Research Division
Federal Reserve Bank of St. Louis
Working Paper Series

Low Real Interest Rates and the Zero Lower Bound

Stephen D. Williamson

Working Paper 2017-010A
https://doi.org/10.20955/wp.2017.010

April 2017

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
P.O. Box 442
St. Louis, MO 63166

Low Real Interest Rates and the Zero Lower Bound

Stephen D. Williamson*
Federal Reserve Bank of St. Louis

April 2017

Abstract

How do low real interest rates constrain monetary policy? Is the zero lower bound optimal if the real interest rate is sufficiently low? What is the role of forward guidance? A model is constructed that can incorporate sticky price frictions, collateral constraints, and conventional monetary distortions. The model has neo-Fisherian properties. Forward guidance in a liquidity trap works through the promise of higher future inflation, generated by a higher future nominal interest rate. With very tight collateral constraints, the real interest rate can be very low, but the zero lower bound need not be optimal.

*The views expressed are those of the author and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. The author thanks seminar participants at the Federal Reserve Bank of St. Louis, Washington University in St. Louis, and the Bank of Portugal, for helpful comments and suggestions.

1 Introduction

The purpose of this paper is to develop a simple macroeconomic model that can include some key frictions necessary to evaluate the effects of monetary policy in low-real-interest-rate environments. The model can include sticky price distortions, a role for secured credit, multiple assets, and open market operations. An important theme is that there can be differences in optimal monetary policy depending on the cause of the low real interest rate, what frictions are present, and how we model the detail of the financial structure of the economy. More often than not, New Keynesian conventional wisdom is turned on its head. For example, In a liquidity trap, when zero lower bound is a binding constraint on monetary policy, inefficiencies are reflected in excessive inflation and high output, rather than deflation and low output. As well, a central bank that can commit to future policy in a liquidity trap may choose to promise higher future inflation by committing to a higher future nominal interest rate. Further, if the real interest rate is low because of a scarcity of safe collateral, optimal monetary policy may entail a nominal interest rate that is not only greater than zero, but higher than if the scarcity did not exist.

By any measure, real rates of interest on government debt have been declining in the world since the early 1980s. For example, Figure 1 shows the three-month Treasury bill rate minus the 12-month rate of increase in the personal consumption deflator, for the period 1980-2017. The figure shows that the real interest rate, by this measure, has decreased on trend since the 1981-1982 recession, and has been persistently low following the 2008-2009 recession.

[Figure 1 here.]

There is now an extensive New Keynesian literature that analyzes monetary policy at the zero lower bound (ZLB) – a "liquidity trap." Two key (and closely-related) papers in this literature are Eggertsson and Woodford (2003) and Werning (2011). These authors use sticky price frameworks to model a low-real-interest-rate environment that results from a temporary fall in the subjective rate of time preference – a fall in the "natural rate of interest." The key findings are:

1. If the central banker in the model cannot commit, this creates an inefficiency in the low-real-interest-rate liquidity trap state. With the nominal interest rate at zero, the extent of the inefficiency at the beginning of the liquidity trap period rises as prices become *less* sticky, and as the length of the liquidity trap period increases.

2. If the central banker in the model can commit, then forward guidance – a promise concerning the path for the nominal interest rate after the liquidity trap period ends – is effective.

3. Optimal forward guidance takes the form of promises to keep the nominal interest rate low after the liquidity trap period ends. This forward

guidance produces higher inflation and output in the future than what the central banker would choose if he or she could not commit.

While Eggertsson and Woodford (2003) and Werning (2011) focus on the use of forward guidance when the ZLB is a binding constraint for the central banker, other solutions to the ZLB problem have been suggested, and some of these have been implemented in practice. Such solutions include: (i) raising the central bank's inflation target; (ii) quantitative easing (QE); (iii) negative nominal interest rates; (iv) helicopter money. It is certainly important to understand the effects of QE, negative nominal interest rates, and helicopter money, but this paper will focus on forward guidance, interest rate policy, and inflation targeting. The effects of QE and large central bank balance sheets are studied in detail in Williamson (2015, 2016a, 2016b).

A primary goal in this paper is to address monetary policy issues in a tractable analytical framework that permits alternative assumptions about frictions, the array of available assets, and how monetary policy works. To this end, we start with a simple representative-household sticky-price model that permits a straightforward analysis of optimal policy. The model has standard New Keynesian features, including a Phillips curve tradeoff. There is a sticky price inefficiency, and zero inflation is optimal, provided that monetary policy is unconstrained by the zero lower bound.

Part of what makes the model simple is quasilinear preferences, which does away with wealth effects. The model then becomes starkly Fisherian, in that the real interest rate is exogenous, and the current nominal interest rate determines the expected inflation rate. This highlights a feature of all New Keynesian models, as pointed out in particular by Cochrane (2016, 2017) and Rupert and Sustek (2016). That is, inflation dynamics in these models are essentially Fisherian – higher nominal interest rates tend to increase inflation. The idea that central bankers have the sign wrong – that inflation increases if the nominal interest rate goes up – is sometimes called "neo-Fisherism."

The first step in the analysis is to subject the model to "natural real interest rate" shocks of the type studied by Eggertsson and Woodford (2003) and Werning (2011). In this model, we get a similar characterization of optimal policy whether the real interest rate is low because the discount factor is temporarily high (as in the New Keynesian literature), or because productivity growth is anticipated to be temporarily low. Much as in the related literature, the allocation is inefficient in the liquidity trap state if the central banker cannot commit, and welfare increases if the central banker can commit to higher inflation once the economy reverts to the non-liquidity trap state. However, in the absence of commitment, actual inflation and output exceed optimal inflation and output, respectively, in the liquidity trap state. As well, if the central banker provides credible forward guidance, this takes the form of a higher promised nominal interest rate in the post-liquidity trap state than would hold without commitment. These results have a neo-Fisherian tone, and are the opposite of what we find in Eggertsson and Woodford (2003) and Werning (2011), where the liquidity trap problem is too-low inflation, and forward guidance is about promising low nominal interest rates in the future non-liquidity trap state. Further, in contrast to what Werning (2011) obtains, the liquidity trap problem is not at its worst when prices are close to perfectly flexible.

New Keynesian models have been criticized for not being explicit about how monetary policy works (e.g. Williamson and Wright 2010, 2011). For the problem considered in this paper, being explicit about monetary policy seems particularly important, since it will in general matter for monetary policy *why* the real interest rate is low. It certainly seems unsatisfactory to model the temporarily low real interest rate as a discount factor shock, given what is currently known about the reasons for low real interest rates. In particular, Krishnamurthy and Vissing-Jorgensen (2012), Andolfatto and Williamson (2015), and Caballero et al. (2016) lend empirical and theoretical support to the idea that low real interest rates on government debt can be explained by a shortage of safe collateral. To do justice to the problem at hand, it seems important to capture this, along with an explicit account of the monetary policy mechanism, in a model with a sufficiently rich set of assets.

So that we can understand what is going on, it helps to develop the model by adding detail in steps. We first take the basic model with sticky prices, which is a Woodford-type "cashless" economy, and assume that all transactions are conducted using secured credit. For convenience, the only available collateral is government debt. If the real quantity of government debt is sufficiently small, collateral constraints bind and the real interest rate is low. This then leads to a similar liquidity trap problem to what occurs in the baseline model, but there are two sources of inefficiency, sticky prices and binding collateral constraints. While the sticky price friction causes an inefficiency only in the market for sticky-price goods, a binding collateral constraint also results in inefficiency in the market for flexible-price goods. Further, the problem is complicated by the existence of multiple equilibria. Given a particular monetary policy, if the collateral constraint binds in the temporary liquidity trap state, there can be two equilibria – one with a high inflation rate, a tighter collateral constraint, and a lower real interest rate, and one with a lower inflation rate, a less-tight collateral constraint, and a not-so-low real interest rate. Thus, if we are explicit about the source of the low real interest rate, our results do not look like the ones we get by taking a reduced form approach (e.g. modeling the liquidity trap state as arising from a high discount factor).

The final version of the model includes money, government debt, and secured credit, which allows us to be explicit about the source of the low real interest rate, *and* about open market operations. Now, we have three sources of inefficiency to worry about: (i) the sticky price friction, common to the first two setups; (ii) scarce collateral, as in the second setup; (iii) a Friedman-rule type inefficiency, i.e. scarcity of cash. Here, it helps to first consider inefficiencies (ii) and (iii), and then add the sticky prices. In this context, open market operations are non-neutral – an open market purchase of government bonds will in general lower the real interest rate permanently, as this tightens the collateral constraint. But, without sticky prices, New Keynesian results are turned on their head. When the collateral constraint does not bind, the nominal interest rate is zero at the optimum – a Friedman rule. But, when the collateral constraint binds and the real interest rate is low, the nominal interest rate should be positive. This is because the central bank should optimally trade off the two sources of liquidity scarcity. In particular, a higher nominal interest rate makes cash more scarce, in real terms, but the open market sale that raises the nominal interest rate mitigates the scarcity-of-safe-collateral problem.

Further, once we include all three sources of inefficiency in the model, this need not overturn the flexible price results, even if prices are highly sticky. Indeed, we show in an example that, if scarce collateral matters for optimal monetary policy, the central bank responds to the problem by raising the nominal interest rate, which has the effect of raising the real interest rate.

The remainder of the paper is organized as follows. The baseline cashless New-Keynesian-style model is constructed in the second section. Then, in the third section, secured credit is added, and the fourth section contains an analysis of the full-blown model with money, bonds, secured credit, sticky prices, and open market operations. The final section is a conclusion.

2 Baseline Model

There is a continuum of households with unit mass, with each maximizing

$$E_0 \sum_{t=0}^{\infty} \prod_{s=0}^{t} (\beta_s) \left[u(c_t^f) + u(c_t^s) - (n_t^f + n_t^s) \right] \tag{1}$$

Here, β_t is the discount factor for period $t+1$ utility relative to period t utility. As well, c_t^f is consumption by the household of the flexible-price good, c_t^s is consumption of the sticky-price good, and n_t^f and n_t^s denote, respectively, household labor supplied to produce the flexible-price and sticky-price good. A household cannot consume its own output. It purchases and sells goods on competitive markets.

As in mainstream New Keynesian cashless models, goods are denominated in terms of money, and money does not serve as a medium of exchange, only as a unit of account. Let P_t denote the price of flexible-price goods in units of money. The spot market in flexible-price goods clears every period, but households are technologically constrained to sell sticky-price goods at the price P_{t-1}, and must satisfy whatever demand arises for these goods at that price. Demand is assumed to be distributed uniformly among households in the sticky-price goods market. This setup is equivalent to a world in which there are two physically distinct goods. For a given good, the price is determined competitively one period, and remains fixed at that value through the following period. Then in the next period, the price is again competitively determined, etc. There is staggered price-setting, in that the prices for one good are set on competitive markets in even periods, while the prices for the other good are set in odd periods. This yields the setup we have specified, with this period's flexible-price good being next period's sticky-price good.

The household produces goods using a linear technology, identical for the two goods. Output of flexible-price and fixed-price goods is $\gamma_t n_t^f$ and $\gamma_t n_t^s$, respectively, where productivity γ_t follows a first-order Markov process. The household's period t budget constraint is

$$q_t B_{t+1} + P_t c_t^f + P_{t-1} c_t^s = P_t \gamma_t n_t^f + P_{t-1} \gamma_t n_t^s + B_t, \tag{2}$$

where B_{t+1} denotes the quantity of one-period bonds acquired in period t at price q_t. Each of these bonds is a promise to deliver one unit of money in period $t+1$, so B_t denotes the payoffs from bonds acquired in period $t-1$.

