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Abstract

What is the prescription of Ramsey capital taxes for the heterogeneous-agent incomplete-market economy in the long run? Aiyagari (1995) addressed the question, showing that a positive capital tax should be imposed to implement the steady-state allocation that satisfies the so-called modified golden rule. In his analysis of the Ramsey problem, a critical assumption implicitly made is the existence of steady-state allocations at the optimum. This paper revisits the issue and finds sharply different results. We demonstrate that the optimal Ramsey allocation may feature no steady state. The key to our results is embedded in the hallmark of incomplete-market models that the risk-free rate is lower than the time discount rate at the steady state in competitive equilibrium.

JEL Classification: C61; E22; E62; H21; H30

Key Words: Capital Taxation; Modified Golden Rule; Ramsey Problem; Incomplete Markets

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1 Introduction

The heterogeneous-agent incomplete-market (HAIM hereafter) model considers an environment in which households are subject to uninsurable idiosyncratic shocks and borrowing restrictions. In response, households buffer their consumption against adverse shocks via precautionary savings. During the past two decades, the HAIM model has become a standard workhorse for policy evaluations in the current state-of-the-art macroeconomics that jointly addresses aggregate and inequality issues.[1]

Given the importance and popularity of the HAIM model, it is natural to ask: What is the prescription of Ramsey capital taxation for the HAIM economy in the long run? The first attempt to answer this question is the work of Aiyagari (1995). Assuming its existence, he showed that the so-called "modified golden rule" (MGR hereafter) has to hold in the Ramsey steady state.[2] On the other hand, in the steady state, the after-tax gross return on capital, which is equated to the risk-free gross interest rate, R, is always less than the time discount rate, $1/\beta$. Aiyagari (1995) thus reached the conclusion that a positive capital tax should be imposed to implement the steady-state allocation that satisfies the MGR. Agents overaccumulate capital relative to the level implied by the MGR because of their precautionary savings motive. The imposition of positive capital taxation provides a remedy to restore production efficiency — the MGR. The finding by Aiyagari (1995) is important in the optimal taxation literature and it represents a distinct departure from the celebrated result of no capital tax in the long run prescribed by Chamley (1986) and Judd (1985).

Aiyagari (1995) addressed the issue mainly under the setting of endogenous government spending. However, he argued that his finding remains intact under the setting of exogenous rather than endogenous government spending.[3] In both settings, Aiyagari (1995) implicitly assumed that the shadow price of resources converges to a finite limit in the steady state.

This paper revisits the same issue and finds sharply different results. Working with the power utility function, we demonstrate at the optimum: (i) There is no Ramsey steady state with $R < 1/\beta$ if the elasticity of intertemporal substitution (EIS) is weakly less than 1;[4] (ii) if the EIS is larger

[1]It is also known as the Bewley-Huggett-Aiyagari model. For surveys of the literature, see Heathcote, Storesletten, and Violante (2009), Guvenen (2011), and Quadrini and Ríos-Rull (2015).

[2]The Ramsey steady state is defined as a situation where the optimal Ramsey allocation features the steady-state property in the long run. Our definition of the steady state is an interior one. See Definitions 3 and 4 for details.

[3]See his footnote 15.

[4]The non-existence of a Ramsey steady state does not imply the non-existence of a steady state in competitive equilibrium. It simply means that the Ramsey planner forgoes the steady-state allocation at the optimum.

than 1, a Ramsey steady state with $R < 1/\beta$ is possible, but the shadow price of resources must diverge in the steady state. Result (i) questions the existence of a Ramsey steady state, the basic premise of the Aiyagari (1995) analysis. Result (ii) contradicts Aiyagari's (1995) implicit assumption on the convergence of the shadow price of resources. Both results cast doubt on the implementation of the MGR at the optimum.

Our analysis departs from Aiyagari (1995) in one important way — the primal Ramsey approach, which allows us to explicitly account for the first-order condition with respect to aggregate consumption. This margin over aggregate consumption is overlooked by the analysis of Aiyagari (1995).[5] After counting the margin, we show that the social benefit of having one extra unit of aggregate consumption must diverge if $R < 1/\beta$ holds in the steady state. This very diverging feature could make the Ramsey steady state fail to exist. Put differently, we show that once the additional margin over aggregate consumption is reckoned in the analysis, the assumed Ramsey steady state in Aiyagari (1995) can be at odds with the margin over aggregate consumption at the optimum. In addition, we demonstrate that our results remain robust, regardless of whether government spending is endogenously determined or exogenously given.

It is well known that the steady-state outcome in competitive equilibrium, $R < 1/\beta$, represents the hallmark of the HAIM model.[6] The fundamental force that drives the divergence described above is exactly embedded in this hallmark. Indeed, we show that the divergent force will exist (vanish) if and only if $R < 1/\beta$ ($R = 1/\beta$) holds in the steady state. Intuitively, unlike individual households in the face of uncertain labor income, the Ramsey planner in the HAIM economy faces no uncertainty in allocating aggregate resources. The strict inequality of $R < 1/\beta$ then dictates that the market discounts resources at a lower rate than the planner discounts utility, implying the existence of room for the planner to improve welfare by front-loading aggregate consumption and/or back-loading aggregate labor through policy tools. This existence of room for improving welfare persists as long as $R < 1/\beta$ holds in the steady state. Thus, unless there is a counterbalance to offset it, the persistence can lead to "extremes" to upset the Ramsey steady state. Interestingly, we find whether the counterbalance exists has to do with the value of the EIS. Thanks to our primal Ramsey approach, the effect of optimally adjusting aggregate consumption/labor and the

[5] Aiyagari (1995) formulated the Ramsey problem in terms of the dual approach and thus did not explore this margin.

[6] Ljungqvist and Sargent (2012, p.9) explained that the outcome of $R < 1/\beta$ in the steady state can be thought of as follows: It lowers the rate of return on savings enough to offset agents' precautionary savings motive so as to make their asset holdings converge rather than diverge in the limit.

role of the EIS can be clearly addressed in our analysis.[7]

The existence of a Ramsey steady state is commonly assumed for the Ramsey problem in the extant literature. However, this assumption is not innocuous for the HAIM environment according to our finding. The warning is particularly relevant and strong since the key to our results exactly underlies the hallmark feature of the HAIM economy — the steady-state risk-free rate is lower than the time discount rate.

1.1 Methodology — Primal Ramsey Approach

In order to consider the social benefit of having one extra unit of aggregate consumption, the primal approach to the Ramsey problem is adopted. As aforementioned, the explicit accounting for the margin over aggregate consumption is critical to our analysis. Importantly, the primal approach will enable us to directly compare our model with the representative-agent (RA hereafter) model and make our results transparent and more clear.

One difficulty encountered in formulating the Ramsey problem in the HAIM model is in regard to how to properly formulate the implementability condition. Werning (2007) extends the Ramsey primal approach from the RA to the heterogeneous-agent framework. However, agent types are permanently fixed in the Werning model, while agent types evolve stochastically over time in the HAIM model. Park (2014) extended the work of Werning (2007) to a complete-market environment in which agent types evolve stochastically. We extend her approach to the incomplete-market environment, or more specifically, to the HAIM economy.

Our primal approach formulates the household problem as a time zero trading problem as in the Arrow-Debreu complete-market economy; however, there is the imposition of two additional constraints — one for incomplete markets and the other for borrowing constraints — to account for the key features of the HAIM economy. This approach of modeling incomplete markets is pioneered by Aiyagari, Marcet, Sargent, and Seppala (2002), who named the additional constraints for incomplete markets as measurability conditions. The later work by Chien, Cole, and Lustig (2011) extends this approach to heterogeneous-agent models in the context of asset pricing. The advantage of this time zero setting allows us to trace the evolution of stochastic agent types over time through the Lagrangian multipliers associated with these additional constraints. Moreover, it helps us to set up the primal approach to the Ramsey problem. Similar to an RA model or a complete-market model, there is only a single implementability condition. However, due to the

[7] We provide a more detailed explanation after presenting our main result, Proposition 2.

fact that the Ramsey planner also encounters the same incomplete-market frictions as faced by households, the single implementability condition is not sufficient for the characterization of the Ramsey problem. This causes our HAIM Ramsey problem to fundamentally depart from the RA Ramsey problem.

1.2 Related Literature

The literature on optimal capital taxation is vast. Here we focus only on a subset of the studies framed in a heterogeneous-agent environment with incomplete markets or market frictions.

Our work is closely related to the recent study by Chien and Wen (2017), who utilized an analytically tractable heterogeneous-agent model to address the same issue. They demonstrated that the planner's desire to relax agents' borrowing constraints may lead to an ever-increasing accumulation of government debt, resulting in a dynamic path featuring no steady state at the optimum. Hence, our results of no Ramsey steady state are consistent with their findings. However, in order to have an analytical solution, their model makes a few special assumptions and deviates from the standard HAIM model. Our paper adopts the standard HAIM model exactly.

The nature of our finding of no Ramsey steady state is to some extent the same as the work of Straub and Werning (2014). They pointed out that the common assumption that endogenous multipliers associated with the Ramsey problem converge in the limit is not necessarily true and could thus lead to incorrect optimal policy prescriptions in the long run. Aiyagari's (1995) assumption on the Ramsey steady state may be subject to the same problem. We explicitly address it. It should be noted that the mechanism for our non-convergence of endogenous multipliers originates from incomplete markets. Such a mechanism is absent from the environment studied by Straub and Werning (2014).

