead, in this model the government should withdraw as much bond as possible from the market to make the nominal interest rate as low as possible. The reason is simple: In our model, lower interest rate implies better risk sharing across households. Namely, since the essential role of the banking system here is to channel credit

resources from savers to borrowers so as to eliminate inefficiency due to borrowing constraints, social welfare would increase by lowering the nominal interest rate. When the government issue bonds in the money market, it crowds out the amount of credit resources flowing to the households. As a result, the market interest rate rises and the social welfare declines.

Formally, suppose the amount of government bond is denoted by B_t^g. In the credit market, the aggregate supply of nominal credit (deposits) is M_t, and the aggregate demand for credit is $B_t + B_t^g$. Note credit demand cannot exceed supply because the loan rate will always rise to clear the market, and the nominal loan rate cannot be lower than the deposit rate because the bank has the option not to lend. Hence, the credit market-clearing conditions are characterized by the following complementarity conditions:

$$(M_t - B_t - B_t^g)\left(\tilde{\imath}_t - i_t^d\right) = 0; \quad M_t \geq B_t + B_t^g, \tilde{\imath}_t \geq i_t^d. \tag{37}$$

The bank's balance sheet is given by

$$\underbrace{M_t}_{\text{deposit}} + \underbrace{(1 + \tilde{\imath}_{t-1})\left(B_{t-1} + B_{t-1}^g\right)}_{\text{loan payment from H \& G}} \implies \underbrace{\left(1 + i_{t-1}^d\right) M_{t-1} + \tau_t +}_{\text{withdraw}} \underbrace{B_t + B_t^g}_{\text{new loan to H \& G}} + \underbrace{\Pi_t}_{\text{profit income}}, \tag{38}$$

where the left-hand side is total inflow of liquidity in period t and the right-hand side is total outflow of liquidity in period t. That is, in the beginning of period t the bank accepts deposit M_t, makes new loans of B_t and B_t^g to the households (H) and the government (G), respectively; and at the beginning of period t it receives loan payment $(1 + \tilde{\imath}_{t-1})\left(B_{t-1} + B_{t-1}^g\right)$ from households and government with interest, and faces withdrawal of $M_{t-1} + \tau_t$ (on the right hand side) plus interest payment on deposits. Any profits are distributed back to households lump sum in the amount Π_t.

Clearly, increasing bond supply B_t^g would increase the nominal interest rate $\tilde{\imath}_t$. But efficiency (or perfect risk sharing) requires that the liquidity premium $R\left(\underline{\theta}, \overline{\theta}\right) = 1$, or $\underline{\theta} = \overline{\theta}$ and $\tilde{\imath}_t = i_t^d \leq 0$. Hence, bond supply should be kept at its minimum to improve social welfare.

In addition, the Friedman rule is no longer the optimal policy of money supply. Instead, we have the modified Friedman rule: $1 = \beta \frac{1+i^d}{1+\pi}$. It suggests that if the deposit rate $i^d < 0$, the optimal inflation rate is $\pi^* = \beta\left(1 + i^d\right) - 1$, which is less that traditional Friedman rule of $\beta - 1$. The reason is that the lower bound of the nominal interest rate now is negative at $\iota^d < 0$, instead of zero. This can be seen from the Euler equation for optimal money demand:

$$1 = \beta \frac{1+i^d}{1+\pi} R\left(\underline{\theta}, \overline{\theta}\right). \tag{39}$$

On the other hand, at the traditional Friedman rule $\pi = \beta - 1$, the above equation implies $R > 1$

and $\underline{\theta} < \overline{\theta}$ (since $i^d < 0$) as well as $\tilde{i} > i^d$. In this case, a positive measure of the households are still borrowing constrained, implying inefficiency in credit resource allocation.