In period t, the household observes γ_t and market prices, and then chooses c_t^f, c_t^s, n_t^f, and B_{t+1}. Again, the labor input for production of the sticky-price good, n_t^s, is determined by the the household's share of demand for the sticky-price good at market prices. From the first-order conditions for an optimum and market clearing in the bond market (on which the supply of bonds is zero) and in the market for flexible-price goods,

$$u'(c_t^f) = \frac{1}{\gamma_t}, \tag{3}$$

$$u'(c_t^f) = \pi_t u'(c_t^s), \tag{4}$$

$$q_t u'(c_t^f) = \beta_t E_t \left[\frac{u'(c_{t+1}^f)}{\pi_{t+1}} \right]. \tag{5}$$

Here, $\pi_t = \frac{P_t}{P_{t-1}}$ is the intratemporal relative price of the two goods, and it is also the intertemporal relative price of flexible price goods. Thus, as in Woodford-type New Keynesian models, price stickiness will lead to intratemporal and intertemporal distortions, but this is captured in a very simple fashion in this specification. Though π_t is the economically relevant relative price in the model, we can also calculate the measured gross inflation rate as

$$\mu_t = \frac{P_t c_t^f + P_{t-1} c_t^s}{P_{t-1} c_{t-1}^f + P_{t-2} c_{t-1}^s}. \tag{6}$$

Equation (3) states that exchange in the market for the flexible-price good is efficient, (4) that the marginal rate of substitution of flexible price goods for sticky-price goods is equal to their relative price, and (5) is a standard Euler equation that prices a nominal bond. In line with the New Keynesian literature, we might call (5) the IS curve, and from (3) and (4) we get

$$\frac{1}{\gamma_t} = \pi_t u'(c_t^s), \tag{7}$$

Then, from (3), output in the flexible-price sector is tied down by fundamentals (technology and preferences), and (7) specifies a Phillips curve relationship – a positive relationship between π_t and consumption in the sticky-price goods sector.

In standard New Keynesian fashion, assume that the central bank determines q_t, with $R_t = \frac{1}{q_t} - 1$ denoting the one-period nominal interest rate.

From (3) and (5) we obtain

$$q_t = \beta_t E_t \left[\frac{\gamma_t}{\gamma_{t+1}\pi_{t+1}} \right] \tag{8}$$

Equation (8) determines the stochastic process $\{\pi_{t+1}\}_{t=0}^{\infty}$ given a central bank policy specifying a rule for q_t. Then we can determine consumption of the sticky-price good from (7). Thus, this economy behaves in a neo-Fisherian fashion. In a world in which the monetary policy instrument is a market nominal interest rate, the nominal interest rate determines anticipated future inflation. Our assumption of quasilinearity in the household's utility function implies an absence of wealth effects (the marginal utility of wealth is constant), so the real interest rate is exogenous. In particular, if s_t denotes the price, in units of the current flexible price of good, of a claim to one unit of the flexible price good next period, then

$$s_t = \beta_t E_t \left[\frac{\gamma_t}{\gamma_{t+1}} \right]. \tag{9}$$

That is, the real interest rate is determined by the discount factor and anticipated productivity growth.

2.1 Optimal Policy When the Zero Lower Bound is not Binding

We will first characterize an optimal monetary policy rule in the case where the zero lower bound (ZLB) constraint on the nominal interest rate does not bind, as a benchmark case, before we go on to examine cases where the ZLB matters.

A social planner seeking to maximize household utility in this economy need only solve a sequence of one-period static problems, i.e.

$$\max_{c_t^f, c_t^s} \left[u(c_t^f) + u(c_t^s) - \frac{(n_t^f + n_t^s)}{\gamma_t} \right],$$

for $t = 0, 1, 2, ...$, and the first-order conditions for an optimum are

$$u(c_t^i) = \frac{1}{\gamma_t}, \text{ for } i = f, s, \text{ and } t = 0, 1, 2, ... \tag{10}$$

What is an optimal monetary policy rule? Any such rule must satisfy the zero lower bound (ZLB) constraint

$$q_t \leq 1 \tag{11}$$

for all t. If we ignore the ZLB constraint, then the solution is straightforward. From (3) and (4), if

$$\pi_t = 1$$

for all t, then this will support the social planner's optimum in equilibrium. From (8), a policy rule that can support the optimal allocation as a unique equilibrium is

$$q_t = \min\left[1, \beta_t E_t\left(\frac{\gamma_t}{\gamma_{t+1}\pi_{t+1}}\right)\pi_t\right]. \tag{12}$$

From (12) and (8), $\pi_t = 1$ in equilibrium, so the policy rule observed in equilibrium is

$$q_t = \beta_t E_t\left[\frac{\gamma_t}{\gamma_{t+1}}\right]. \tag{13}$$

A necessary and sufficient condition for the optimality of the policy rule (12) is that the zero lower bound constraint (11) be satisfied in all states of the world in equilibrium, or from (13).

$$\beta_t E_t\left[\frac{\gamma_t}{\gamma_{t+1}}\right] \leq 1.$$

Readers may wonder why the equilibrium policy rule (13) is not sufficient for optimality. A problem is that there exist stochastic processes for the technology shock (for example, the i.i.d. case) that will yield multiple equilibria if the central bank follows the rule given by (13). But, the rule (12) kills off any undesired equilibria.

The equilibrium policy rule (13) should be familiar from the New Keynesian literature, as it states that the nominal interest rate is equal to the "natural real rate of interest." That is, from (13) and (9), $q_t = s_t$. A difference here from some of the literature is that the natural real rate of interest is equal to the actual real rate of interest, which is invariant to monetary policy.

2.1.1 Technology Shocks

Next, consider a special case. Assume that the discount factor is $\beta_t = \beta$, a constant, for all t, and that productivity is currently high, and is expected to revert to a low level forever sometime in the future. Therefore, in this setup current expected productivity growth is low, so that the real interest rate is low, but productivity growth is expected to revert to a permanently higher level sometime in the future, with a correspondingly higher real interest rate. To be more specific, there are two states for productivity, γ^h and γ^l with $\gamma^h > \gamma^l$. Assume that the initial state is $\gamma_0 = \gamma^h$, and that

$$\Pr[\gamma_{t+1} = \gamma^h \mid \gamma_t = \gamma^h] = \rho,$$

where $0 < \rho < 1$, and that the low productivity state is an absorbing state, i.e.

$$\Pr \Pr[\gamma_{t+1} = \gamma^l \mid \gamma_t = \gamma^l] = 1.$$

This stochastic process was chosen specifically to capture, in a straightforward way, a temporary – but possibly highly-persistent – state of the world in which the central bank is faced with a low real interest rate and the possibility that the ZLB constraint could bind. First, though, we want to characterize an optimal policy when the central bank does not encounter the ZLB.

Solving for relative prices in the high-productivity and low-productivity states, π^h and π^l, respectively, from (8) we obtain:

$$\pi^h = \frac{\beta\rho}{q^h - (1-\rho)q^l\frac{\gamma^h}{\gamma^l}}, \tag{14}$$

$$\pi^l = \frac{\beta}{q^l}, \tag{15}$$

where q^i, $i = h, l$, denotes the policy choice of the bond price in high and low productivity states, respectively. As in the general case, if policy can achieve an equilibrium in which $\pi^h = \pi^l = 1$, then such a policy is optimal, provided that the policy satisfies the ZLB constraints

$$q^i \leq 1 \text{ for } i = h, l. \tag{16}$$

From (14) and (15) we can solve for the optimal policy, which is

$$q^h = \beta\left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l}\right] = q^{h*} \tag{17}$$

$$q^l = \beta = q^{l*} \tag{18}$$

For this stochastic process for the technology shock, a sophisticated policy rule of the form (12) is not necessary, and we have specified the rule as in (13). That is, the optimal policy described by (17) and (18) supports a unique efficient equilibrium, so long as the ZLB constraints (16) are satisfied. Since $\gamma^h > \gamma^l$, the ZLB constraints are satisfied if and only if

$$\frac{\gamma^h}{\gamma^l} \leq \frac{1-\beta\rho}{\beta(1-\rho)}. \tag{19}$$

Thus, the ZLB constraint does not bind if and only if the drop in productivity when reversion takes place is sufficiently small. So, assuming that (19) holds, the nominal interest rate is low while productivity is high, and then reverts to a higher level. Basically, the inflation rate is zero at the optimum, so the optimal nominal interest rate must track the real interest rate, which is low when productivity is high. From (9), the "natural" real interest rate is given by $\frac{1}{s_t} - 1$, with $s_t = s^i$, $i = h, l$, where s^i denotes the price of a real bond when $\gamma_t = \gamma^i$, for $i = h, l$. From (9),

$$s^h = \beta\left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l}\right],$$

and $s^l = \beta$. Therefore, the natural rate is low in the high-productivity state, and then reverts to a higher value.

2.1.2 Discount Factor Shocks

The low natural rate could also arise because of a preference shock, i.e. a high discount factor, as in some of the New Keynesian literature, e.g. Eggertsson and Woodford (2003) and Werning (2011). Suppose for example that $\gamma_t = \gamma$, a constant, for all t, and that, rather than a high-productivity state that reverts to a low-productivity state, there is a high-discount-factor state with $\beta_t = \beta^h$ that ultimately reverts to a low-discount-factor state with $\beta_t = \beta^l$, where $\beta^h > \beta^l$. Then, in a similar fashion to the solution with productivity shocks, the solution for relative prices in each state is:

$$\pi^h = \frac{\beta^h \rho}{q^h - (1-\rho) q^l \frac{\beta^h}{\beta^l}}, \tag{20}$$

$$\pi^l = \frac{\beta^l}{q^l}, \tag{21}$$

In this case, an optimal monetary policy when the ZLB constraints (16) do not bind is

$$q^i = \beta^i, \text{ for } i = h, l, \tag{22}$$

so the ZLB constraints do not bind if and only if

$$\beta^h \leq 1. \tag{23}$$

As with technology shocks, the optimal policy is a low nominal interest rate when the natural rate is low.

So far this is straightforward, and in line with typical New Keynesian models. Price stability is optimal, as this implies that there are no relative price distortions arising from sticky prices. But, in the context of aggregate shocks, active monetary policy is necessary to induce price stability. Further, for either productivity shocks or preference shocks, the optimal policy implies that the nominal interest rate is low when the natural real interest rate is low. Finally, note that the optimal monetary policy is time consistent when the ZLB constraint does not bind. That is, once the natural interest rate reverts to its long-run higher value, the central bank has no incentive to deviate from its promise to increase the nominal interest rate.

2.2 Optimal Policy with a Binding ZLB Constraint

In this section, we will determine optimal monetary policies with productivity shocks and preference shocks, for cases in which the ZLB constraint binds. First, suppose that $\beta_t = \beta$, and there are productivity shocks as specified in the previous section. Also assume that (19) does not hold, so that the ZLB constraint binds in the high-productivity state, i.e. $q^h = 1$.

First, consider the case in which the central bank cannot commit. Then, when productivity reverts to the low state, $\gamma_t = \gamma^l$, it is optimal for the central bank to choose $q^l = \beta$, so from (14) and (15), $\pi^l = 1$ and

$$\pi^h = \frac{\beta\rho}{1 - (1-\rho)\beta\frac{\gamma^h}{\gamma^l}}. \tag{24}$$

Since (19) does not hold, we have $\pi^h > 1$, i.e. the binding ZLB constraint and lack of commitment implies that the inflation rate is higher than it would be at the optimum if the ZLB constraint did not bind. As well, output is higher than if the constraint did not bind. This is quite different from Werning (2011), who argues that a binding ZLB constraint in related circumstances will lead to inflation below the central bank's target, and low output. Here, the low real interest rate creates inflation that is too high, through the Fisher effect, given the binding ZLB constraint.