Gottardi, Kajii, and Nakajima (2015) considered an environment deviating from the standard HAIM economy, in that there is risky human capital in addition to physical capital. They derived qualitative and quantitative properties for the solution to the Ramsey problem, showing that the interaction between market incompleteness and risky human capital accumulation provides a justification for taxing physical capital. Instead, we stick to the standard HAIM economy and show that assuming the existence of a Ramsey steady state could be problematic due to incomplete-market frictions.

Conesa, Kitao, and Krueger (2009) considered optimal capital taxation in a HAIM-type economy but put it in a life-cycle framework. The quantitative part of their study largely focuses on the steady-state welfare. In an overlapping generations model with two-period-lived households, Krueger and Ludwig (2018) characterized the optimal capital tax of the Ramsey problem in a HAIM-type economy. They provided a full analytical solution for logarithmic utility. Our results indicate that the Ramsey steady state might fail to exist within the framework where agents are infinitely-lived.

Acikgoz (2013) and Dyrda and Pedroni (2016) numerically solved optimal fiscal policy for the transition and the steady state of the HAIM economy. In contrast to our findings, the numerical results of both papers feature a Ramsey steady state in the long run. It is not clear what explains such a contrasting result exactly. However, according to our analysis, the sources of the difference could be the implicit assumptions of the existence of a steady state in their numerical analyses. In particular, our paper signals a warning about the existence of Ramsey steady state, which is commonly assumed in the literature.

Dávila, Hong, Krusell, and Ríos-Rull (2012) characterized constrained efficiency for the HAIM economy. To decentralize the constrained efficient allocation, the planner is required to know each agent's realized shocks in order to impose individual-specific capital taxes. We consider flat tax rates applied uniformly to all agents as in the typical Ramsey problem and, as such, the constrained efficient allocation is infeasible to the Ramsey planner.

The rest of the paper is organized as follows. Section 2 introduces our model economy, and Section 3 characterizes its competitive equilibrium. Section 4 formulates the Ramsey problem, and Section 5 addresses the Ramsey steady state. Section 6 checks the robustness of our results, and Section 7 concludes.

2 Model Economy

The model economy mainly builds on Aiyagari (1994). There is a unit measure of infinitely-lived households that are subject to idiosyncratic labor productivity shocks. There are no aggregate shocks. Markets are incomplete in that there are no state-contingent securities available for households to insure their idiosyncratic shocks. In addition, all households are subject to exogenous borrowing constraints at all times.

Time is discrete and the horizon is infinity, indexed by $t = 0, 1, 2, ...$ Time 0 is a planning period and actions begin in time 1. All households are ex ante identical and endowed with the same asset holdings at time 0. Ex post heterogeneity arises from time 1 on because households experience different histories of the idiosyncratic shock realization. Let $\theta_t \in \Theta$ (a finite set) denote the incidence of the idiosyncratic labor productivity shock at time t, and let θ^t denote the history of events for the idiosyncratic shock of a household up through and until time t. The shock θ_t is independently and identically distributed across households, and the sequence $\{\theta_t\}$ follows a first-order Markov process over time. We let $\pi_t(\theta^t)$ denote the unconditional probability of θ^t being realized as of time zero and $\pi(\theta_t|\theta_{t-1})$ denote the conditional probability of the Markov process. Note that $\pi_t(\theta^t) = \pi(\theta_t|\theta_{t-1})\pi_{t-1}(\theta^{t-1})$. Because of the independence of productivity shocks across households, a law of large numbers applies so that the probability $\pi_t(\theta^t)$ also represents the fraction of the population that experiences θ^t at time t. We call a household that has the history θ^t simply "the household θ^t." We also introduce additional notations: $\theta^{t+1} \succ \theta^t$ means that the left-hand-side node is a successor node to the right-hand-side node; and for $s > t$, $\theta^s \succeq \theta^t$ ($\theta^s \succ \theta^t$) represents the set of successor shocks after θ^t up to θ^s including (excluding) θ_t.

Households maximize their lifetime utility

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right] \pi_t(\theta^t),$$

where $\beta \in (0,1)$ is the discount factor, $c_t(\theta^t)$ and $l_t(\theta^t)$ denote the consumption and the labor supply for a household θ^t at time t, and $l_t(\theta^t)/\theta_t$ is the corresponding "raw" labor supply. The assumptions on the functions $u(.)$ and $v(.)$ are standard.

There is a standard neoclassical constant returns to scale production technology, denoted by $F(K,L)$, that is operated by a representative firm, where K and L are aggregate capital and labor. The firm produces output by hiring labor and renting capital from households. The firm's optimal conditions for profit maximization at time t satisfy

$$\begin{aligned} w_t &= F_L(K_t, L_t), \\ q_t &= F_K(K_t, L_t), \end{aligned}$$

where w_t and q_t are the wage rate and the capital rental rate, and F_L and F_K denote the marginal product of labor and capital. All markets are competitive.

The government is required to finance an exogenous stream of government spending $\{G_t\}$ and it can issue one-period government bonds and levy flat-rate, time-varying labor and capital taxes at rates $\tau_{l,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint at time t is read as

$$\tau_{l,t} w_t L_t + \tau_{k,t}(q_t - \delta) K_t + B_{t+1} = G_t + R_t B_t, \tag{1}$$

where R_t is the risk-free gross interest rate between time $t-1$ and t, and B_t is the amount of government bonds issued at time $t-1$. The government is assumed to fully commit to a sequence of taxes imposed and debts issued, given the initial amount of government bond B_1 at time 0. This setup for the government is standard for the Ramsey problem. Section 6 considers an alternative setup where $\{G_t\}$ becomes endogenously determined rather than exogenously given.

Given (1), the time 0 government budget constraint is given by

$$B_1 = \sum_{t \geq 1} P_t \left[\tau_{l,t} w_t L_t + \tau_{k,t} \left(q_t - \delta \right) K_t - G_t \right].$$

3 Characterization of Competitive Equilibrium

This section characterizes the competitive equilibrium of the model economy, paving the way for the formulation of the Ramsey problem in the next section. We first describe the household problem.

3.1 Household Problem

We express the household problem as a time zero trading problem as in an Arrow-Debreu complete-market economy but with the imposition of additional constraints to account for the key features of the HAIM economy. As noted in the Introduction, this method facilitates the formulation of the primal Ramsey problem for the HAIM economy.

3.1.1 Measurability Conditions and Borrowing Constraints

Two key features of the HAIM economy are (i) incomplete markets — no state-contingent claims on idiosyncratic shocks, and (ii) borrowing constraints — a lower bound on household asset holdings. We show how to embed these two features into a time zero trading problem for the household.

Let $p_t(\theta^t) = P_t \pi_t(\theta^t)$ denote the Arrow-Debreu price for a state-contingent claim that delivers one unit of consumption in the event of θ^t at time t, where P_t is the time zero price of one unit of consumption delivered at time t. We set $P_0 = 1$ as a normalization. Given the history of shocks θ^t

at time t, the asset holdings with complete markets can be written as

$$p_t(\theta^t)a_t(\theta^t) = \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right], \tag{2}$$

where $\widehat{w}_s \equiv (1-\tau_{l,s})w_s$ is defined as the after-tax wage rate at time s and $a_t(\theta^t)$ is the amount of the state-contingent claim held by the household θ^t at the beginning of time t. Equation (2) embodies the property that at each time t and each history θ^t, the value of a household's net asset holdings is equal to the present value of current and future consumption net of after-tax labor income earned.

However, markets are incomplete rather than complete and households do not have access to state-contingent markets in the HAIM economy. This implies that the asset holdings at time $t+1$ are only measurable up to the events prior to the realization of shock θ_{t+1}. Formally, households face the following measurability conditions: for $\forall t \geq 0$ and θ^t,

$$a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \widetilde{\theta}_{t+1}) \text{ for all } \widetilde{\theta}_{t+1}, \theta_{t+1} \in \Theta,$$

which practically impose constraints on a household's asset holdings.

For ease of exposition, we rewrite the measurability condition as: for $\forall t \geq 0$ and θ^t,

$$\frac{a_{t+1}(\theta^t, \theta_{t+1})}{R_{t+1}} = \frac{a_{t+1}(\theta^t, \widetilde{\theta}_{t+1})}{R_{t+1}} \equiv \widehat{a}_{t+1}(\theta^t) \text{ for all } \widetilde{\theta}_{t+1}, \theta_{t+1} \in \Theta, \tag{3}$$

where R_{t+1} is the risk-free gross interest rate between time t and $t+1$. That is, $\widehat{a}_{t+1}(\theta^t)$ is defined so that $R_{t+1}\widehat{a}_{t+1}(\theta^t) = a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \widetilde{\theta}_{t+1})$ for all $\widetilde{\theta}_{t+1}$, $\theta_{t+1} \in \Theta$. This makes sense because households can only hold a one-period risk-free asset; and their asset holdings at the beginning of time $t+1$ deflated by their asset return, the risk-free gross rate, must be equal to the end of time t asset holdings, which is denoted by $\widehat{a}_{t+1}(\theta^t)$. The property holds, regardless of whether the realized idiosyncratic shock at time $t+1$ is θ_{t+1} or $\widetilde{\theta}_{t+1}$. Note that the one-period risk-free asset held by households can be either K (capital) or B (government bond), or both; K and B are perfect substitutes in the view of households.