4.1 Fisherian Decomposition

Standard model implies the following Fisherian relationship:

$$1 = \beta \frac{1 + \tilde{i}}{1 + \pi}, \tag{40}$$

where $\tilde{i}$ is nominal interest rate and π is the inflation rate. This relationship suggests that the real interest rate can be defined as $\frac{1+\tilde{i}}{1+\pi}$; hence, we often say that the real interest rate is the inverse of the time preference β. Economic intuition behind this definition of the real interest rate is as follows: If a creditor lends out one dollar today, she would expect to be paid $(1 + \pi)$ dollars back tomorrow to break even in a competitive environment. Since the dollar's purchasing power shrinks by inflation, the lender expect a compensation of $\tilde{i} = \pi$ dollars by charging an interest at the inflation rate. Hence, the real rate of return is $1 + r = \frac{1+i}{1+\pi}$ or approximately $r = \tilde{i} - \pi$.

But such a conventional linear Fisherian relationship between the real interest rate and the inflation rate as well as the economic rational do not hold here in this model. In this model, (for simplicity and without loss of generality, we set the deposit rate $i^d = 0$), equation (39) becomes

$$1 = \beta \frac{R\left(\underline{\theta}, \overline{\theta}\right)}{1 + \pi}, \tag{41}$$

where the liquidity premium

$$R \equiv \int_{\theta(i)<\underline{\theta}} dF(\theta) + \int_{\underline{\theta}\leq\theta(i)\leq\overline{\theta}} [\theta(i)/\underline{\theta}]\, dF(\theta) + \int_{\theta(i)>\overline{\theta}} \left[\overline{\theta}/\underline{\theta}\right] dF(\theta) > 1, \tag{42}$$

and the nominal interest rate is determined by

$$1 + \tilde{i} = \frac{\overline{\theta}}{\underline{\theta}}. \tag{43}$$

Clearly, the relationship between the liquidity premium R and the nominal interest rate $1 + \tilde{i}$ is highly non-linear. In particular, as the inflation rate π increases, the nominal interest rate $\tilde{i}$ increases more than one-for-one with inflation. In other words, the ratio $\frac{1+\tilde{i}}{1+\pi}$ is an increasing function of the inflation rate, as shown in Figure 2.
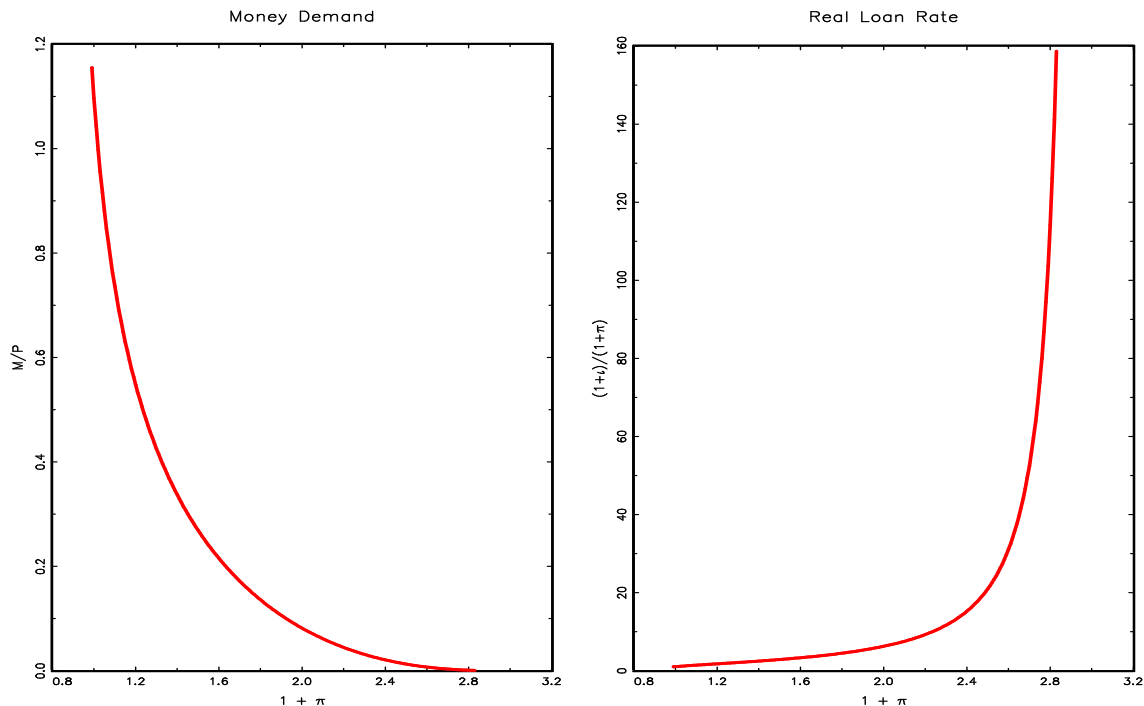
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Figure 2. Money Demand $\frac{M}{P}$ and "Real" Interest Rate $\frac{1+\tilde{\imath}}{1+\pi}$.