Next, assume that the central bank can commit in period 0 to a policy q^l when productivity reverts to its lower value. This is forward guidance, which of course is not feasible if the central bank cannot commit. The central bank's problem is to commit to a policy that maximizes the expected utility of the household at the first date. That is, the central bank solves

$$\max_{q^l} \left\{ (1-\beta)\left[u(c^h) - \frac{c^h}{\gamma^h}\right] + \beta(1-\rho)\left[u(c^l) - \frac{c^l}{\gamma^l}\right]\right\}, \tag{25}$$

where c^h and c^l denote, respectively, consumption of sticky-price goods in the high and low-productivity states (note that flexible price consumption is always efficient). From (7), (14), (15), and (17), c^h and c^l solve

$$1 = \beta\left[\rho\gamma^h u'(c^h) + (1-\rho)\frac{\gamma^h q^l}{\gamma^l \beta}\right], \tag{26}$$

$$u'(c^l) = \frac{q^l}{\gamma^l \beta}. \tag{27}$$

Thus, the central bank's problem is to solve (25) subject to (26) and (27). From (26) and (27) it is clear that c^h is an increasing function of q^l, while c^l is a decreasing function of q^l. If we let $\tilde{q} \equiv \frac{\gamma^l(1-\beta\rho)}{\gamma^h(1-\rho)}$, then from (19), (**??**), (26), and (27), equilibrium welfare is strictly increasing in q^l for $q^l \in (0, \tilde{q}]$, and welfare is strictly decreasing for $q^l \in [\beta, 1]$. Further, since (19) does not hold, therefore $\tilde{q} < \beta$. Thus, welfare is maximized for $q^l \in (\tilde{q}, \beta)$.

The optimal policy therefore implies, from (3) and (4), that $\pi^h > 1$ and $\pi^l > 1$, so the best policy at the zero lower bound is a promise of high inflation in the future when productivity reverts to its lower value. Note that this higher inflation is achieved with a higher nominal interest rate than the central bank would choose if it could not commit. That is, the optimal policy for the central bank, without commitment, once productivity reverts to its low value, is $q^l = \beta$, but in the high-productivity state, the central bank wants to commit to a future policy $q^l < \beta$. Thus, the optimal policy for the central bank is achieved with a neo-Fisherian commitment: a promise of high future inflation brought about by a higher future nominal interest rate.

In some analyses of monetary policy at the ZLB, for example Werning (2011), the degree of price flexibility matters in an important way. In the ZLB problem that Werning (2011) specifies, the welfare loss due to the binding ZLB constraint increases as prices become more flexible. While perfect price flexibility makes the ZLB problem go away in Werning (2011), a very small amount of price flexibility is a very bad thing.

In our model, the degree of price flexibility is determined by the length of the period, but in adjusting the length of the period, we have to make appropriate adjustments in the discount factor and the probability of reversion to the low-productivity state. That is, shortening the period implies that the discount factor should increase and the probability of reversion to the low-productivity state should decrease. Letting Δ denote the length of a period, we let $\beta = e^{-r\Delta}$, where r is the discount rate per unit time. As well, if reversion to the low-productivity state is a Poisson arrival with arrival rate α, then $\rho = e^{-\alpha\Delta}$. From (19) the zero lower bound on the nominal interest rate is a binding constraint for the central banker if and only if

$$1 < \beta \left[\rho + (1-\rho)\frac{\gamma^h}{\gamma^l} \right], \tag{28}$$

or

$$1 < \frac{\gamma^h}{\gamma^l} e^{-r\Delta} - \left(\frac{\gamma^h}{\gamma^l} - 1 \right) e^{-(r+\alpha)\Delta}. \tag{29}$$

Note, from (9), that (28) and (29) state that the ZLB constraint is binding for central banker if and only if the real interest rate is negative.

Define the right-hand side of (29) by

$$\phi(\Delta) = \frac{\gamma^h}{\gamma^l} e^{-r\Delta} - \left(\frac{\gamma^h}{\gamma^l} - 1 \right) e^{-(r+\alpha)\Delta}$$

Then,

$$\phi(0) = 1$$

$$\lim_{\Delta\to\infty} \phi(\Delta) = 0$$

$$\phi'(\Delta) = e^{-r\Delta} \left[-r\frac{\gamma^h}{\gamma^l} + (r+\alpha)\left(\frac{\gamma^h}{\gamma^l} - 1 \right) e^{-\alpha\Delta} \right]$$

Therefore,

$$\phi'(0) = \alpha\left(\frac{\gamma^h}{\gamma^l} - 1 \right) - r$$

So, if

$$\alpha\left(\frac{\gamma^h}{\gamma^l} - 1 \right) - r \leq 0,$$

then (29) does not hold for any Δ, and the ZLB is not a problem. However, if

$$\alpha\left(\frac{\gamma^h}{\gamma^l}-1\right)-r>0, \tag{30}$$

then $\phi'(\Delta) > 0$ for $\Delta \in [0,\Delta^*)$, and $\phi'(\Delta) < 0$ for $\Delta > \Delta^*$, and (29) holds for $\Delta \in (0,\bar{\Delta})$, where $\phi(\bar{\Delta}) = 1$. Further, price stickiness is worst for $\Delta = \Delta^*$, where $\phi'(\Delta^*) = 0$. From (**??**), we can solve for Δ^* to obtain

$$\Delta^* = \frac{\ln\left(1+\frac{\alpha}{r}\right)+\ln\left(1-\frac{\gamma^l}{\gamma^h}\right)}{\alpha}$$

The function $\phi(\Delta)$ is depicted in Figure 2.

[Figure 2 here.]

These results are quite different from Werning (2011). To see this, note that $\phi(\Delta)$ is essentially a measure of the welfare loss due to the binding ZLB constraint, and $\phi(\Delta)$ is also the price of a real bond. Thus, higher $\phi(\Delta)$ implies a lower real interest rate. First, $\phi(\Delta) = 0$, so in contrast to Werning (2011) there is no discontinuity at zero – a small amount of price stickiness implies a small potential welfare loss from the binding ZLB constraint. Second, if (30) holds, so that the ZLB constraint is a problem, then $\phi(\Delta)$ is increasing, then decreasing in price stickiness. In Werning (2011), the welfare loss from the binding ZLB constraint is monotonically decreasing in the degree of price stickiness. In our model, the left-hand side of inequality (30) is strictly increasing in α, strictly increasing in the ratio $\frac{\gamma^h}{\gamma^l}$, and strictly decreasing in r. An increase in α reduces ρ, which lowers the real interest rate; an increase in $\frac{\gamma^h}{\gamma^l}$ lowers the real interest rate, and an increase in r raises the real interest rate. Thus, a third difference from Werning (2011) is that the severity of the ZLB problem increases as the expected length of time at the ZLB falls. In Werning (2011) the welfare loss at the ZLB increases with the length of time over which the ZLB is a binding constraint.

Next, consider the case in which $\gamma_t = \gamma$, a constant, for all t, and there is a high discount factor instead of high productivity in the low natural real interest rate state. That is, $\beta_t \in \{\beta^h, \beta^l\}$, with $\beta_0 = \beta^h$ and β_t follows the same stochastic process as γ_t does in the example considered above. In the New Keynesian literature, this is a typical approach to constructing a scenario in which the real interest rate is low (see for example Werning 2011). In the case in which the central bank cannot commit and the ZLB constraint binds in state h, we have $\pi^l = 1$ and

$$\pi^h = \frac{\beta^h\rho}{1-(1-\rho)\beta^h}.$$

The ZLB constraint binds if and only if $\beta^h > 1$. From equation (9), $s_t = \beta_t$, so the ZLB constraint binds if and only if the real interest rate is negative in the high-discount-factor state.

In the case where the central bank can commit at the first date, we can write the central bank's problem as

$$\max_{q^l} \left\{ (1-\beta^l) \left[u(c^h) - \frac{c^h}{\gamma^h} \right] + \beta^h (1-\rho) \left[u(c^l) - \frac{c^l}{\gamma^l} \right] \right\}, \tag{31}$$

Suppose that $\beta^h > 1$, so that $q^h = 1$ at the optimum (the ZLB constraint binds in the high-discount-factor state). Then, from (20), (21), (7), and (8), the two equations

$$1 = \beta^h \left[\rho\gamma u'(c^h) + \frac{(1-\rho)\, q^l}{\beta^l} \right], \tag{32}$$

$$u'(c^l) = \frac{q^l}{\gamma\beta^l}, \tag{33}$$

solve for c^h and c^l given a monetary policy q^l. We can then obtain a result that is qualitatively identical to that with productivity shocks. In particular, if $\beta^h > 1$, this implies that the optimal policy is $q^l \in (\bar{q}, \beta^l)$, where

$$\bar{q} = \frac{(1-\beta^h\rho)\beta^l}{(1-\rho)\beta^h},$$

and optimal policy implies $\pi^h > 1$, $\pi^l > 1$, and a commitment that the nominal interest rate will be higher in the low-discount-factor state than would be the case without commitment.

In order to evaluate how price stickiness matters for this setup, the discount factor in the high state is $\beta^h = e^{-r^h\Delta}$, where r^h is the discount rate in state h, and Δ denotes period length. For the ZLB problem to exist, we require $r^h < 0$, which implies that the real interest rate in state h decreases as Δ increases, and $e^{-r^h\Delta} = 1$ when $\Delta = 0$. Thus, the ZLB problem goes away as price stickiness disappears, and the severity of the problem is monotonically increasing in price stickiness.

There are some differences here from the case in which the real interest rate is low because of low productivity growth. In the case here, with a temporarily high discount factor, price stickiness matters in terms of how households discount the future, in which prices are flexible, relative to the present, in which some prices are fixed. Greater price stickiness implies that this future is discounted at a lower rate. However, if the real interest rate is low because of low productivity growth, what matters is anticipated productivity growth between the current period, when prices are fixed, and the future period, when prices are flexible. With greater price stickiness, anticipated productivity growth is lower between the sticky-price present and the flexible-price future, which tends to lower the real interest rate . However, greater price stickiness implies that the flexible-price future is discounted at a higher rate, which tends to increase the real interest rate. At low levels of price stickiness, the first effect dominates, while at high levels of price stickiness, the second effect dominates. Thus, the distortion from price stickiness when the ZLB binds is nonmonotonic when the real interest rate is low because of low productivity growth.

Other than some differences with regard to the effects of price flexibility, this baseline model yields essentially identical conclusions, whether the real interest rate is temporarily low because of low productivity growth, or because of a temporarily-high discount factor. However, the results are very different from those in standard New Keynesian models, for example Werning (2011). Why the difference? In our model, the equilibrium real interest rate is exogenous, which gives the model neo-Fisherian properties. When the real interest rate is low, this makes the inflation rate high, given the nominal interest rate. So, if the ZLB constraint is binding for a benevolent central banker, then the inflation rate is too high. If the central banker is able to commit in such circumstances, then optimal forward guidance comes in the form of a promise to keep the nominal interest rate and inflation rate higher in the future than they would be without commitment.

3 Credit, Collateral, and Low Real Interest Rates

Given that our goal is to understand the role of monetary policy in the context of low real interest rates, it will be productive to consider more explicitly the causes of such low real interest rates. One potential explanation for the low real interest rates that have been observed recently in the world is that there is a low supply of safe assets relative to the demand for such assets (see Andolfatto and Williamson 2015 and Caballero et al. 2016). We can model this safe asset shortage by including an explicit role for government debt in our model. For now, we will retain the assumption that money is in zero supply – the economy is cashless. However, we will assume that all transactions in the goods market use secured credit, with government debt serving as collateral. An interpretation is that we are capturing, in a stylized way, the key roles of government debt in credit markets. For example, government debt is the primary asset used as collateral in the market for repurchase agreements, and government debt is an important asset backing the deposit liabilities of banks.

Assume that the representative household receives a lump-sum transfer τ_t from the government in period t, so we can write the government's budget constraints as

$$q_0 b_0 = \tau_0.$$

$$q_t b_t = \frac{b_{t-1}}{\pi_t} + \tau_t, \text{ for } t = 1, 2, ... \tag{34}$$

Here, b_t denotes the bonds issued in period t, in units of the flexible-price good. Also assume that the fiscal authority sets exogenously the total value of government debt, v_t, so

$$v_t = q_t b_t, \tag{35}$$

which implies that transfers are endogenous in periods $t = 1, 2, ...$. Re-write the household's budget constraint to incorporate the transfer:

$$q_t b_{t+1} + c_t^f + \frac{c_t^s}{\pi_t} = \gamma_t n_t^f + \frac{\gamma_t n_t^s}{\pi_t} + \frac{b_t}{\pi_t} + \tau_t \tag{36}$$

We will assume that, within the period, goods must be purchased with secured credit. Suppose that each household is a buyer/seller pair. At the beginning of the period, the buyer in the household purchases goods with IOUs, while the seller exchanges goods for IOUs. Then, within-period debts are settled at the end of the period. Also assume that households cannot commit, and that there is no memory. In particular, no records can be kept of past defaults. This implies that there can be no unsecured credit. But, there exists a technology which allows households to post government debt as collateral. Then, the following incentive constraint must be satisfied

$$c_t^f + \frac{c_t^s}{\pi_t} \leq \breve{q}_t b_{t+1}, \tag{37}$$

where $\breve{q}_t$ denotes the price of government debt at the end of the period. The inequality (37) states that the value of purchases of consumption goods (in units of the flexible-price good) cannot exceed the value of the collateral posted by the household. Here the value of the collateral is assessed as the value to the household at the end of the period. In other words, the household must post sufficient collateral that it has the incentive to pay off its debts at the end of the period rather than absconding.