Households also face the following borrowing restrictions for $\forall t \geq 0$:

$$a_{t+1}(\theta^{t+1}) \geq \underline{a} = 0\,, \text{ for all } \theta^{t+1},$$

where $\underline{a}$ is an exogenously given borrowing limit. As far as this paper is concerned, we assume $\underline{a} = 0$.

Since there is no aggregate uncertainty, the following must hold:

$$\frac{P_t}{P_{t+1}} = R_{t+1} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta), \tag{4}$$

where the after-tax gross rate of return on capital is equal to the risk-free gross rate, which constitutes a no-arbitrage condition for trades in capital and government bonds.

3.1.2 Formulating and Solving the Household Problem

At each time and history, the value of a household's net asset holdings is equal to the present value of current and future consumptions net of labor incomes earned. Therefore, we can represent the asset holding restrictions, such as measurability and borrowing constraints, equivalently as the restrictions imposed on the whole sequence of consumption and labor choices. This is exactly what we do next.

Using (2), we can restate the measurability conditions as

$$P_{t-1}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t) = \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right], \ \forall t \geq 1, \theta^t, \tag{5}$$

where we have replaced $a_t(\theta^t)$ with $R_t\widehat{a}_t(\theta^{t-1})$ as defined in (3) and used $p_t(\theta^t) = P_t\pi_t(\theta^t)$ and the result of $P_{t-1} = P_tR_t$ in (4). If markets were complete, then households would only face a single constraint (7) below. The presence of the additional constraints represented by (5) is due to the incompleteness of markets. As to the borrowing constraints, they can be expressed as

$$\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right] \geq 0, \ \forall t \geq 1, \theta^t. \tag{6}$$

Finally, the household's time 0 budget constraint is expressed as[8]

$$\widehat{a}_1 = \sum_{t\geq 1}\sum_{\theta^t} p_t(\theta^t)\left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right], \tag{7}$$

with $\widehat{a}_1 = K_1 + B_1$, where K_1 and B_1 are the economy's initial capital and government bond,

[8]When $t = 1$, the left side of (5) equals $P_0\widehat{a}_1\pi_1(\theta^1) = \widehat{a}_1\pi_1(\theta^1)$, where we have utilized $P_0 = 1$. It then gives $\widehat{a}_1 = \sum_{\theta^1}\widehat{a}_1\pi_1(\theta^1)$ from the ex ante viewpoint at time 0.

respectively. All households by assumption have the same initial asset holdings $\widehat{a}_1$.

Given prices $\{\widehat{w}_t, p_t(\theta^t)\}$, the household chooses a sequence of consumption $\{c_t(\theta^t)\}$, labor $\{l_t(\theta^t)\}$, and asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$ to maximize the lifetime utility as of time zero, subject to the time 0 budget constraint (7), the measurability conditions (5), and the borrowing constraints (6). Let χ be the multiplier on the time 0 budget constraint, $\nu_t(\theta^t)$ the multiplier on the measurability condition in the event of θ^t at time t, and $\varphi_t(\theta^t)$ the multiplier on the borrowing constraint in the event of θ^t at time t. Incorporating all the constraints through these multipliers gives the household's Lagrangian:

$$\begin{aligned}
\tilde{L} = & \min_{\{\chi,v,\varphi\}} \max_{\{c,l,\widehat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t} \right) \right] \pi_t(\theta^t) \\
& + \chi \left\{ \widehat{a}_1 - \sum_{t=1}^{\infty} \sum_{\theta^t} p_t(\theta^t) \left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \right] \right\} \\
& + \sum_{t=1}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) \left\{ \sum_{s \geq t} \sum_{\theta^\tau \succeq \theta^t} p_s(\theta^s) \left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s) \right] - P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \right\} \\
& + \sum_{t=1}^{\infty} \sum_{\theta^t} \varphi_t(\theta^t) \left\{ \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} p_s(\theta^s) \left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s) \right] \right\}.
\end{aligned}$$

Using Abel's summation formula, the Lagrangian $\tilde{L}$ can be rewritten as[9]

$$\begin{aligned}
L = & \min_{\{\chi,v,\varphi\}} \max_{\{c,l,\widehat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t} \right) \right] \pi_t(\theta^t) \\
& - \sum_{t=1}^{\infty} \sum_{\theta^t} \zeta_t(\theta^t) p_t(\theta^t) \left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \right] \\
& + \chi \widehat{a}_1 - \sum_{t=1}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t),
\end{aligned}$$

where $\zeta_t(\theta^t)$ is called the "cumulative multiplier," and its motion is given by

$$\zeta_{t+1}(\theta^{t+1}) = \zeta_t(\theta^t) - \nu_{t+1}(\theta^{t+1}) - \varphi_{t+1}(\theta^{t+1}) \text{ with } \zeta_0 = \chi > 0, \ \forall t \geq 0, \theta^t. \tag{8}$$

Obviously, $\zeta_t(\theta^t)$ is a cumulative sum of all Lagrangian multipliers in the past history from the measurability conditions and the borrowing constraints; it encodes the frequency and severity of

[9] See Ljungqvist and Sargent (2012, p.821) for the formula.

both types of constraints over time.[10]

From the Lagrangian L, the FOCs (first-order conditions) with respect to consumption $c_t(\theta^t)$ and labor $l_t(\theta^t)$ are given by

$$\beta^t u'(c_t(\theta^t)) = \zeta_t(\theta^t) P_t, \tag{9}$$

$$\beta^t v'\left(\frac{l_t(\theta^t)}{\theta_t}\right)\frac{1}{\theta_t} = \zeta_t(\theta^t)\widehat{w}_t P_t, \tag{10}$$

while the FOC with respect to asset holdings $\widehat{a}_{t+1}(\theta^t)$ is given by

$$\sum_{\theta^{t+1}\succ\theta^t} \nu_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) = 0. \tag{11}$$

From the FOCs (9)-(10), we see that the value of $\zeta_t(\theta^t)$ cannot be negative.

The last FOC requires that the mean of multipliers on the measurability condition across idiosyncratic states θ^{t+1} be equal to zero, given θ^t. If markets were complete instead, households could have a short position on the consumption claims at time t contingent on shock θ_{t+1} being high at time $t+1$ ("save less for a high state"), and could have a long position on the consumption claims at time t contingent on shock θ_{t+1} being low at time $t+1$ ("save more for a low state"). However, markets are incomplete and households cannot save at time t depending on whether shock θ_{t+1} at time $t+1$ is high or low. As such, the best choice for $\widehat{a}_{t+1}(\theta^t)$ at time t is to satisfy an average — that is, the condition (11). Combining the FOCs (9) and (11) with the motion (8) actually enforces the household's Euler equation

$$u'(c_t(\theta^t)) \geq \beta\frac{P_t}{P_{t+1}}\sum_{\theta^{t+1}\succ\theta^t} u'(c_{t+1}(\theta^{t+1}))\pi(\theta_{t+1}|\theta_t), \tag{12}$$

where the equality holds if the borrowing constraint does not bind for all possible subsequent θ_{t+1} states at time $t+1$.

3.2 Competitive Equilibrium

A competitive equilibrium of the model economy is defined in the standard way.

[10] Note that the household problem is a standard convex programming problem since the constraint set is convex even with the incorporation of the measurability conditions and the borrowing constraints. Thus, the resulting first-order conditions are necessary and sufficient. In addition, this approach of defining recursive multipliers as in (8) was proposed and developed by Marcet and Marimon (1999, 2017) for solving dynamic incentive problems. Both Aiyagari, Marcet, Sargent, and Seppala (2002) and Chien, Cole, and Lustig (2011) adopted this approach.

Definition 1. *Given the initial capital K_1, initial government bond B_1, and sequences of tax rates, government spending, and government bonds $\{\tau_{l,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^{\infty}$, a competitive equilibrium is sequences of prices $\{w_t, q_t, P_t\}_{t=1}^{\infty}$, sequences of aggregate allocations $\{C_t, L_t, K_{t+1}\}_{t=1}^{\infty}$ and individual allocation plans $\{c_t(\theta^t), l_t(\theta^t), \widehat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, such that:*

1. *Given the sequence $\{w_t, P_t, \tau_{l,t}\}$, $\{c_t(\theta^t), l_t(\theta^t), \widehat{a}_{t+1}(\theta^t)\}$ solve the household problem.*

2. *Given the sequence $\{w_t, q_t\}$, $\{L_t, K_t\}$ solve the representative firm's problem.*

3. *The no-arbitrage condition holds: $\frac{P_t}{P_{t+1}} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta)$.*

4. *The time 0 government budget constraint holds:*

$$B_1 = \sum_{t=1}^{\infty} P_t \left[\tau_{l,t} w_t L_t + \tau_{k,t} (q_t - \delta) K_t - G_t\right].$$

5. *All markets are clear for all t:*

$$\begin{aligned} B_{t+1} + K_{t+1} &= \sum_{\theta^t} \widehat{a}_{t+1}(\theta^t)\pi_t(\theta^t), \\ L_t &= \sum_{\theta^t} l_t(\theta^t)\pi_t(\theta^t), \\ C_t &= \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t), \\ F(K_t, L_t) &= C_t + G_t + K_{t+1} - (1-\delta)K_t. \end{aligned}$$

3.3 Characterizing Competitive Equilibrium

This subsection characterizes the competitive equilibrium in terms of the aggregate allocations and the cumulative multipliers of the household problem. This step is critical for the primal Ramsey approach in the HAIM economy. To facilitate the characterization, we work with the popular power utility function:

Assumption 1.