Thus, if a creditor calculates the real rate of return to a dollar by $\frac{1+\tilde{\imath}}{1+\pi}$, she would make losses when making loans in this model. The intuition is as follows: The nominal interest rate in this model is determined in the money-credit market by equating the total supply of credit (M/P) with total demand for credit (B/P). As the inflation rate rises, households opt to hold less money to buffer their consumption shocks because the cost of holding cash increases. Instead, they prefer borrowing money. As a result, not only the banking system's total deposit (credit supply) declines but total credit demand (loans) also rises, pushing up the nominal loan rate. In the limiting case when the inflation rate approaches $\pi_{\max}$, for example, households opt to hold zero balances, which drives the nominal loan rate to infinity. Thus, as Figure 2 shows, the equilibrium nominal interest rate rises rapidly (right window) while real money demand approaches zero (left window).[5]

Hence, in this model, a meaningful definition of the real interest rate that makes the purchasing power of a dollar constant is not $\frac{1+\tilde{\imath}}{1+\pi}$, but something else (i.e., $\frac{R(\theta,\bar{\theta})}{1+\pi}$).

5 The Non-neutrality of Money

This section studies the real effects of money injections on the economy under negative nominal interest rate $\tilde{\imath}_t < 0$. We close the model adding production and capital accumulation in the

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[5]To illustrate (and magnify) the non-linear effect, we have assumed a Pareto distribution for $F(\theta) = 1 - \theta^{-\sigma}$, with support $\theta \in [1, \infty)$ and the shape parameter $\sigma > 1$.

benchmark model. Let $Y_t = A_t(e_t K_t)^\alpha N_t^{1-\alpha}$, $\delta_t = \frac{1}{1+\omega}e_t^{1+\omega}$, $w_t = (1-\alpha)\frac{Y_t}{N_t}$ and $r_t - \delta_t = \alpha\frac{Y_t}{K_t}$ denote output, capital depreciation rate, real wage and the user cost of capital, respectively. The goods market-clearing condition is given by

$$C_t + K_{t+1} - (1 - \delta_t)K_t = Y_t. \tag{44}$$

We consider the following monetary policy shock τ_t:

$$M_t = M_{t-1} + \tau_t, \tag{45}$$

$$\log \tau_t = \rho \log \tau_{t-1} + \varepsilon_t, \tag{46}$$

where money injection τ_t follows an AR(1) process. This implies that money injection is transitory and money growth is zero in the long run. Hence, the steady-state inflation rate is zero, $\pi = 0$. With zero inflation rate and negative deposit rate $i^d < 0$, equation (39) becomes

$$1 = \beta \left(1 + i^d\right) R\left(\underline{\theta}, \overline{\theta}\right). \tag{47}$$

Consider the Pareto distribution $F(\theta) = 1 - \theta^{-\sigma}$, with support $\theta \in [1, \infty)$ and the shape parameter $\sigma > 1$. The model is calibrated according to Dong and Wen (2017). In particular, we set the value of σ low enough so that equilibrium nominal interest rate $\tilde{i}$ is negative but greater than i^d. We show that conventional monetary injection has real stimulating effects on the economy, despite negative nominal interest rate.