For simplicity, assume that $\gamma_t = 1$ and $\beta_t = \beta$, a constant, for all t. Then, letting μ_t and λ_t denote, respectively, the multipliers associated with (36) and (37), the following must be satisfied:

$$u'(c_t^f) - \mu_t - \lambda_t = 0, \tag{38}$$

$$u'(c_t^s) - \frac{(\mu_t + \lambda_t)}{\pi_t} = 0, \tag{39}$$

$$-1 + \mu_t = 0, \tag{40}$$

$$-q_t \mu_t + \lambda_t \breve{q}_t + \beta E_t \left[\frac{\mu_{t+1}}{\pi_{t+1}} \right] = 0. \tag{41}$$

The value of government bonds at the end of the period is

$$\breve{q}_t = \beta E_t \left[\frac{\mu_{t+1}}{\pi_{t+1}} \right]. \tag{42}$$

Then, (40), (41) and (42) give

$$q_t = (1 + \lambda_t)\, \breve{q}_t, \tag{43}$$

so the price of government debt at the beginning of the period exceeds its value at the end of the period if and only if the collateral constraint binds ($\lambda_t > 0$).

From (35), (37), (38)-(41), and (43), if the collateral constraint (37) does not bind in period t, then

$$u'(c_t^f) = 1, \tag{44}$$

$$u'(c_t^s) = \frac{1}{\pi_t} \tag{45}$$

$$-q_t + \beta E_t\left[\frac{1}{\pi_{t+1}}\right] = 0 \tag{46}$$

$$c_t^f + \frac{c_t^s}{\pi_t} \leq v_t \tag{47}$$

However, if the collateral constraint binds in period t, then

$$u'(c_t^f) - \pi_t u'(c_t^s) = 0, \tag{48}$$

$$-q_t + u'(c_t^f)\beta E_t\left[\frac{1}{\pi_{t+1}}\right] = 0 \tag{49}$$

$$c_t^f + \frac{c_t^s}{\pi_t} = \frac{v_t}{u'(c_t^f)} \tag{50}$$

$$u'(c_t^f) - 1 \geq 0 \tag{51}$$

Note, in (49), that the price of government debt reflects a liquidity premium, which increases with the inefficiency wedge in the market for the flexible price good. The inefficiency wedge is $u'(c_t^f) - 1 = \lambda_t$, the multiplier on the household's collateral constraint. Thus, the tighter is the collateral constraint, the larger is the inefficiency wedge, and the higher is the liquidity premium on government debt.

3.1 Optimal Monetary Policy When the Collateral Constraint is Tight

Since this model potentially has quite different implications from the one in the previous section, we will start with the simplest case.
Suppose that $v_t = v$, a constant, for all t, and $q_t = q$ for all t. Look for an equilibrium in which all quantities and the relative price of sticky-price and flexible price goods are constant for all t, and suppose that v is sufficiently small that the collateral constraint always binds. Then, from (48)-(51), an equilibrium consists of c^f, c^s, and π satisfying

$$u'(c^f) - \pi u'(c^s) = 0, \tag{52}$$

$$-q + \frac{u'(c^f)\beta}{\pi} = 0, \tag{53}$$

$$c^f + \frac{c^s}{\pi} = \frac{v}{u'(c^f)}, \tag{54}$$

$$u'(c^f) - 1 \geq 0, \tag{55}$$

given monetary policy q. Simplifying, from (52)-(54), the consumption allocation (c^f, c^s) solves

$$q = \beta u'(c^s), \tag{56}$$

$$c^f u'(c^f) + c^s u'(c^s) = v. \tag{57}$$

Next, restrict attention to constant-relative-risk-aversion utility, where α denotes the coefficient of relative risk aversion. Then, from (56) and (57), we can write the monetary policy problem as

$$\max_q \left[\frac{\left(c^f\right)^{1-\alpha}}{1-\alpha} - c^f + \frac{(c^s)^{1-\alpha}}{1-\alpha} - c^s \right]$$

subject to

$$q = \beta (c^s)^{-\alpha}, \tag{58}$$

$$q \leq 1, \tag{59}$$

$$\left(c^f\right)^{1-\alpha} + (c^s)^{1-\alpha} = v. \tag{60}$$

If the ZLB constraint (59) does not bind, then the solution is

$$c^s = c^f = \left(\frac{v}{2}\right)^{\frac{1}{1-\alpha}},$$

$$q = \beta \left(\frac{2}{v}\right)^{\frac{\alpha}{1-\alpha}}, \tag{61}$$

$$\pi = 1,$$

Then, from (61), the ZLB constraint does not bind if and only if

$$v \geq 2\beta^{\frac{1-\alpha}{\alpha}},$$

Also, note that, from (51), the collateral constraint binds if and only if

$$v < 2,$$

so for a nonbinding ZLB constraint to be compatible with a binding collateral constraint,

$$2\beta^{\frac{1-\alpha}{\alpha}} \leq v < 2.$$

But if

$$v < 2\beta^{\frac{1-\alpha}{\alpha}}$$

then the zero lower bound constraint binds, and optimal policy is given by $q = 1$, with

$$c^s = \beta^{\frac{1}{\alpha}},$$

$$c^f = \left(v - \beta^{\frac{1-\alpha}{\alpha}}\right)^{\frac{1}{1-\alpha}},$$

$$\pi = \beta\left(v - \beta^{\frac{1-\alpha}{\alpha}}\right)^{\frac{-\alpha}{1-\alpha}} > 1$$

Therefore, if v is sufficiently large, then zero inflation is optimal, though there is an inefficiency wedge associated with the binding collateral constraint, i.e. $u'(c^s) = u'(c^f) > 1$. But if v is small, which implies a large inefficiency wedge and a large liquidity premium on government debt, the ZLB constraint binds, and $\pi > 1$ at the optimum. So, the inefficiency related to the binding ZLB constraint is reflected in too-high inflation. Further, a decline in v when the ZLB constraint binds at the optimum increases the inflation rate. That is, a tighter collateral constraint raises inflation at the optimum.

Next, consider a scenario like the one we considered with the previous version of the model. Suppose that there are two states, $v_t = v^l$ and $v_t = v^h$, with $v^l < v^h$. Assume that the economy is initially in the state with low v, and assume that the state evolves as in the previous sections. That is

$$\Pr[v_{t+1} = v^l \mid v_t = v^l] = \rho,$$

$$\Pr[v_{t+1} = v^h \mid v_t = v^h] = 1.$$

Suppose first that, given optimal monetary policy, the collateral constraint does not bind in the high-v state, and binds in the low-v state. Let c^{hf} and c^{hs} denote, respectively, consumption of flexible-price and fixed-price goods in the high-v state, while c^{lf} and c^{ls} are the corresponding quantities in the low-v state. Also let $\pi_t = \pi^i$, and $q_t = q^i$, where $i = h, l$ denote the high and low-v states, respectively. We will continue to assume that the utility function has constant coefficient of relative risk aversion α.

From (46), $\pi^h = \frac{\beta}{q^h}$, so given monetary policy (q^l, q^h), from (48)-(50), consumption quantities in the low-v state are determined by:

$$q^l = \beta\rho(c^{ls})^{-\alpha} + q^h(1-\rho)(c^{lf})^{-\alpha} \tag{62}$$

$$v^l = (c^{ls})^{1-\alpha} + (c^{lf})^{1-\alpha} \tag{63}$$

First, consider the case where the central bank cannot commit. Then, once v reverts to v^h, so that the collateral constraint does not bind, the central bank will choose $q^h = \beta$, implying $\pi^h = 1$ and $c^{hf} = c^{hs} = 1$.Similar to the case with constant v and a binding collateral constraint, if

$$2\beta^{\frac{1-\alpha}{\alpha}} \leq v^l < 2, \tag{64}$$

then there is an optimal monetary policy in the low-v state given by

$$q^l = \beta\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}} \tag{65}$$

which implies that, in equilibrium,67

$$c^{lf} = c^{ls} = \left(\frac{v^l}{2}\right)^{\frac{1}{1-\alpha}} \tag{66}$$

In this case, the collateral constraint binds in the low-v state, but a policy supporting an optimal allocation (given a lack of commitment) does not imply a binding ZLB constraint.

A complication is that, given the policy $\left(q^l, q^h\right) = \left(\beta\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}}, \beta\right)$ there may exist a second, suboptimal equilibrium. To see this, $\left(q^l, q^h\right) = \left(\beta\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}}, \beta\right)$ implies that we can write (62) as

$$\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}} = \rho(c^{ls})^{-\alpha} + (1-\rho)(c^{lf})^{-\alpha} \tag{67}$$

Then, change variables, letting $x^l = u'(c^{ls})$ and $x^f = u'(c^{lf})$, so we can rewrite (67) and (63), respectively, as

$$\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}} = \rho x^s + (1-\rho)x^f \tag{68}$$

$$v^l = (x^s)^{1-\frac{1}{\alpha}} + (x^f)^{1-\frac{1}{\alpha}} \tag{69}$$

Clearly, $x^s = x^f = \left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}}$ is a solution to (68) and (69), which is the equilibrium allocation (66) we used to back out the optimal policy. While (68) is linear in x^s and x^f, (69) is strictly concave for any $\alpha > 0$, $\alpha \neq 1$. If $0 < \alpha < 1$, then (69) is depicted in Figure 3, where point A is the desired equilibrium allocation, but there potentially exists another equilibrium solution to (68) and (69), which is point B if $\rho > \frac{1}{2}$ and point C if $\rho < \frac{1}{2}$. Note that, if $\rho = \frac{1}{2}$ there is only one solution. In equilibrium (51) must hold, which it does at point A and at point B if $\rho > \frac{1}{2}$ in Figure 3. However, there is no guarantee that (51) holds if $\rho < \frac{1}{2}$ for the equilibrium at point C. Alternatively, if $\alpha > 1$, then (69) is depicted in Figure 4. Point A is the desired equilibrium allocation, and points B and C have exactly the same interpretations as in Figure 3, with the same conclusions.

[Figures 3,4 here.]

The bottom line is that we can construct an optimal policy with no commitment, for this case where the ZLB constraint does not bind and the collateral constraint binds in the low-v state. But there are cases for which the central bank's policy supports the optimal allocation as an equilibrium, and also supports a suboptimal allocation. An equilibrium with a high liquidity premium on government debt, a low real interest rate, and a high inflation rate in the low-v

state, can coexist with a second equilibrium with a lower liquidity premium, a higher real interest rate, and a lower inflation rate.

If $v^l < 2\beta^{\frac{1-\alpha}{\alpha}}$, and still assuming no commitment, the policy given by (65) is not feasible, so in that case $(q^l, q^h) = (1, \beta)$ supports an optimal equilibrium allocation without commitment – the ZLB constraint binds at the optimum. But, just as in the case in which the ZLB constraint does not bind, it is straightforward to show (just as above) that an undesired equilibrium can coexist with the desired optimal equilibrium allocation, given the optimal policy.

Next, suppose that the central bank can commit to a policy $\left(q^l, q^h\right)$. Assume for now that the collateral constraint does not bind in the high-v state (below, we will determine conditions under which the collateral constraint does not bind in the high-v state). If (64) holds, then the solution is the same as when the central bank cannot commit. The ZLB constraint does not bind in the low-v state, and the optimal policy is given by $\left(q^l, q^h\right) = \left(\beta\left(\frac{v^l}{2}\right)^{-\frac{\alpha}{1-\alpha}}, \beta\right)$. Again, this policy implies the potential existence of a second, suboptimal equilibrium.