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha},\ \alpha > 0;\ v\left(\frac{l}{\theta}\right) = \frac{1}{\gamma}\left(\frac{l}{\theta}\right)^{\gamma},\ \gamma > 1.$$

It is known that $1/\alpha$ represents the consumption elasticity of intertemporal substitution (EIS). As will be seen, the value of the consumption EIS plays an important role for our main result.

With the imposition of Assumption 1, the FOC for consumption (9) yields

$$c_t(\theta^t) = (\frac{\zeta_t(\theta^t)P_t}{\beta^t})^{-\frac{1}{\alpha}}.$$

Summing $c_t(\theta^t)$ over θ^t gives the aggregate consumption at time t:

$$C_t = \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t) = \sum_{\theta^t}(\frac{\zeta_t(\theta^t)P_t}{\beta^t})^{-\frac{1}{\alpha}}\pi(\theta^t) = (\frac{P_t}{\beta^t})^{-\frac{1}{\alpha}}\sum_{\theta^t}\zeta_t(\theta^t)^{-\frac{1}{\alpha}}\pi_t(\theta^t).$$

So, the consumption of a household with the history of events θ^t at time t can be expressed as

$$c_t(\theta^t) = \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t, \tag{13}$$

where

$$H_t = \sum_{\theta^t}\zeta_t(\theta^t)^{\frac{-1}{\alpha}}\pi_t(\theta^t).$$

The formula (13) characterizes the consumption sharing rule, given $\zeta_t(\theta^t)$ and C_t. We refer to H_t as the "consumption aggregate multiplier," which is a specific moment of the distribution of the individual cumulative multiplier $\zeta_t(\theta^t)$.[11]

From (10), we have

$$l_t(\theta^t) = (\frac{\theta_t^\gamma\zeta_t(\theta^t)\widehat{w}_tP_t}{\beta^t})^{\frac{1}{\gamma-1}}.$$

Similarly, the labor supply of a household with the history of events θ^t at time t can be expressed as

$$l_t(\theta^t) = \frac{\theta_t^{\frac{\gamma}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t, \tag{14}$$

where

$$J_t = \sum_{\theta^t}\theta_t^{\frac{\gamma}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}\pi_t(\theta^t).$$

We refer to J_t as the "labor aggregate multiplier."

Utilizing (9) and (13), we obtain

$$P_t = \beta^tC_t^{-\alpha}H_t^{\alpha}, \tag{15}$$

[11]Similar expressions for consumption can be seen in Nakajima (2005), Werning (2007), and Park (2014).

and hence from (4) we have

$$\frac{P_{t+1}}{P_t} = \frac{1}{R_{t+1}} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} \left(\frac{H_{t+1}}{H_t}\right)^{\alpha}. \tag{16}$$

From (10), (14), and (15), the after-tax wage rate is equal to

$$\widehat{w}_t = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^{\alpha}}. \tag{17}$$

Equations (13) through (17) show that one can express the individual allocations $\{c_t(\theta^t), l_t(\theta^t)\}$ and the market prices $\{P_t, \widehat{w}_t\}$ of the competitive equilibrium in terms of the aggregate allocations $\{C_t, L_t\}$, the individual cumulative multipliers $\{\zeta_t(\theta^t)\}$, and the aggregate multipliers $\{H_t, J_t\}$. The following proposition demonstrates that the Ramsey planner can pick a competitive equilibrium by choosing aggregate allocations plus asset holdings and individual multipliers that satisfy a set of conditions.

Proposition 1. *Impose Assumption 1. Given the initial capital K_1, government bond B_1, capital tax rate $\tau_{k,1}$, and a stream of government spending $\{G_t\}$, sequences of aggregate allocations $\{C_t, L_t, K_{t+1}\}$, asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$, and cumulative multipliers $\{\zeta_t(\theta^t)\}$ (with the associated aggregate multipliers, H_t and J_t) can be supported as a competitive equilibrium if and only if they satisfy the following conditions:*[12]

1. *Resource constraints:* $F(K_t, L_t) + (1-\delta)K_t - K_{t+1} \geq C_t + G_t,\ \forall t \geq 1.$

2. *Implementability condition:*

$$\sum_{t=1}^{\infty} \sum_{\theta^t} \beta^t \pi_t(\theta^t) \left[C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} - L_t^{\gamma} J_t^{-\gamma} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}\right] \geq \widehat{a}_1.$$

3. *Measurability conditions:*

$$\begin{aligned} & \sum_{s \geq t} \sum_{\theta^s \succeq \theta^t} \beta^s \pi_s(\theta^s) \left[C_s^{1-\alpha} H_s^{\alpha-1} \zeta_s(\theta^s)^{\frac{-1}{\alpha}} - L_s^{\gamma} J_s^{-\gamma} \theta_s^{\frac{\gamma}{\gamma-1}} \zeta_s(\theta^s)^{\frac{1}{\gamma-1}}\right] \\ = \ & \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t),\ \forall t \geq 1, \theta^t. \end{aligned}$$

[12] The initial capital tax rate, $\tau_{k,1}$ should be a choice variable for the Ramsey planner. However, given that the initial capital is pre-installed and that households are homogeneous at time zero, taxing the initial capital is essentially the same as allowing a lump-sum tax. As is standard in the literature, we restrict the planner's ability of choosing $\tau_{k,1}$.

4. *Borrowing constraints:*

$$\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t}\beta^s\pi_s(\theta^s)\left[C_s^{1-\alpha}H_s^{\alpha-1}\zeta_s(\theta^s)^{\frac{-1}{\alpha}}-L_s^{\gamma}J_s^{-\gamma}\theta_s^{\frac{\gamma}{\gamma-1}}\zeta_s(\theta^s)^{\frac{1}{\gamma-1}}\right]\geq 0,\ \forall t\geq 1,\theta^t.$$

5. *The evolution of* $\zeta_t(\theta^t)$ *satisfies* $\sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t)\leq\zeta_t(\theta^t)$, $\forall t\geq 1,\theta^t$.

6. *Conditional on* θ^t *at time* t*, if the borrowing constraint does not bind for all possible subsequent* θ_{t+1} *states at time* $t+1$*, then*

$$\sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t)=\zeta_t(\theta^t),$$

   and this property holds for all θ^t *and all* $t\geq 1$.

The proofs of our results, including Proposition 1, are all relegated to the Appendix. Results similar to Proposition 1 but in different contexts can be seen in Aiyagari, Marcet, Sargent, and Seppala (2002, Proposition 1), Werning (2007, Proposition 1), and Park (2014, Proposition 1). The first paper considers an RA economy without capital; the second paper focuses on an economy without idiosyncratic shocks; and the third paper envisions a complete-markets framework in the presence of market frictions according to Kehoe and Levine (1993). None of them address the HAIM economy, which is the focus of this paper.

4 Ramsey Problem

Different government policies result in different competitive equilibria. We define the Ramsey problem formally:

Definition 2. *The Ramsey problem is to choose a competitive equilibrium that attains the maximization of the household's lifetime utility* U*.*

On the basis of Proposition 1, the Ramsey problem can be represented as

$$\max_{\{C_t,L_t,K_{t+1},\{\widehat{a}_{t+1}(\theta^t)\},\{\zeta_t(\theta^t)\}\}}\sum_{t\geq 1}\beta^t\sum_{\theta^t}\left[\frac{1}{1-\alpha}\left(\frac{\zeta(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t\right)^{1-\alpha}-\frac{1}{\gamma}\left(\frac{\theta_t^{\frac{1}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t\right)^{\gamma}\right]\pi_t(\theta^t)$$

subject to Conditions 1 to 6 stated in Proposition 1 and to H_t and J_t defined earlier. The objective of the Ramsey problem is derived by substituting the consumption sharing rule (13) and the labor sharing rule (14) into $U(.)$.

Note that we have formulated the Ramsey problem in terms of the sequences of the aggregate allocations, $\{C_t, L_t, K_{t+1}\}$, the asset holdings, $\{\widehat{a}_{t+1}(\theta^t)\}$, and the cumulative multipliers, $\{\zeta_t(\theta^t)\}$.

4.1 Relaxed Ramsey Problem

The optimal allocation chosen by the Ramsey planner that satisfies Conditions 1-5 of Proposition 1 will also satisfy Condition 6 of Proposition 1. Namely, Condition 6 is not a binding constraint in the maximization of the Ramsey problem. The detail of the proof for the claim is in Appendix A.2. The proof is done by contradiction and here we briefly explain the basic logic of the proof.