The impulse responses of the model to a one-percent transitory increase in the money stock (with $\rho = 0.9$) are graphed in Figure 3. It shows that transitory monetary shocks are expansionary. In particular, a one-percent increase in τ_t can raise output by about 0.6 percent, consumption by 0.3 percent, and investment by 1.4 percent. The price level remains sluggish, as if it is sticky as in the New Keynesian model despite flexible prices. Also, the nominal interest rate decreases for a prolonged period (see the bottom middle window in Figure 3), capturing the so-called liquidity effect found in the data (see, e.g., Christiano, Eichenbaum, and Evans, 1995).
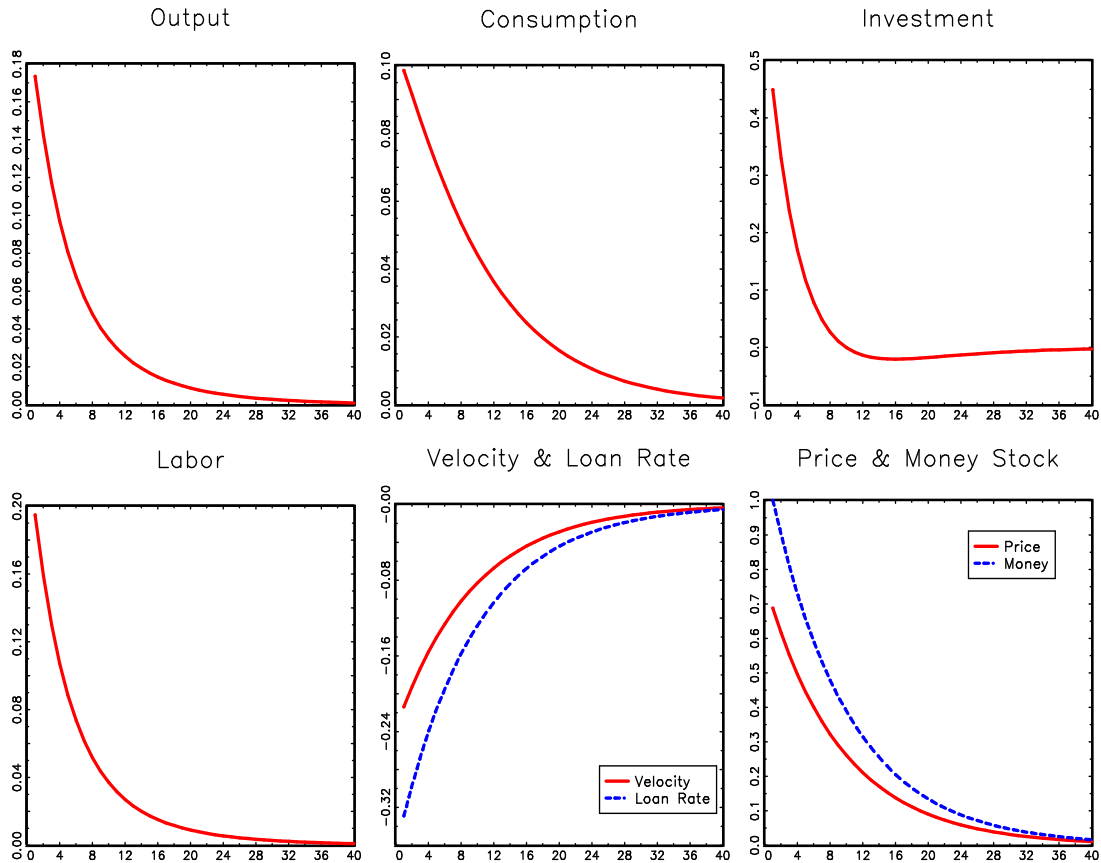
Figure 3. Impulse Responses to 1% Money Injection.