However, there are circumstances under which forward guidance can work to increase welfare. An optimal monetary policy with commitment is the solution to

$$\max_{q^l, q^h}\left\{(1-\beta)\left[\frac{\left(c^{lf}\right)^{1-\alpha}}{1-\alpha} - c^{lf} + \frac{\left(c^{ls}\right)^{1-\alpha}}{1-\alpha} - c^{ls}\right] + \beta(1-\rho)\left[\frac{\left(c^{hs}\right)^{1-\alpha}}{1-\alpha} - c^{hs}\right]\right\} \tag{70}$$

subject to (62), (63), and

$$q^l \leq 1, \tag{71}$$

$$q^h \leq 1, \tag{72}$$

$$c^{hs} = \left(\frac{\beta}{q^h}\right)^{\frac{1}{\alpha}} \tag{73}$$

Further, note that the optimal monetary policy problem must have a solution satisfying (51), i.e. $c^{lf} < 1$, to be consistent with our assumption that the collateral constraint binds in the low-v state.

Suppose that

$$v^l < 2\beta^{\frac{1}{\alpha}-1}, \tag{74}$$

which is the case of interest, as otherwise there is no value to forward guidance. That is, if (74) does not hold, then the solution to the central bank's problem has the feature that the ZLB constraint does not bind in the low-v state, and the no-commitment and commitment solutions are the same. So, given (74), $q^l = 1$ at the optimum.

Then, given $q^l = 1$, (62), (63), and (73), we can substitute in the objective function (70), and write the central bank's problem as

$$\max_{q^h \in [0,1]}\left[(1-\beta)U^l(q^h) + \beta(1-\rho)U^h(q^h)\right], \tag{75}$$

where $U^i(q^h)$ denotes period utility in state i given q^i, for $i = l, h$, in the desired equilibrium. First, if

$$2\,(\beta\rho)^{\frac{1}{\alpha}-1} < v^l < 2\beta^{\frac{1}{\alpha}-1},$$

then $U^l(q^h)$ is strictly increasing for $q^h \in [0, \check{q})$ and strictly decreasing for $q^h > \check{q}$, while $U^h(q^h)$ is strictly increasing for $q^h \in [0, \beta)$, and strictly decreasing for $q > \beta$, where

$$\check{q} = \frac{\left(\frac{V}{2}\right)^{\frac{\alpha}{1-\alpha}} - \beta\rho}{1-\rho}$$

Further, given (74), $\check{q} < \beta$. Therefore, the objective function in (75) attains a maximum for $q^h \in (\check{q}, \beta)$. Thus $q^h < \beta$, so if the central bank can commit when the ZLB binds in the low-v state, it should commit to a higher nominal interest rate and higher inflation in the high-v state than if it cannot commit. But, again, it is easy to show that there may exist an equilibrium allocation other than the desired one, given the commitment policy.

A key difference between this example and the two examples without secured credit, is that the real interest rate is inefficiently low in this example. Indeed, the low real interest rate results from a low supply of government debt, and the fiscal authority could raise the real interest rate to its efficient level by supplying more government debt. This would then eliminate the central bank's ZLB problem. Further, even if we treat the inefficiency caused by fiscal policy as given, we do not get the same conclusions about optimal monetary policy as in the previous version of the model in which the real interest rate was low for reasons outside the control of policymakers. In particular, the policymaker may face a multiple equilibrium problem. In general, a monetary policy designed to achieve an optimal allocation, given the constraints faced by the central banker, can also support a second suboptimal equilibrium.

4 Money, Collateral, and Credit

The next step is to analyze a full-blown model that includes a rich enough set of assets to capture the essentials of monetary policy, and that can also explain why the real interest rate can be low in equilibrium. In this section, we add monetary exchange to our model, along with secured credit, captured in the same way as in the last section. There will now potentially be three distortions to be concerned with: (i) a standard Friedman-rule distortion under which there is a suboptimally low quantity of currency, in real terms; (ii) a shortage of interest-bearing debt, reflected in a low real rate of interest; (iii) a sticky price friction. To understand how this version of the model works, it will help to first consider a setup with flexible prices, which includes only the first two distortions. After we have done that analysis, we will consider the sticky price case, which includes all three distortions.

4.1 Flexible Prices

This case will work in a manner similar to Andolfatto and Williamson (2015), though a key difference from that work is in the role that government debt plays in the model. In particular, in this model government debt serves as collateral rather than being traded directly, as in Andolfatto and Williamson (2015).

We want to be explicit about how exchange works. Assume that a household consists of a continuum of consumers with unit mass, and a producer. Each consumer in the household has a period utility function $u(c_t)$, and there are two markets on which goods are sold. In the *cash-only market*, sellers of goods accept only money, as there is no technology available to verify collateral if the consumer attempts to make a credit transaction. In the *cash-and-credit market*, sellers are able to verify the ownership of government debt posted as collateral in a credit transaction, and sellers will also accept money. Unsecured credit is not a possiblility in purchasing goods, as the memory (recordkeeping) needed to support this does not exist. The household would always default on unsecured credit, so none is extended. Each consumer in a household receives a shock which determines the market he or she participates in. With probability θ the consumer goes to the cash-only market, and with probability $1-\theta$, he or she goes to the cash-and-credit market. The household allocates assets to each consumer in the household – money and any government debt to be posted as collateral – and consumers consume on the spot in the markets they go to. That is, consumption cannot be shared within the household.

The producer in the household supplies labor n_t, and can produce one unit of output for each unit of labor input (there are no technology shocks). Output is perfectly divisible and can be sold on either the cash-only market or the cash-and-credit market, or both.

A household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta u(c_t^m) + (1-\theta) u(c_t^b) - n_t, \right]$$

where c_t^m denotes the consumption of each consumer who goes to the cash-only market, while c_t^b is consumption of each consumer in the cash-and-credit market.

At the beginning of the period, the household trades on the asset market and faces the constraint

$$q_t b_{t+1} + \theta c_t^m + m_t' \leq \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{76}$$

On the right-hand side of inequality (76), the household has wealth at the beginning of the period consisting of the payoffs on money and bonds held over from the previous period and the lump-sum transfer from the fiscal authority. Here, m_t denotes beginning-of-period money balances in units of the period $t-1$ cash market consumption good. The left-hand side of (76) includes purchases of one-period nominal government bonds, money (in units of the period t cash market good) the household requires for cash market goods purchases, and money, m_t', that is sent with consumers to the cash-and-credit market.

In the cash-and-credit market, consumers from the household can purchase goods with cash m'_t, or with credit secured by government debt, so the following constraint must hold:

$$(1-\theta)c_t^b \leq b_{t+1} + m'_t. \tag{77}$$

In inequality (77), note that the IOUs issued by the household (by way of consumers in the household) are settled at the end of the period, at which time the bonds the household acquired at the beginning of the period are worth b_{t+1}. That is, at the end of the period, government bonds which pay off at the beginning of the subsequent period are equivalent to cash. Inequality (77) states that, for cash-and-credit purchases in excess of what is paid for with cash, the household will prefer to pay its debt at the end of the period rather than enduring seizure of the bonds posted as collateral.

Finally, the household must satisfy its budget constraint

$$\theta c_t^m + (1-\theta)c_t^b + q_t b_{t+1} + m_{t+1} \leq n_t + \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{78}$$

We can summarize the first order conditions from the household's problem by

$$u'(c_t^b) = q_t u'(c_t^m), \tag{79}$$

$$1 = \underbrace{\beta E_t\left[\frac{u'(c_{t+1}^m)}{u'(c_t^m)\pi_{t+1}}\right]}_{\text{fundamental}} + \underbrace{\frac{u'(c_t^m)-1}{u'(c_t^m)}}_{\text{liquidity premium}}, \tag{80}$$

$$q_t = \underbrace{\beta E_t\left[\frac{u'(c_{t+1}^m)}{u'(c_t^m)\pi_{t+1}}\right]}_{\text{fundamental}} + \underbrace{\frac{u'(c_t^b)-1}{u'(c_t^m)}}_{\text{liquidity premium}}. \tag{81}$$

First, (79) reflects intratemporal optimization. Note that the price of goods purchased on the cash-and-credit market relative to the price of goods on the cash market is q_t, which is also the price of a nominal bond. Second, equations (80) and (81) have been written so as to show the similarities in asset pricing between money and government debt, respectively. In equation (80), the left-hand side is the current price of money, normalized, while the right hand side consists of the fundamental and a liquidity premium. The fundamental is the expected payoff on money in the next period, appropriately discounted, while the liquidity premium is related to the inefficiency in the market for goods purchased with cash. That is, $u'(c_t^m) - 1$ is an inefficiency wedge in this market. The liquidity premium on money is something we observe in most mainstream monetary models, and it typically disappears if the central bank runs a Friedman rule. Similarly, in equation (81), the price of government debt, on the left-hand side, is equal to the sum of a fundamental plus a liquidity premium, on the right-hand side. The fundamental is identical to the one in equation (80), since the explicit payoff on the asset is the same as for money. But government debt has a different liquidity premium, which is related to the inefficiency wedge $u'(c_t^b) - 1$, in the cash-and-credit market.

The consolidated government budget constraints are:

$$m_1 + q_0 b_1 = \tau_0, \tag{82}$$

$$m_{t+1} + q_t b_{t+1} - \frac{m_t + b_t}{\pi_t} = \tau_t, \tag{83}$$

where m_t denotes the real quantity of money outstanding at the beginning of period t, before government intervention occurs. Here, we will assume that the fiscal authority fixes exogenously the path for the real value of the consolidated government debt, i.e.

$$v_t = m_{t+1} + q_t b_{t+1}, \tag{84}$$

where v_t is exogenous. Then, solving for an equilibrium, in any period t, (79) and (80) hold and either

$$u'(c_t^b) = 1$$

and

$$\theta c_t^m + (1-\theta) q_t c_t^b \leq v_t$$

or

$$u'(c_t^b) > 1$$

and

$$\theta c_t^m + (1-\theta) q_t c_t^b = v_t$$

Thus, in period t either exchange is efficient in the cash-and-credit market and the collateral constraint (77) does not bind, or exchange is inefficient in the cash-and-credit market and the collateral constraint binds.

4.1.1 Optimality

Note that the model solves period-by-period for c_t^m and c_t^b, and thus for labor supply, as

$$n_t = \theta c_t^m + (1-\theta) c_t^b$$

Letting c^* denote the solution to $u'(c^*) = 1$, if

$$v_t \geq c^*,$$

then $q_t = 1$ at the optimum, and $c_t^m = c_t^b = c^*$. This is essentially a Friedman rule result. If the collateral constraint does not bind, then exchange will be efficient in the cash-and-credit market. Therefore, if $q_t = 1$, and the collateral constraint does not bind, exchange is efficient in both markets in period t.

However, if

$$v_t < c^*,$$

then the household's collateral constraint binds for $q_t = 1$. When the collateral constraint binds, then

$$\theta c_t^m + (1-\theta) q_t c_t^b = v_t, \tag{85}$$

$$u'(c_t^b) = q_t u'(c_t^m), \tag{86}$$

and equations (85) and (86) solve for (c_t^m, c_t^b) given policy (v_t, q_t). Then, differentiating (85) and (86), and dropping t subscripts for convenience,

$$\frac{dc^m}{dq} = \frac{-(1-\theta)u'(c^b)\left[\frac{c^b u''(c^b)}{u'(c^b)} + 1\right]}{\theta u''(c^b) + (1-\theta)qu''(c^m)}, \tag{87}$$

$$\frac{dc^b}{dq} = \frac{\theta u'(c^m) - (1-\theta)qu''(c^m)c^b}{\theta u''(c^b) + (1-\theta)qu''(c^m)}, \tag{88}$$

We can evaluate welfare period-by-period, with the period utility of the household equal to $W(q)$, where

$$W(q) = \theta\left[u(c^m) - c^m\right] + (1-\theta)\left[u(c^b) - c^b\right] \tag{89}$$

If $q = 1$, then from (85) and (86), we have $c^m = c^b = v$. Then, if we evaluate the derivative of welfare at $q = 1$, from (87)-(89), we get

$$W'(1) = -\left[u'(v) - 1\right](1-\theta)v < 0,$$

so increasing the nominal interest rate above zero (reducing q below 1) is optimal for the central bank if the collateral constraint binds when the nominal interest rate is zero.