Suppose not. That is, the borrowing constraints do not bind for all states θ^{t+1} at time $t+1$, but for some state θ^t at time t Condition 6 fails with

$$\sum_{\theta^{t+1}\succ\theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) < \zeta_t(\theta^t).$$

Applying the FOC (9) to the above inequality yields

$$u'(c_t(\theta^t)) > \beta\frac{P_t}{P_{t+1}} \sum_{\theta^{t+1}\succ\theta_t} u'(c_{t+1}(\theta_{t+1}))\pi(\theta_{t+1}|\theta_t),$$

which indicates that the θ^t household's marginal payoff from time t consumption is higher than the marginal payoff from time $t+1$ consumption. Since the borrowing constraints do not bind for all states θ^{t+1} at time $t+1$, we show that it is feasible for the planner to tighten the borrowing constraints to increase $c_t(\theta^t)$ at the expense of a decrease in $c_{t+1}(\theta_{t+1})$ so as to enhance the θ^t household's overall payoff and, at the same time, still meet Conditions 1-5 of Proposition 1. This then leads to a contradiction, implying that the optimal allocation chosen by the Ramsey planner that satisfies Conditions 1-5 of Proposition 1 must also satisfy Condition 6 of Proposition 1; otherwise, there will be an alternative allocation to improve the household's lifetime utility U.

Thus, we can ignore Condition 6 and consider the following relaxed Ramsey problem:

$$\max_{\{C_t,L_t,K_{t+1},\{\widehat{a}_{t+1}(\theta^t)\},\{\zeta_t(\theta^t)\}\}} \sum_{t\geq 1}\beta^t \sum_{\theta^t}\left[\frac{1}{1-\alpha}\left(\frac{\zeta(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t\right)^{1-\alpha} - \frac{1}{\gamma}\left(\frac{\theta_t^{\frac{1}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t\right)^{\gamma}\right]\pi_t(\theta^t),$$

subject to

$$\{\beta^t \mu_t\} : F(K_t, L_t) + (1-\delta)K_t \geq C_t + G_t + K_{t+1}, \ \forall t \geq 1,$$

$$\chi^P : \sum_{t\geq 1}\sum_{\theta^t} \beta^t \pi_t(\theta^t)\left[C_t^{1-\alpha}H_t^{\alpha-1}\zeta(\theta^t)^{\frac{-1}{\alpha}} - L_t^{\gamma}J_t^{-\gamma}\theta_t^{\frac{\gamma}{\gamma-1}}\zeta(\theta_t)^{\frac{1}{\gamma-1}}\right] \geq \widehat{a}_1,$$

$$\begin{aligned}\{v_t^P(\theta^t)\} \quad &: \quad \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} \beta^s \pi_s(\theta^s)\left[C_s^{1-\alpha}H_s^{\alpha-1}\zeta_s(\theta^s)^{\frac{-1}{\alpha}} - L_s^{\gamma}J_s^{-\gamma}\theta_s^{\frac{\gamma}{\gamma-1}}\zeta_s(\theta^s)^{\frac{1}{\gamma-1}}\right] \\ &= \quad \beta^{t-1}C_{t-1}^{-\alpha}H_{t-1}^{\alpha}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t), \ \forall t \geq 1, \theta^t,\end{aligned}$$

$$\{\varphi_t^P(\theta^t)\} : \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} \beta^s \pi_s(\theta^s)\left[C_s^{1-\alpha}H_s^{\alpha-1}\zeta_s(\theta^s)^{\frac{-1}{\alpha}} - L_s^{\gamma}J_s^{-\gamma}\theta_s^{\frac{\gamma}{\gamma-1}}\zeta_s(\theta^s)^{\frac{1}{\gamma-1}}\right] \geq 0, \ \forall t \geq 1, \theta^t,$$

$$\{\beta^t \xi_t(\theta^t)\} : \sum_{\theta^{t+1}\succ\theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t), \ \forall t \geq 1, \theta^t,$$

where $\{\beta^t\mu_t\}, \chi^P, \{v_t^P(\theta^t)\}, \{\varphi_t^P(\theta^t)\}$ and $\{\beta^t\xi_t(\theta^t)\}$ are the multipliers on the aggregate resource constraints, the implementability condition, the measurability conditions, the borrowing constraints, and the law of motion for the household's cumulative multipliers, respectively.

Forming the Lagrangian for the relaxed Ramsey problem and using Abel's summation formula gives

$$\begin{aligned}\mathcal{L} \quad = \quad & \max_{\{C_t, L_t, K_{t+1}, \{\widehat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t\geq 1}\beta^t W(t) + \sum_{t\geq 1}\beta^t\mu_t\left[F(K_t, L_t) + (1-\delta)K_t - K_{t+1} - C_t - G_t\right] \\ & + \sum_{t\geq 1}\sum_{\theta^t}\beta^t\xi_t(\theta^t)\left[\zeta_t(\theta^t) - \sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t)\right] - \chi^P\widehat{a}_1 \\ & - \sum_{t\geq 1}\beta^t C_t^{-\alpha}H_t^{\alpha}\left[\sum_{\theta^t}\widehat{a}_{t+1}(\theta^t)\sum_{\theta^{t+1}\succ\theta^t}v_{t+1}^P(\theta^{t+1})\pi(\theta_{t+1}|\theta_t)\right]\pi_t(\theta^t),\end{aligned}$$

with

$$W(t) \equiv \sum_{\theta^t}\pi_t(\theta^t)\left[\begin{array}{c}\underbrace{\frac{1}{1-\alpha}\left(\frac{\zeta(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t\right)^{1-\alpha} - \frac{1}{\gamma}\left(\frac{\theta_t^{\frac{1}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t\right)^{\gamma}}_{Part\ 1} \\ \underbrace{+\eta_t(\theta^t)\left(C_t^{-\alpha}H_t^{\alpha}\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t - L_t^{\gamma-1}J_t^{1-\gamma}\frac{\theta_t^{\frac{\gamma}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t\right)}_{Part\ 2}\end{array}\right], \qquad (18)$$

where

$$\eta_{t+1}(\theta^{t+1}) = \eta_t(\theta^t) + \nu^P_{t+1}(\theta^{t+1}) + \varphi^P_{t+1}(\theta^{t+1}),\ \eta_0 = \chi^P > 0, \forall t \geq 0, \theta^t, \tag{19}$$

which is defined as the Ramsey planner's cumulative multiplier. The Ramsey planner cannot complete the market as typically assumed and is thereby subject to the same market structure of the HAIM economy — that is, the same measurability conditions and borrowing constraints as those facing the household. These market frictions are summarized by the multipliers $\nu_{t+1}(\theta^{t+1})$ and $\varphi_{t+1}(\theta^{t+1})$ in the household problem and by $\nu^P_{t+1}(\theta^{t+1})$ and $\varphi^P_{t+1}(\theta^{t+1})$ in the planner problem. However, note that while we have the term $\chi\widehat{a}_1$ in the household Lagrangian L, we have the term $-\chi^P\widehat{a}_1$ in the planner Lagrangian $\mathcal{L}$. The opposite sign is due to the fact that the implementability condition in the Ramsey problem represents the government budget constraint rather than the household budget constraint. As such, while increasing $\widehat{a}_1$ relaxes the household budget constraint, it tightens the government budget constraint.

4.2 Comparison with the Representative-Agent Model

To gain insights into the pseudo-utility $W(t)$ defined in (18), we compare it with the analogous one derived in the RA model:[13]

$$\begin{aligned} W^{RA}(t) &= \underbrace{u(C_t) - v(L_t)}_{Part\ 1} + \underbrace{\chi^P\left(u'(C_t)C_t - v'(L_t)L_t\right)}_{Part\ 2} \\ &= \underbrace{\frac{1}{1-\alpha}C_t^{1-\alpha} - \frac{1}{\gamma}L_t^{\gamma}}_{Part\ 1} + \underbrace{\chi^P\left(C_t^{-\alpha}C_t - L_t^{\gamma-1}L_t\right)}_{Part\ 2}, \end{aligned}$$

where W^{RA} denotes the corresponding pseudo-utility in the RA model and the second equality holds under Assumption 1. Part 1 of $W^{RA}(t)$ represents the current period utility. Its Part 2, in terms of $\beta^t W^{RA}(t)$, is given by

$$\beta^t\chi^P\left(u'(C_t)C_t - v'(L_t)L_t\right) = \chi^P\beta^t u'(C_t)\left(C_t - \frac{v'(L_t)}{u'(C_t)}L_t\right) = \chi^P P_t^{RA}(C_t - \widehat{w}_t^{RA}L_t),$$

where $P_t^{RA} = \beta^t u'(C_t)$, that is, the time zero price of one unit of consumption delivered at time t in the RA model, and $\widehat{w}_t^{RA} = \frac{v'(L_t)}{u'(C_t)}$, that is, the after-tax wage rate at time t in the RA model. Thus, the term $P_t^{RA}(C_t - \widehat{w}_t^{RA}L_t)$ shown in the above equation represents the time t net savings

[13] Please refer to equation (16.6.7) in Ljungqvist and Sargent (2012, p. 627).

evaluated at the time zero price in the RA model. The implementability condition multiplier, χ^P, "measures the utility costs of raising government revenues through distorting taxes"(Ljungqvist and Sargent (2012, p. 629)) in the RA framework. The taxes imposed by the Ramsey planner alter the time t consumption and labor supply of the RA in competitive equilibrium and, consequently, distort the time t net savings. The shadow price of this distortion on net savings is given by the multiplier χ^P.