6 Conclusion

In responding to the extremely weak global economy after the financial crisis in 2008, many industrial nations have implemented negative nominal interest rate policy. This situation raises two important questions for monetary theories: (i) Given the widely held doctrine of the zero lower bound on nominal interest rate, how is a negative interest rate (NIR) policy possible? (ii) Will NIR be effective in stimulating aggregate demand? (iii) Are there any new theoretical issues emerging under NIR policies? This article builds a model to show that

1. Negative nominal interest rate is possible if holding money by the private sector is costly.

2. Conventional monetary policies remain effective under negative nominal interest rate as long as the money-market interest rate is above its lower bond. In fact, it is a good policy to keep the nominal interest rate as low as possible in a recession and as long as possible.

3. The conventional wisdom on the notion of the liquidity trap and the Fisherian decomposition between the nominal and real interest rate can be invalid.

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References

[1] Akyol, A., 2004. Optimal monetary policy in an economy with incomplete markets and idiosyncratic risk. *Journal of Monetary Economics*, 51(6), pp.1245-1269.

[2] Alvarez, F., Atkeson, A. and Edmond, C., 2009. Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand. *The Quarterly Journal of Economics*, 124(3), pp.911-967.

[3] Alvarez, F., Atkeson, A. and Kehoe, P.J., 2002. Money, interest rates, and exchange rates with endogenously segmented markets, *Journal of Political Economy*, 110(1), pp.73-112.

[4] Andolfatto, D. and Williamson, S., 2015. Scarcity of safe assets, inflation, and the policy trap. *Journal of Monetary Economics*, 73, pp.70-92.

[5] Berentsen, A., Camera, G. and Waller, C., 2007. Money, credit and banking. *Journal of Economic Theory*, 135(1), pp.171-195.

[6] Bewley, T., 1980, "The Optimum Quantity of Money," in Models of Monetary Economics, ed. by John Kareken and Neil Wallace. Minneapolis, Minnesota: Federal Reserve Bank, 1980.

[7] Bewley, T., 1983, A difficulty with the Optimum Quantity of Money, *Econometrica* 51(5), 1485-1504.

[8] Christiano, L.J. and Eichenbaum, M., 1995. Liquidity effects, monetary policy, and the business cycle. *Journal of Money, Credit and Banking*, 27(4), pp.1113-1136.

[9] Dong, F. and Wen, Y., 2017. Optimal Monetary policy under Negative Nominal Interest Rate. Mimeo.

[10] Grandmont, J.M. and G. Laroque, 1976. The liquidity trap. *Econometrica*, 44 (1), pp.129-135.

[11] Holmström, B. and Tirole, J., 1998. Private and public supply of liquidity. *Journal of Political Economy*, 106(1), pp.1-40.

[12] Krugman, P.R., 1998. It's baaack: Japan's slump and the return of the liquidity trap. *Brookings Papers on Economic Activity*, 1998(2), pp.137-205.

[13] Lagos, R. and Wright, R., 2005. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3), pp.463-484.

[14] Lucas, R.E., 1980. Equilibrium in a pure currency economy. *Economic inquiry*, 18(2), pp.203-220.

[15] Wen, Y., 2009. Liquidity and welfare in a heterogeneous-agent economy. Working Papers 2009-019, Federal Reserve Bank of St. Louis.

[16] Wen, Y., 2010. Optimal money demand in a heterogeneous-agent cash-in-advance economy. Working Papers 2010-014, Federal Reserve Bank of St. Louis.

[17] Wen, Y., 2014. When and how to exit quantitative easing?. *Federal Reserve Bank of St. Louis Review*, 96(3), pp.243-265.

[18] Wen, Y., 2015. Money, liquidity and welfare. *European Economic Review*, 76, pp.1-24.

[19] Williamson, S.D., 2012. Liquidity, monetary policy, and the financial crisis: A new monetarist approach. *The American Economic Review*, 102(6), pp.2570-2605.