But, what is the optimal monetary policy? If the collateral constraint does not bind, then (c^m, c^b) is determined by $u'(c^b) = 1$ and $u'(c^m) = \frac{1}{q}$. Therefore, in standard fashion, increasing q must increase welfare when the collateral constraint does not bind, since consumption of goods in the cash market increases and consumption in the cash-and-credit market does not change. Therefore, if the collateral constraint binds when $q = 1$, then $q < 1$ at the optimum and (**??**) holds with equality at the optimum.

Then, if $-\frac{c^b u''(c^b)}{u'(c^b)} > 1$, from (87) and (88), $\frac{dc^m}{dq} < 0$ and $\frac{dc^b}{dq} < 0$, so reducing q until the collateral constraint does not bind, but (85) holds with equality is optimal. In this case it is optimal to eliminate the inefficiency in the cash-and-credit market, at the expense of greater inefficiency in the cash market. Alternatively, if $-\frac{c^b u''(c^b)}{u'(c^b)} < 1$, then from (87) and (88), $\frac{dc^m}{dq} > 0$ and $\frac{dc^b}{dq} < 0$, so welfare is increasing at the margin in q when $u'(c^b) = 1$. Therefore, the collateral constraint binds at the optimum in this case.

Therefore, so long as the supply of government debt is sufficiently low, the central bank should conduct open market operations so as to raise the nominal interest rate above zero. With enough curvature in the utility function, it is optimal in these circumstances for the central bank to sell enough government debt so as to relax the collateral constraint. This then reverses the implications of the sticky price model we started with. With sticky prices, shocks that lower the real interest rate can make a zero nominal interest rate optimal, in which case forward guidance in the form of commitments to high future inflation (and high nominal interest rates) are also part of optimal policy. But here, forward guidance does not play a role, and the nominal interest rate is zero when the collateral constraint is not binding and the real interest rate is high. But, the nominal interest rate should be greater than zero in states of the world in which the collateral constraint binds and the real interest rate is low.

We obtain these results because the low real interest rate is caused by a shortage of safe collateral – a shortage of government debt. In the context of such a shortage, an increase in the nominal interest rate is accomplished if the central bank swaps government debt for money, thus relieving the collateral shortage. While this may also have the effect of making cash more scarce, so that the inefficiency wedge goes up in cash-only market, the increase in efficiency in this respect is more than offset by an increase in efficiency in the cash-and-credit market, evaluated in welfare terms.

4.2 Sticky Prices

Next, we will extend this model of money and credit to include sticky prices, as in the baseline model. Assume, as in the previous subsection, that there exists a continuum of consumers in each household. Each period, an individual consumer in a household receives a shock that determines whether he or she receives utility from flexible-price or sticky-price goods. With probability $\frac{1}{2}$ the consumer gets utility only from the flexible price good, and with probability $\frac{1}{2}$ the consumer receives utility only from the sticky price good. As well, goods are sold in the cash-only market, and the cash-and-credit market. Each consumer in a household receives a shock each period determining their goods market participation. With probability θ the consumer goes to the cash-only market, and with probability $1-\theta$, he or she goes to the cash-and-credit market. Further, the preference shock and the shock determining market participation are independent of each other and are also independent across consumers.

On the production side, households can choose the quantities of flexible price goods to supply in each market. However, as before, the demand for sticky price goods is distributed uniformly among households, and each household must supply the quantity of sticky price goods demanded at market prices. Assume in this section that there are no technology shocks – one unit of labor input produces one unit of any good.

A household then maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta \left[u(c_t^{mf}) + u(c_t^{ms}) \right] + (1-\theta) \left[u(c_t^{bf}) + u(c_t^{bs}) \right] - (n_t^f + n_t^s) \right\}. \tag{90}$$

Thus, there are now four different goods: c_t^{mf} (c_t^{ms}) denotes consumption of flexible-price (fixed-price) goods that are purchased in the cash-only market, while c_t^{bf} (c_t^{bs}) denotes consumption of flexible-price (fixed-price) goods that can be purchased in the cash-and-credit market. At the beginning of the period, the household faces a financing constraint

$$q_t b_{t+1} + \theta \left[c_t^{mf} + \frac{c_t^{ms}}{\pi_t} \right] + m_t' \leq \frac{m_t + b_t}{\pi_t} + \tau_t. \tag{91}$$

The constraint (91) is a modification of (76) that includes both flexible-price and sticky-price goods purchases using cash on the left-hand side. Similarly, we can adapt (77) to include flexible-price and sticky-price goods so that the household's collateral constraint in the cash-and-credit market is

$$(1-\theta)\left[c_t^{bf}+\frac{c_t^{bs}}{\pi_t}\right]\leq m_t'+b_{t+1}. \tag{92}$$

Finally, the household's budget constraint is

$$q_t b_{t+1}+\theta\left[c_t^{mf}+\frac{c_t^{ms}}{\pi_t}\right]+(1-\theta)\left[c_t^{bf}+\frac{c_t^{bs}}{\pi_t}\right]+m_{t+1}\leq\frac{m_t+b_t}{\pi_t}+\tau_t+n_t^f+\frac{n_t^s}{\pi_t} \tag{93}$$

The government's budget constraints are the same as in the flexible-price version of the model, i.e. (82) and (83) hold. As well, the fiscal authority follows the rule (84), i.e. the real value of the consolidated government debt is set exogenously at v_t in period t.

Given optimization and market clearing, we can characterize an equilibrium as follows. In each period, the following hold:

$$1=\beta E_t\left[\frac{u'(c_{t+1}^{mf})}{\pi_{t+1}}\right], \tag{94}$$

$$\pi_t u'(c_t^{ms})=u'(c_t^{mf}), \tag{95}$$

$$\frac{u'(c_t^{bf})}{q_t}=u'(c_t^{mf}), \tag{96}$$

$$\frac{\pi_t}{q_t}u'(c_t^{bs})=u'(c_t^{mf}), \tag{97}$$

As well, either the collateral constraint does not bind, so

$$q_t=\beta E_t\left[\frac{u'(c_{t+1}^{mf})}{u'(c_t^{mf})\pi_{t+1}}\right], \tag{98}$$

$$u'(c_t^{bf})=1,$$

and

$$\theta\left[c_t^{mf}+\frac{c_t^{ms}}{\pi_t}\right]+(1-\theta)q_t\left[c_t^{bf}+\frac{c_t^{bs}}{\pi_t}\right]\leq v_t \tag{99}$$

or the collateral constraint binds so

$$q_t=\frac{u'(c_t^{bf})-1}{u'(c_t^{mf})}+\beta E_t\left[\frac{u'(c_{t+1}^{mf})}{u'(c_t^{mf})\pi_{t+1}}\right], \tag{100}$$

$$u'(c_t^{bf})>1, \tag{101}$$

$$\theta\left[c_t^{mf}+\frac{c_t^{ms}}{\pi_t}\right]+(1-\theta)q_t\left[c_t^{bf}+\frac{c_t^{bs}}{\pi_t}\right]=v_t. \tag{102}$$

Thus, in period t, the collateral constraint (92) may not bind, in which case (98) holds – government debt sells at its fundamental price, the appropriately discounted value of the payoff stream on the asset – and (99) holds in equilibrium, i.e. the value of the consolidated government debt is large enough to finance all consumption purchases. Alternatively, (92) binds, so that there is a liquidity premium on government debt, reflected in a tight collateral constraint and a resulting inefficiency in the market for flexible price goods in the cash-and-credit market (inequality (101)). As well, in (102), the value of consolidated government debt is just sufficient to purchase all goods.

4.2.1 Unconstrained Equilibrium

As a first step, consider the case where $v_t = v$ for all t, with v sufficiently large that (92) does not bind. Then, solving for a stationary equilibrium from (94)-(98), and dropping t subscripts,

$$\pi=\frac{\beta}{q}, \tag{103}$$

$$u'(c^{mf})=\frac{1}{q}, \tag{104}$$

$$u'(c^{ms})=\frac{1}{\beta}, \tag{105}$$

$$u'(c^{bf})=1, \tag{106}$$

$$u'(c^{bs})=\frac{q}{\beta}. \tag{107}$$

The period utility of the household is given by

$$W=\theta\left[u(c^{mf})-c^{mf}\right]+\theta\left[u(c^{ms})-c^{ms}\right]+(1-\theta)\left[u(c^{bf})-c^{bf}\right]+(1-\theta)\left[u(c^{bs})-c^{bs}\right].$$

The central bank's problem is to choose q to maximize welfare in equilibrium. From the equilibrium solution (103)-(107), it is straightforward to show that welfare is strictly increasing in q for $q \leq \beta$, and strictly decreasing in q for $q = 1$. Therefore, the optimal monetary policy satisfies $q \in (\beta, 1)$. When the collateral constraint does not bind in this model, there are in general two inefficiencies at work. The first is a standard monetary friction, which is corrected if the nominal interest rate is zero, i.e. $q = 1$, that is a Friedman rule. The second is the sticky price friction, which is corrected when $q = \beta$, which implies $\pi = 1$. The optimal monetary policy then trades off these two frictions. A zero nominal interest rate is not optimal, and neither is price stability, as $\pi < 1$ at the optimum.

4.2.2 Constrained Equilibrium

The purpose of this subsection is to analyze an equilibrium and optimal policy under the same scenario as considered above in cashless economies. Assume that, in the current state, $v_t = v^l$, and that the state will revert permanently to $v_{t+1} = v^h$ with probability $1-\rho$. Here, $v^l < v^h$, and assume for now that the collateral constraint binds (at least for some monetary policies) in the low-v state and does not bind in the high-v state.

From (94)-(102), an equilibrium consists of consumption quantities c^{mf}, c^{ms}, c^{bf}, and c^{bs}, and relative price π in the low-v state solving

$$1 = \beta\left[\frac{\rho u'(c^{mf})}{\pi} + \frac{(1-\rho)}{\beta}\right], \tag{108}$$

$$\pi u'(c^{ms}) = u'(c^{mf}), \tag{109}$$

$$\frac{u'(c^{bf})}{q} = u'(c^{mf}), \tag{110}$$

$$\frac{\pi}{q}u'(c^{bs}) = u'(c^{mf}), \tag{111}$$

and either

$$u'(c_t^{bf}) = 1, \tag{112}$$

and

$$\theta\left[c^{mf} + \frac{c^{ms}}{\pi}\right] + (1-\theta)q\left[c^{bf} + \frac{c^{bs}}{\pi}\right] \leq v^l \tag{113}$$

or

$$u'(c_t^{bf}) < 1, \tag{114}$$

and

$$\theta\left[c^{mf} + \frac{c^{ms}}{\pi}\right] + (1-\theta)q\left[c^{bf} + \frac{c^{bs}}{\pi}\right] = v^l, \tag{115}$$

given monetary policy q. Note, in (108)-(115), that monetary policy in the future state in which reversion to the high-V state occurs has no bearing on the determination of quantities in the current period. That is, in this monetary model forward guidance is irrelevant.