Part 1 of $W(t)$ in our HAIM model also represents the current period utility. Its Part 2, in terms of $\beta^t W(t)$, is given by

$$\eta_t(\theta^t)\beta^t C_t^{-\alpha} H_t^{\alpha} \left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t - \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^{\alpha}} \frac{\theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right) = \eta_t(\theta^t) P_t \left(c_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \right), \quad (20)$$

where $P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}$ according to (15) and $\widehat{w}_t = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^{\alpha}}$ according to (17). Thus, the term $P_t \left(c_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \right)$ shown in the above equation represents the time t net savings of the household θ^t evaluated at the time zero price in the HAIM economy. Note that the structure of $W(t)$ is basically the same as that of $W^{RA}(t)$. However, $W(t)$ deviates from $W^{RA}(t)$ in two important respects.

First, the multiplier in (20), $\eta_t(\theta^t)$, is no longer a constant as χ^P in the RA model. The Ramsey planner has to consider the impact of its policies on each household's net savings under incomplete rather than complete markets. The associated shadow price is captured by the multiplier $\eta_t(\theta^t)$. From the evolution of $\eta_t(\theta^t)$ governed by (19), we see that $\eta_t(\theta^t)$ starts from χ^P ($\eta_0 = \chi^P$) but in a sequence it varies not only across households but also over time, meaning that the utility cost of implementing policy for households is not only household-specific but also time-varying. The following lemma shows that the average value of $\eta_t(\theta^t)$ tends to increase over time, implying that it could stochastically diverge to infinity in the limit.

Lemma 1. *The average of $\eta_t(\theta^t)$, $\sum_{\theta^t} \eta_t(\theta^t)\pi_t(\theta^t)$, is positive and, moreover, it is non-decreasing and becomes strictly increasing if $\varphi_t^P(\theta^t) > 0$ for some θ^t.*

The second important deviation stems from the behavior of intertemporal prices. While the time zero price of consumption delivered at time t is $P_t^{RA} = \beta^t C_t^{-\alpha}$ in the RA model, this price becomes $P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}$ in the HAIM model. As such, in the steady state, the market discounting rate implied by P_t^{RA} is consistent with the time discount factor β, whereas the market discounting rate implied by P_t is lower than β, provided that H_t is increasing over time. Now consider the steady-state version of equation (16):

$$1 = \beta R \left(\frac{H_{t+1}}{H_t} \right)^{\alpha}, \tag{21}$$

which tells us that H_t is increasing over time and must diverge to infinity in the limit if $R < 1/\beta$ holds in the steady state.[14] Thus, the feature of an increasing and divergent H_t exactly underlies the hallmark of the HAIM model — the risk-free rate is lower than the time discount rate in the steady state.

The divergent tendency of both $\eta_t(\theta^t)$ and H_t, all else equal, makes Part 2 of $W(t)$ converge more slowly than Part 1. As will be seen, this asymmetric convergence between Part 1 and Part 2 of $W(t)$ is the key to our result of no Ramsey steady state.

These two deviations of $W(t)$ from $W^{RA}(t)$ are in fact the two sides of one coin. Both are rooted in the frictions of the HAIM economy and both will vanish once markets are complete, as in the RA model.

4.3 Optimal Conditions of the Ramsey Problem

From the Lagrangian $\mathcal{L}$, the necessary FOCs with respect to $\widehat{a}_{t+1}(\theta^t), C_t, L_t,$ and K_{t+1} for $t \geq 1$ yield, respectively,

$$\sum_{\theta^{t+1} \succ \theta^t} v_{t+1}^P(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) = 0, \tag{22}$$

$$W_C(t) = \mu_t, \tag{23}$$

$$-W_L(t) = \mu_t F_L(K_t, L_t), \tag{24}$$

$$\mu_t = \beta \mu_{t+1} \left[F_K \left(K_{t+1}, L_{t+1} \right) - \delta + 1 \right], \tag{25}$$

where the derivation of (23) has made use of (22), and $W_C(t)$ and $W_L(t)$ denote the derivatives of $W(t)$ with respect to C_t and L_t, respectively.[15]

The explicit expressions of $W_C(t)$ and $W_L(t)$ in the FOCs of the Ramsey problem are crucial to our analysis later. One can derive them from the pseudo-utility $W(t)$ defined in (18). However, to facilitate the proof and discussion hereafter, it is convenient to express W_C and W_L in the following

[14] The converse should hold as well. When H_t diverges, it means that households suffer from the frictions of incomplete markets as in the standard scenario of the HAIM economy. Then, according to footnote 6, the outcome of $R < 1/\beta$ will result in the steady state.

[15] The FOC with respect to $\zeta_t(\theta^t)$ will not be needed for the derivation of our main results.

way. First, using the consumption sharing rule (13), $W_C(t)$ in (23) is read as

$$W_C(t) = C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)}_{Part\ 1} + \underbrace{(1-\alpha) H_t^{\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t} \right) \eta_t(\theta^t) \pi_t(\theta^t)}_{Part\ 2} \right]. \quad (26)$$

Second, using (13)-(15) and (17), $W_L(t)$ in (24) is read as

$$-W_L(t) = \widehat{w}_t C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \left(\frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t)}_{Part\ 1} + \underbrace{\gamma H_t^{\alpha} \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t} \right) \eta_t(\theta^t) \pi_t(\theta^t)}_{Part\ 2} \right]. \quad (27)$$

Part 1 of $W_C(t)$ (resp. Part 1 of $W_L(t)$) denotes the sum of households' "normalized" marginal utility of consumption, $\left(\frac{c_t(\theta^t)}{C_t}\right)^{-\alpha}$, weighted by their consumption shares (resp. labor shares). They represent the planner's social evaluation of increasing C_t and L_t respectively under the utilitarian objective. We next explain the meaning of the weighted sum of $\eta_t(\theta^t)$ shown in Part 2 of $W_C(t)$ and of $W_L(t)$. Summing up (20) across all households gives

$$P_t \sum_{\theta^t} \eta_t(\theta^t) \left[\frac{c_t(\theta^t)}{C_t} C_t - \frac{l_t(\theta^t)}{L_t} \widehat{w}_t L_t \right] \pi_t(\theta^t).$$

Contrasting the above with the corresponding term in the RA model, namely, $P_t^{RA} \chi^P (C_t - \widehat{w}_t^{RA} L_t)$, we see that the role of χ^P has been replaced by $\sum_{\theta^t} \eta_t(\theta^t) \frac{c_t(\theta^t)}{C_t} \pi_t(\theta^t)$ or $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t)$. In other words, they represent the shadow prices of distorting net savings in the aggregate in the HAIM economy. Since the issue is about net savings in the aggregate, it is intuitive that these shadow prices are weighted rather than unweighted as given by $\sum_{\theta^t} \eta_t(\theta^t) \pi_t(\theta^t)$. Note that, unlike the common χ^P in the RA model, these shadow prices may differ, depending on whether distorting the net aggregate savings, $C_t - \widehat{w}_t L_t$, takes place via changing consumption C_t or changing labor income $\widehat{w}_t L_t$.

5 Ramsey Steady State

After the formulation of the Ramsey problem in the previous section, we come to the central focus of the paper: The Ramsey planner's prescription for the allocation of the HAIM economy in the long run and its implication for optimal capital taxation. The HAIM model is not a representative-agent economy but a heterogeneous-agent economy; as a result, its steady state is more involved. We have the following definition:

Definition 3. *The steady state of the HAIM economy meets two conditions:*

1. *Each aggregate variable converges to a non-zero finite limit.*
2. *The distributions of the consumption share $c_t(\theta^t)/C_t$ and of the labor share $l_t(\theta^t)/L_t$ across θ^t are time invariant with finite bounded support.*

If either of the two conditions fails to hold in the long run, we state that the economy has "no steady state."[16] We also have:

Definition 4. *The optimal solution to the Ramsey problem is defined as a Ramsey steady state if it features the steady state of the HAIM economy.*

We are ready to state our main finding.

Proposition 2. *Impose Assumption 1.*

1. *If $\alpha \geq 1$, there is no Ramsey steady state with $R < 1/\beta$.*
2. *If $\alpha < 1$, a Ramsey steady state with $R < 1/\beta$ is possible, but (i) the corresponding shadow price of resources, μ_t, must diverge in the limit, and (ii) the planner may not implement the MGR.*

Our main finding contrasts sharply with the result obtained by Aiyagari (1995), who argued for the imposition of a positive capital tax to restore the MGR in the assumed Ramsey steady state. Our first result listed in Proposition 2 is clearly at odds with the basic premise of Aiyagari (1995) that a Ramsey steady state with $R < 1/\beta$ exists. We show in the proof that if there exists a Ramsey steady state with $\alpha \geq 1$, then in this steady state both the MGR and $R = 1/\beta$ must hold; otherwise, the Ramsey steady state does not exist. The second result of Proposition 2 also deviates from that prescribed by Aiyagari (1995). As shown in the proof of Proposition 2, the only possibility for the existence of a Ramsey steady state with $R < 1/\beta$ has to come along with the

[16] Straub and Werning (2014) made the distinction between interior and non-interior steady states. Their interior steady states correspond to Condition 1 in our Definition 3. It is more convenient for us to conduct the analysis under our definition.

divergence of μ_t, the shadow price of resources. This contradicts the implicit assumption made by Aiyagari (1995) that μ_t converges to a finite limit. At any rate, Proposition 2 shows that the Ramsey steady state described by Aiyagari (1995) does not arise at the optimum.