To see what the possibilities are, it is useful to consider a simple example. Let $u(c) = \log(c)$. Then, from (108)-(115), if $q \leq \frac{v^l}{2}$, then the collateral constraint does not bind in the low-v state, and we can solve for consumption quantities and inflation in the low-v state as follows:

$$c^{mf} = q \tag{116}$$

$$c^{ms} = \beta \tag{117}$$

$$c^{bf} = 1 \tag{118}$$

$$c^{bs} = \frac{\beta}{q} \tag{119}$$

$$\pi = \frac{\beta}{q} \tag{120}$$

To evaluate the welfare effects of policy in the low-v state we need only be concerned with how q affects period utility when v is low. If $q \leq \frac{v^l}{2}$ so that the collateral constraint does not bind, then from (116)-(120) we can evaluate policy in terms of the function $W(q)$, where

$$W(q) = \theta\left(\log q - q\right) + (1-\theta)\left(-\log q - \frac{\beta}{q}\right).$$

Differentiating, we get

$$W'(q) = \theta\left(\frac{1}{q} - 1\right) + \frac{(1-\theta)}{q}\left(-1 + \frac{\beta}{q}\right).$$

Therefore, there is some $\hat{q} \in (\beta, 1)$, such that $W(q)$ is strictly increasing for $q < \hat{q}$, and strictly decreasing for $q > \hat{q}$, with $W(q)$ attaining a unique maximum for $q = \hat{q}$.

But, if the collateral constraint binds in the low-v state, that is when $q > \frac{v^l}{2}$, from (108)-(115),

$$c^{mf} = \frac{v^l}{2} \tag{121}$$

$$c^{ms} = \beta \tag{122}$$

$$c^{bf} = \frac{v^l}{2q} \tag{123}$$

$$c^{bs} = \frac{\beta}{q} \tag{124}$$

$$q \geq \frac{v^l}{2} \tag{125}$$

$$\pi = \frac{2\beta}{v^l} \tag{126}$$

Then, if the collateral constraint binds, welfare as a function of q is determined by

$$W(q) = (1-\theta)\left(-2\log q - \frac{v^l}{2q} - \frac{\beta}{q}\right).$$

So,

$$W'(q) = \frac{1}{q}\left[-2 + \frac{1}{q}\left(\frac{v^l}{2} + \beta\right)\right]$$

Therefore, if

$$\frac{v^l}{2} \leq \beta,$$

then welfare is maximized for

$$q = \frac{v^l + 2\beta}{4} < 1,$$

so at the optimum, $q \in \left(\frac{v^l}{2}, 1\right)$, and the collateral constraint binds at the optimum. But, if

$$\beta < \frac{v^l}{2} < 1,$$

then at the optimum the collateral constraint is relaxed. Either $\hat{q} < \frac{v^l}{2}$, and $q = \hat{q}$ at the optimum, or $\hat{q} \geq \frac{v^l}{2}$ and $q = \frac{v^l}{2}$ at the optimum.

So, in the case where $\hat{q} < \frac{v^l}{2} < 1$, the collateral constraint could bind if the nominal interest rate were sufficiently low, but optimal policy in the low-v state implies that the collateral constraint does not bind. But a binding collateral constraint matters at the optimum. If $\beta < \frac{v^l}{2} \leq \hat{q}$, then $q = \frac{v^l}{2}$ at the optimum, and the nominal interest rate is higher at the optimum than it would be if the scarcity of collateral did not impinge on optimal monetary policy. Optimal monetary policy in this case is conducted so as to relax the collateral constraint. Finally, if $\frac{v^l}{2} \leq \beta$, then again the scarcity of collateral implies that the optimal nominal interest rate is higher than it would be in the absence of the collateral scarcity, but in this case the collateral constraint binds at the optimum. Figure 5 shows the optimal monetary policy setting for q, in the example, as a function of $\frac{v^l}{2}$.

[Figure 5 here.]

It is useful to see the effects of policy on the real interest rate when the collateral constraint binds. In particular, suppose that there exists a real bond having the same liquidity properties as nominal government bonds. That is, the real bond can be used as collateral in secured credit transactions. Then, from (90)-(93), we can express the price s_t of the real bond as

$$s_t = \frac{u'(c_t^{bf}) - 1}{u'(c_t^{mf})} + \beta E_t \left[\frac{u'(c_{t+1}^{mf})}{u'(c_t^{mf})} \right].$$

Then, for this example, in the case where the collateral constraint binds in the low-v state and the central bank follows an optimal policy of $q_t = \hat{q}$ in the high-v state when the collateral constraint does not bind, from (121)-(126), we get

$$s = q + \beta\rho + \frac{v^l}{2}\left[-1 + \frac{\beta(1-\rho)}{\hat{q}}\right] \tag{127}$$

So, note first that, since $\hat{q} > \beta$, from (127) s is decreasing in v^l. So, in a standard fashion, a reduction in the real value of consolidated government debt implies an increasing scarcity of collateral, and the real interest rate falls. From our analysis of optimal policy, when the collateral constraint is sufficiently tight that $q = \hat{q}$ implies that the collateral constraint binds, the central bank should reduce q below $\hat{q}$, i.e. it should raise the nominal interest rate. This has the effect, from (127), of reducing s and raising the real interest rate.

A typical New Keynesian argument is that factors reducing the real interest rate in the context of sticky prices can produce a situation in which the zero lower bound on the nominal interest rate binds at the optimum. Thus, according to the argument, the real interest rate is *too high.* Our model has sticky prices, and a low real interest rate that results from a scarcity of government debt. But, if this scarcity affects policy, it is because the real interest rate is too low, and an optimal policy response is to raise the nominal interest rate, which serves to increase the real interest rate.

The reason we get these results is the same as in the version of this model with money and secured credit with flexible prices. That is, a higher nominal interest rate is achieved through central bank sales of government debt, which act to mitigate the shortage of collateral. The example shows that the sticky price friction need not trump other frictions in determining monetary policy actions in low-real-interest rate environments.

5 Conclusion

In this paper, we developed a tractable model for the analysis of monetary policy in a low-real-interest-rate context. The model can incorporate various sticky-price, scarce-collateral, and standard monetary frictions, in alternative combinations. The model has neo-Fisherian properties, in that high nominal interest rates tend to induce high inflation.

More often than not, the model turns New Keynesian results on their head. In a baseline model, a temporarily-high discount factor, or temporarily-low productivity growth can result in a binding zero-lower-bound constraint. But this binding constraint implies that inflation and output are higher than they would otherwise be. As well, the binding zero-lower-bound constraint can be relaxed through forward guidance, if the central bank is able to commit. But forward guidance takes the form of promises to increase the future nominal interest rate above what it would otherwise be. These results translate to a secured-credit version of the model in which a low real interest rate results from a tight collateral constraint. But that model also exhibits a multiple equilibrium problem that does not exist in reduced-form explanations for the low real interest rate.

In the full-blown version of the model, which includes sticky prices, secured credit, money, bonds, and explicit open market operations, a binding collateral constraint which impinges on optimal policy typically implies that the central bank should raise the nominal interest rate. A low real interest rate reflects an inefficiency, and this inefficiency is mitigated through an open market sale of government debt by the central bank. This acts to raise the real interest rate, and increase welfare.

6 References

Andolfatto, D. and Williamson, S. 2015. "Scarcity Of Safe Assets, Inflation, And The Policy Trap," *Journal of Monetary Economics*, 73, pp. 70-92.

Caballero, R., Farhi, E., and Gourinchas, P. 2016. "Safe Asset Scarcity and Aggregate Demand," NBER Working Paper 22044.

Cochrane, J. 2016. "Do Higher Interest Rates Raise or Lower Inflation?" working paper, Hoover Institution.

Cochrane, J. 2017. "Michelson-Morley, Occam and Fisher: The Radical Implications of Stable Inflation at Near-Zero Interest Rates," working paper, Hoover Institution.

Eggertsson, G. and Woodford, M. 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, Economic Studies Program, The Brookings Institution, vol. 34(1), pages 139-235.

Krishnamurthy, A. and Vissing-Jorgensen, A. 2012. "The Aggregate Demand for Treasury Debt," *Journal of Political Economy* 120, issue 2, pages 233 - 267.

Rupert, P. and Sustek, R. 2016. "On the Mechanics of New Keynesian Models," working paper, University of California, Santa Barbara.

Werning, I. 2011. "Managing a Liquidity Trap: Monetary and Fiscal Policy," NBER working paper 17344.

Williamson, S. 2015. "Interest on Reserves, Interbank Lending, and Monetary Policy," working paper, Federal Reserve Bank of St. Louis.

Williamson, S. 2016a. "Scarce Collateral, The Term Premium, And Quantitative Easing," *Journal of Economic Theory* 164, pp. 136-165.

Williamson, S. 2016b. "Low Real Interest Rates, Collateral Misrepresentation, and Monetary Policy," working paper.

Williamson, S. and Wright, R. 2010. "New Monetarist Economics: Methods," *St. Louis Federal Reserve Bank Review* 92, 265-302, July/August.

Williamson, S. and Wright, R. 2011. "New Monetarist Economics: Models," *Handbook of Monetary Economics*, vol. 3A, B. Friedman and M. Woodford, eds., Elsevier, pp. 25-96.