As already mentioned, the key driving force of our result stems from the asymmetric converging rate between Part 1 and Part 2 of $W(t)$, which originated from the frictions of the HAIM economy. In the presence of $H_t \to \infty$ associated with $R < 1/\beta$ in the steady state, the limiting behavior of $W_C(t)$ and $-W_L(t)$ shown in (26)-(27) is dominated by their Part 2 so that both $W_C(t)$ and $-W_L(t)$ explode in the limit. In the proof of Proposition 2, this divergence of $W_C(t)$ or $-W_L(t)$ causes the violation of the FOCs (23)-(25) that characterizes the optimal Ramsey allocation if $\alpha > 1$, and it causes the planner to choose a corner rather than an interior aggregate solution if $\alpha = 1$. Either way, it makes the Ramsey steady state fail to exist in the case of $\alpha \geq 1$. Thanks to our primal approach, the divergent force embedded in $W_C(t)$ or $-W_L(t)$ can be seen clearly by means of our derived W_t.

Intuitively, unlike households in the face of idiosyncratic income shocks, the Ramsey planner faces no uncertainty in allocating aggregate resources. The strict inequality $R < 1/\beta$ in the steady state then indicates that the market discounting rate is lower than the preference discounting rate. This feature of asymmetric discounting impels a desire for the planner to front-load aggregate consumption and/or back-load aggregate labor through its policy. Such a desire persists as long as $R < 1/\beta$ holds in the steady state. Thus, unless there exists a counterbalance to offset the planner's desire, the persistence can lead to "extremes" to upset the Ramsey steady state. We explain below that while a counterbalance exists in the case of $\alpha < 1$, there does not in the case of $\alpha \geq 1$; in fact, there is an enforcement to enhance the planner's desire if $\alpha > 1$.

As discussed in Section 4.2, the utility costs of implementing a policy depend on the policy effects on the net savings of households. Let us consider the impact of changing aggregate consumption on the net savings through consumption spending. There is only a term involving C_t in Part 2 of $W(t)$ given by (18). Expressing in $\beta^t W(t)$ and omitting $\eta_t(\theta^t)$, this term equals

$$\beta^t C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} = P_t C_t \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t},$$

which represents the θ^t household's consumption spending at time t according to the consumption sharing rule (13). From (15), $P_t C_t = \beta^t C_t^{1-\alpha} H_t^{\alpha}$ and so $\partial(P_t C_t)/\partial C_t = (1-\alpha)\beta^t C_t^{-\alpha} H_t^{\alpha}$. Thus, all else equal, a drop in aggregate consumption C_t will raise, lower, or not change individual consumption spending via altering $P_t C_t$ if α is larger than, less than, or equal to 1, respectively.

This implies that a reduction in aggregate consumption (front-loading consumption) will make the government constraint associated with $\eta_t(\theta^t)$ in (18) looser, tighter, or unchanged, depending on whether α is larger than, less than, or equal to 1, respectively. Since front-loading aggregate consumption relaxes the government constraint if $\alpha > 1$, it actually enforces the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in the steady sate. By contrast, since front-loading aggregate consumption tightens the government constraint if $\alpha < 1$, it counterbalances the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in the steady state.[17]

When $\alpha = 1$, neither enforcement (associated with $\alpha > 1$) nor counterbalance (associated with $\alpha < 1$) occurs. We then see a clean case of front-loading aggregate consumption in the presence of $R < 1/\beta$ in the steady state. From the proof of Proposition 2, we know that μ_t is increasing and divergent because H_t is increasing and divergent. If $\alpha = 1$, we have $W_C(t) = C_t^{-1}$ from (26). Thus, given that μ_t increases over time, it is apparent that the optimal C_t determined by the FOC (23), namely, $C_t^{-1} = \mu_t$, will decrease over time. In the limit we obtain $C_t \to 0$, which violates Condition 1 of Definition 3 for the steady state of the HAIM economy.

It is important to recognize that the Ramsey prescription suggested by Aiyagari (1995) — a positive capital tax to enforce the MGR in the steady state — is still feasible to the Ramsey planner. However, that prescription is inconsistent with the necessary FOCs of the Ramsey problem, and hence it cannot be a Ramsey steady state. This analysis through the primal Ramsey approach is overlooked by Aiyagari (1995) in his dual approach. Overall, Proposition 2 suggests that the MGR, which obeys production efficiency, might not be the most important margin to the Ramsey planner. This possibility is consistent with the results of Chien and Wen (2017) who found that the Ramsey planner always intends to relax the frictions imposed by incomplete markets, so as to aim for $R = 1/\beta$ in the steady state.

[17]In the presence of $R < 1/\beta$ in the steady state, the planner would like to back-load aggregate labor as well as front-load aggregate consumption. However, note that back-loading aggregate labor always tightens the government constraint associated with $\eta_t(\theta^t)$ in (18), regardless of the value of γ. This implies that, unlike front-loading aggregate consumption, there always exists the counterbalance to offset the planner's desire to back-load aggregate labor in the presence of $R < 1/\beta$ in the steady state. It explains why the violation of the FOCs in the proof of Proposition 2.is through (23) rather than (24).

6 Endogenous Government Spending

This section checks the robustness of our findings by altering the model setup from exogenous to endogenous government spending, which is the main setting considered by Aiyagari (1995). We show here that even with endogenous government spending, our results are robust and remain unchanged.

Following Aiyagari (1995), the household lifetime utility U is modified to

$$U^G = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) + V(G_t) \right] \pi_t(\theta^t),$$

where $V(.)$ is the utility function of public consumption G_t, which is assumed common for all households. The usual assumptions are applied to $V(.)$. This modification of the setup does not change the household problem since the determination of G_t is exogenous to households. However, the Ramsey problem is changed slightly because G_t is now a choice variable to the Ramsey planner. As long as G_t is non-negative (which could be ensured by assuming $V'(0) = \infty$), G_t can be chosen to satisfy the time t resource constraint so that Proposition 1 still applies. The Lagrangian for the Ramsey problem is modified as

$$\begin{aligned}
\mathcal{L}^G \quad = \quad & \max_{\{C_t, L_t, K_{t+1}, \{\widehat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}, G_t\}} \sum_{t\geq 1} \beta^t [W(t) + V(G_t)] \\
& + \sum_{t\geq 1} \beta^t \mu_t \left[F(K_t, L_t) + (1-\delta)K_t - K_{t+1} - C_t - G_t \right] \\
& + \sum_{t\geq 1} \sum_{\theta^t} \beta^t \xi_t(\theta^t) \left[\zeta_t(\theta^t) - \sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \right] - \chi^P \widehat{a}_1 \\
& + \sum_{t\geq 1} \beta^t C_t^{-\alpha} H_t^{\alpha} \left[\sum_{\theta^t} \widehat{a}_{t+1}(\theta^t) \sum_{\theta^{t+1} \succ \theta^t} v_{t+1}^P(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \right] \pi_t(\theta^t),
\end{aligned}$$

which is identical to the Lagrangian $\mathcal{L}$, except for the replacement of $W(t)$ by $W(t) + V(G_t)$. The FOCs with respect to aggregate consumption, labor, and capital remain the same as before. The additional FOC with respect to G_t is given by

$$V'(G_t) = \mu_t, \tag{28}$$

which together with the FOC (25) does imply the MGR if the Ramsey steady state is assumed.

This is essentially the procedure for obtaining the MGR in Aiyagari (1995); see Equation (20) of his Proposition 1 on page 1170.

However, the introduction of endogenous G_t does not alter the fundamental force that drives our main result. The marginal social benefit of having one extra unit of aggregate consumption, namely, $W_C(t)$, could still diverge in the long run given that the Ramsey outcome of $R\beta = 1$ is infeasible in the steady state. With the additional government tool — the endogenous government spending — the extra output can be spent either on government spending or on private consumption, and hence the marginal benefits to the social welfare by exercising these two options have to be equalized at the optimum. Indeed, putting (23) and (28) together gives rise to $V'(G_t) = W_C(t)$. This equality casts doubt on the convergence assumption of G_t to a positive value in the steady state because it is inconsistent with the divergence of $W_C(t)$.

7 Conclusion

This paper revisits the long-standing issue of Ramsey capital taxation in the HAIM economy. Our results show that the conventional wisdom on the issue may be problematic. In particular, we find that the policy prescription of taxing capital to restore the MGR in the steady state may not be optimal for the Ramsey problem. Instead, the Ramsey planner might choose an outcome featuring no steady state if the situation where the risk-free rate is lower than the time discount rate persists. As this persistence is a key feature of the HAIM environment, assuming the existence of a Ramsey steady state is not innocuous.