Figure 1: Real Interest Rate

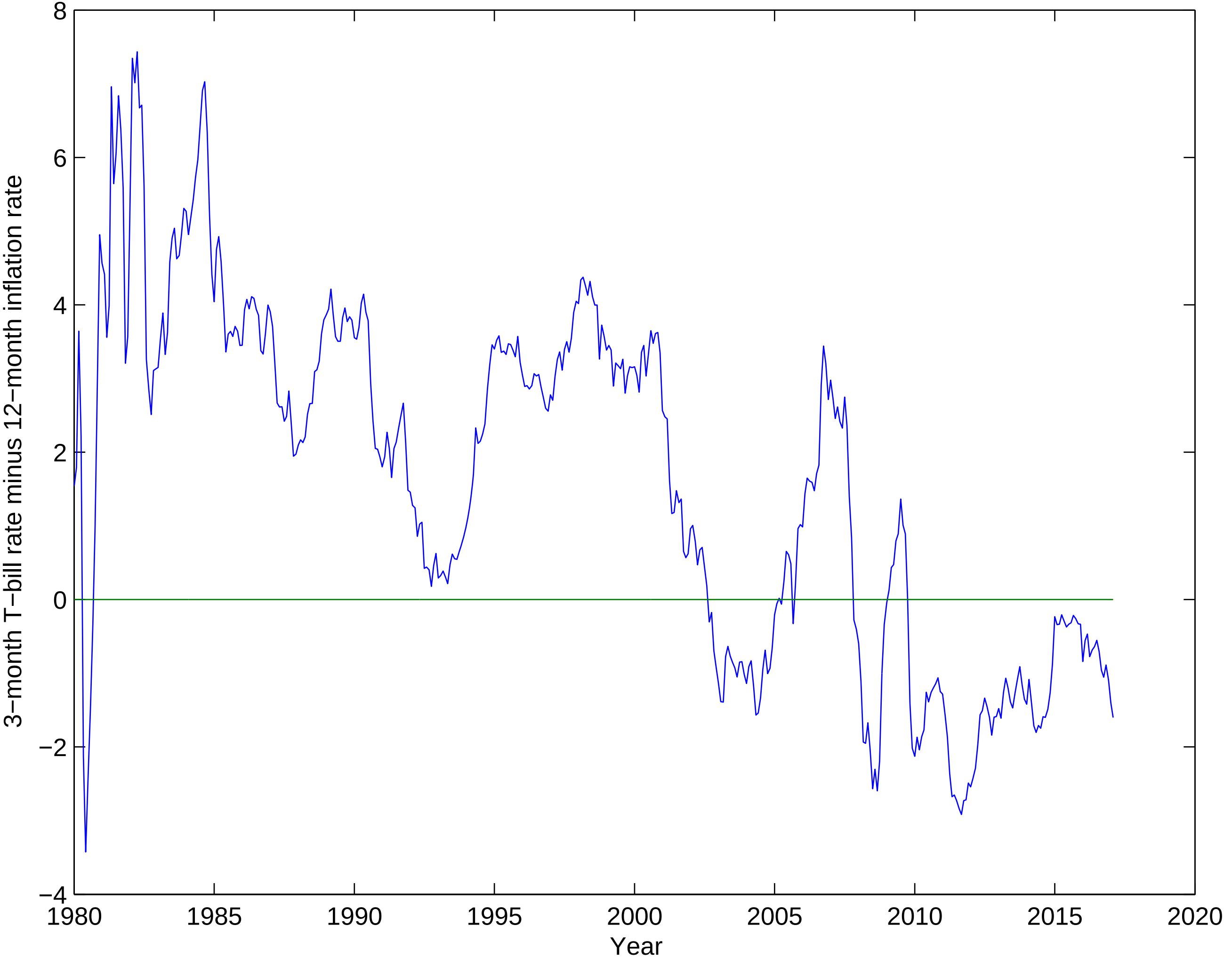

Figure 2: Distortions at the ZLB

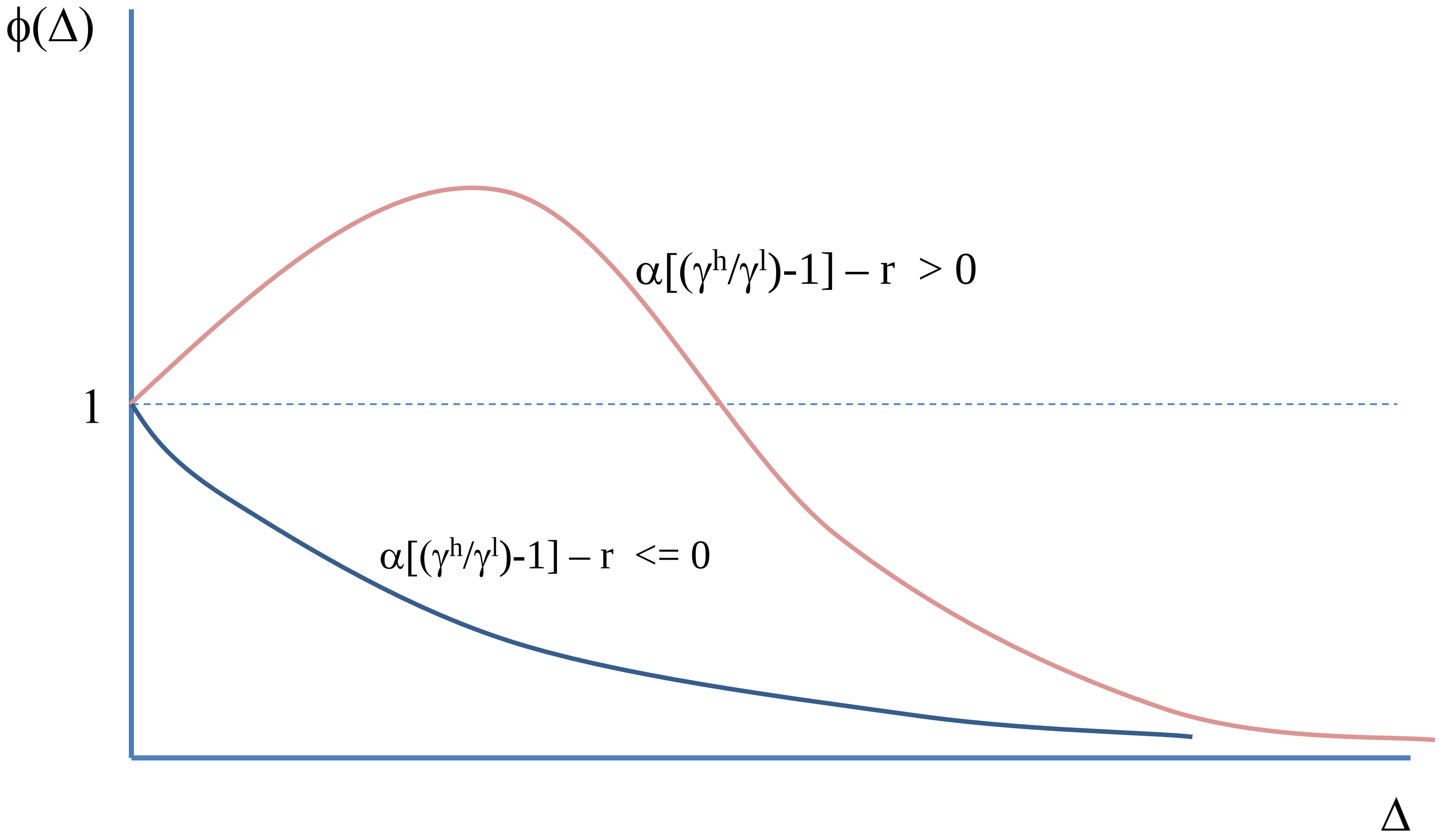

Figure 3: 0 < α < 1

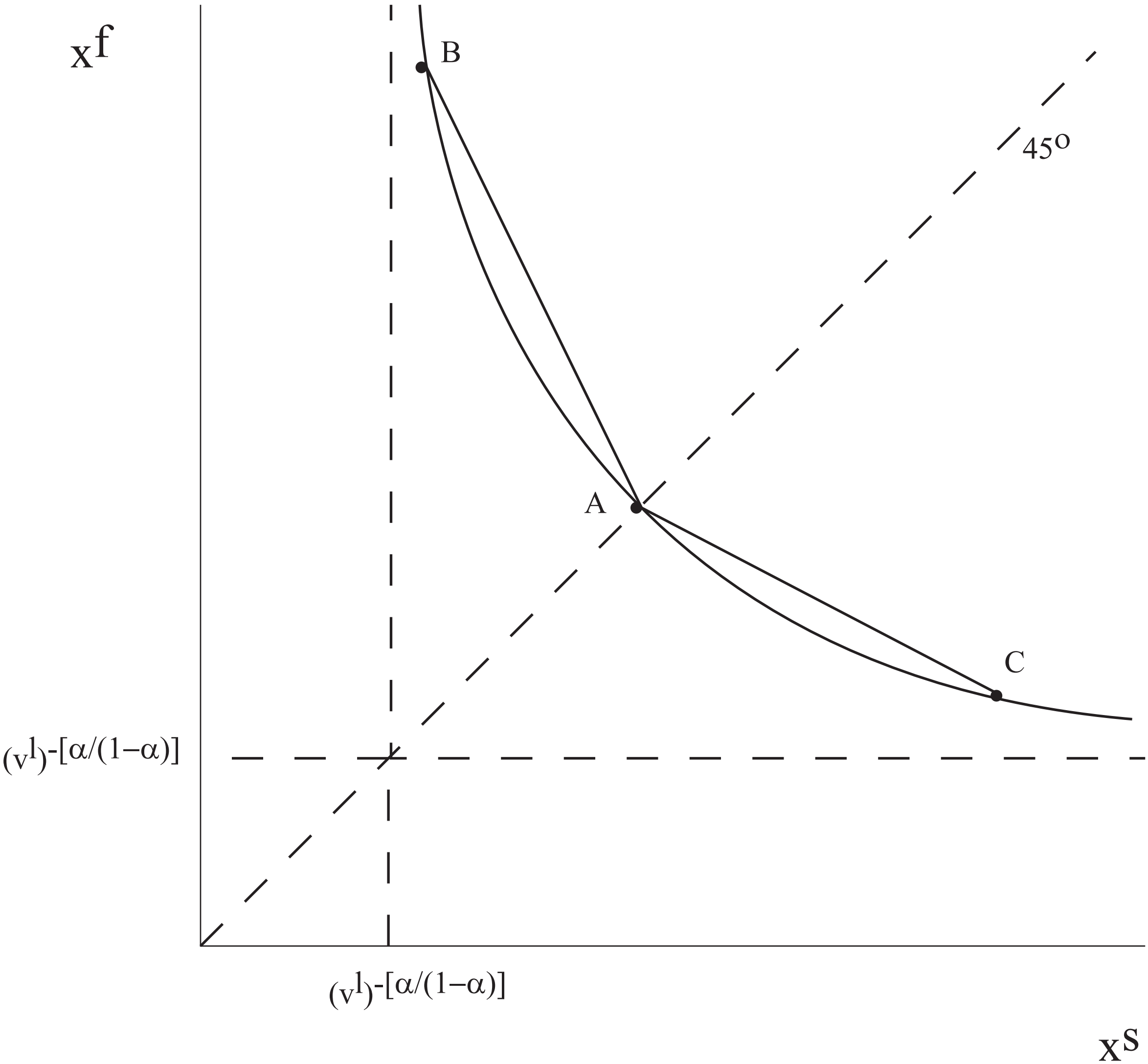

Figure 4: $\alpha > 1$

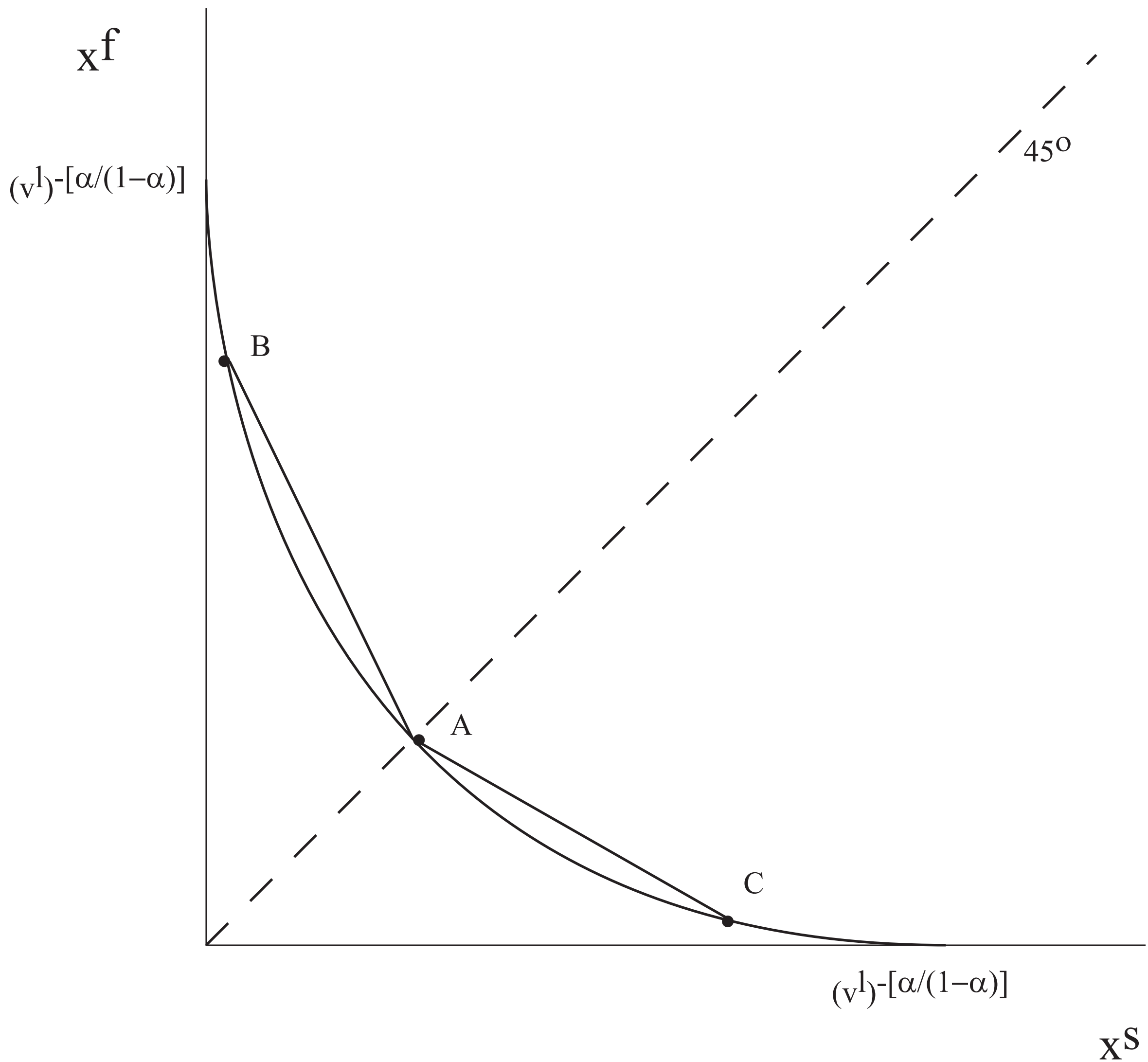

Figure 5: Optimal Policy, Log Example