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A Appendix

A.1 Proof of Proposition 1

"Only if" part: Condition 1 of Proposition 1 — the resource constraints — is implied by a competitive equilibrium since it is part of the definition. Note also that Conditions 5 and 6 of Proposition 1 are implied by (8) and (11) from the household problem in competitive equilibrium.

The remaining proof is to show that the time 0 budget constraint (7), the measurability conditions (5) and the borrowing constraints (6) in the household problem can be re-expressed as Conditions 2-4 of Proposition 1. Substituting (4), (13)-(14) and (16)-(17), all of which build on the household's optimal behavior, into (5)-(7), we obtain Conditions 2-4.

"If" part: Suppose the sequence of asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, aggregate allocations $\{C_t, K_{t+1}, L_t\}_{t=1}^{\infty}$, and cumulative multipliers $\{\zeta_t(\theta^t)\}_{t=1}^{\infty}$ with the associated aggregate multipliers $\{H_t, J_t\}_{t=1}^{\infty}$ satisfy Conditions 1-6 stated in Proposition 1. We show that a competitive equilibrium of the HAIM economy can be constructed in the following way.

First, we pick prices and taxes defined below:

$$q_t = F_K(K_t, L_t), \tag{29}$$

$$w_t = F_L(K_t, L_t), \tag{30}$$

$$P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}, \tag{31}$$

$$1 - \tau_{k,t+1} = \frac{\frac{P_t}{P_{t+1}} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta} = \frac{\frac{1}{\beta}\left(\frac{C_t}{C_{t+1}}\right)^{-\alpha}\left(\frac{H_t}{H_{t+1}}\right)^{\alpha} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta}, \tag{32}$$

$$1 - \tau_{l,t} = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{F_L(K_t, L_t) C_t^{-\alpha} H_t^{\alpha}}. \tag{33}$$

Note that (29)-(30) correspond to the profit-maximization conditions of the representative firm and that (32) ensures that the no-arbitrage condition (4) holds.

Second, we show that the household problem can be solved. Let the individual consumption and labor allocations be given by (13) and (14). Then, individual consumption and labor allocations together with prices defined in (29)-(33) satisfy the first-order conditions, (9) and (10), of the household problem. To derive the household's Euler equation, we combine individual consumption allocations, prices defined in (29)-(33), and Conditions 5-6. The time 0 budget constraint (7), the measurability conditions (5), and the borrowing constraints (6) in the household problem can be obtained by using (29)-(33) plus Conditions 2-4.

Third, we need to make sure that all markets clear. Plugging in individual consumption allocations (13) into Condition 1 implies that the market clearing condition of the good market is satisfied. The labor market clearing condition is achieved by aggregating (14) across all households. For the asset market, we pick $\{B_{t+1}\}_{t=1}^{\infty}$ such that

$$B_{t+1} = \sum_{\theta^t} \widehat{a}_{t+1}(\theta^t) - K_{t+1},$$

which ensures that the asset market clears in each time period.

The last condition to be met in the competitive equilibrium is the government budget constraint. From (7), we have

$$\begin{aligned}
B_1 + K_1 &= \widehat{a}_1 = \sum_{t\geq 1} P_t \sum_{\theta^t} \left[c_t(\theta^t)\pi_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\pi_t(\theta^t)\right] \\
&= \sum_{t\geq 1} P_t \left[C_t - w_t L_t + \tau_{l,t} w_t L_t\right] \\
&= \sum_{t\geq 1} P_t \left[C_t + q_t K_t - F(K_t, L_t) + \tau_{l,t} w_t L_t\right],
\end{aligned}$$

where the derivation has made use of $\widehat{w}_t = w_t(1-\tau_{l,t})$ and $F(K_t, L_t) = w_t L_t + q_t K_t$. Utilizing the resource constraint and the no-arbitrage condition (4) then gives

$$\begin{aligned}
B_1 + K_1 &= \sum_{t\geq 1} P_t \left[q_t K_t - K_{t+1} + (1-\delta)K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= \sum_{t\geq 1} P_t \left[(1 + (1-\tau_{k,t})(q_t - \delta))K_t - K_{t+1} + \tau_{k,t}(q_t-\delta)K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= \sum_{t\geq 1} P_t \left[\frac{P_{t-1}}{P_t} K_t - K_{t+1} + \tau_{k,t}(q_t-\delta)K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= P_0 K_1 + \sum_{t\geq 1} P_t \left[\tau_{k,t}(q_t-\delta)K_t + \tau_{l,t} w_t L_t - G_t\right],
\end{aligned}$$

which leads to the time 0 government budget constraint since we normalize $P_0 = 1$.

A.2 Proof of Relaxed Ramsey

We claim in the text that the optimal allocation chosen by the Ramsey planner that satisfies Conditions 1-5 of Proposition 1 will also satisfy Condition 6 of Proposition 1. This claim enables us to consider the relaxed rather than the original Ramsey problem. We now verify the claim. The proof is done by contradiction.

Suppose there is an original optimal allocation $\{\{\zeta_t(\theta^t)\}, H_t, J_t, C_t, L_t, K_{t+1}, \{\hat{a}_{t+1}(\theta^t)\}\}_{t=1}^{\infty}$ that satisfies Conditions 1-5 but fails Condition 6 of Proposition 1. That is, there exist $\hat{\theta}^t$ at time t and all its possible subsequent $\hat{\theta}_{t+1}$ states at time $t+1$ that borrowing constraints do not bind and

$$x\left(\hat{\theta}^t\right) \equiv \zeta_t(\hat{\theta}^t) - \sum_{\hat{\theta}^{t+1} \succ \hat{\theta}^t} \zeta_{t+1}(\hat{\theta}^{t+1})\pi(\hat{\theta}_{t+1}|\hat{\theta}^t) > 0. \tag{34}$$

We construct an alternative v allocation $\left\{\{\zeta_t^v(\theta^t)\}, H_t^v, J_t^v, C_t^v, L_t^v, K_{t+1}^v, \{\hat{a}_{t+1}^v(\theta^t)\}\right\}_{t=1}^{\infty}$ that deviates from the original allocation such that Conditions 1-5 of Proposition 1 still hold but only tightens (34), i.e.,

$$x\left(\hat{\theta}^t\right) > x^v\left(\hat{\theta}^t\right) \equiv \zeta_t^v(\hat{\theta}^t) - \sum_{\hat{\theta}^{t+1} \succ \hat{\theta}^t} \zeta_{t+1}^v(\hat{\theta}^{t+1})\pi(\hat{\theta}_{t+1}|\hat{\theta}^t) \geq 0. \tag{35}$$

We shall prove that the alternative v allocation surpasses the original optimal allocation in the household's lifetime utility U. This then leads to a contradiction, implying that the optimal allocation of the relaxed Ramsey problem must also satisfy Condition 6 of Proposition 1; otherwise, there will be an alternative allocation to improve U.

To provide a clear exposition, we decompose the whole proof into three parts:

Part 1: Construct the v allocation by making a (ϵ_1, ϵ_2)-variation to the original allocation.

First, we choose $\{\zeta_s^v(\theta^s)\}$ such that

$$\zeta_s^v(\theta^s) = \begin{array}{ll} \zeta_s(\theta^s) - \epsilon_1 & \text{if } \theta^s = \hat{\theta}^t \text{ at } s = t, \\ \zeta_s(\theta^s) + \epsilon_2 & \text{if } \theta^s = \left(\hat{\theta}^t, \hat{\theta}_{t+1}\right) \text{ at } s = t+1, \\ \zeta_s(\theta^s) & \text{otherwise,} \end{array} \tag{36}$$

where $\epsilon_1, \epsilon_2 > 0$. Due to the strict inequality of (34), it is feasible to have (ϵ_1, ϵ_2) so that the chosen $\{\zeta_s^v(\theta^s)\}$ according to (36) satisfies (35). The corresponding aggregate multipliers associated with $\{\zeta_s^v(\theta^s)\}$ are denoted by H_t^v and J_t^v. Given (36), these aggregate multipliers are identical to the original ones except at time t and $t+1$, that is, $H_s^v = H_s$ and $J_s^v = J_s$ if $s \notin \{t, t+1\}$.

Second, we choose C_t^v and L_t^v such that

$$\frac{C_t^v}{H_t^v} = \frac{C_t}{H_t} \text{ and } \frac{L_t^v}{J_t^v} = \frac{L_t}{J_t} \text{ for all } t. \tag{37}$$

Thus, $C_s^v = C_s$ and $L_s^v = L_s$ if $s \notin \{t, t+1\}$. Note that the intratemporal and intertemporal shadow prices, $(L_t^v)^{\gamma-1} (J_t^v)^{1-\gamma} / \left(C_t^v\right)^{-\alpha} (H_t^v)^{\alpha}$ and $\beta^t \left(C_t^v\right)^{1-\alpha} (H_t^v)^{\alpha-1}$, are identical to those in the original allocation for all t by our choice.

Third, we choose K_t^v such that

$$K_{t+1}^v = F\left(K_t^v, L_t^v\right) + (1-\delta) K_t^v - C_t^v - G_t \text{ for all } t. \tag{38}$$

Fourth, we choose asset holdings $\{\widehat{a}_{t+1}^v(\theta^t)\}$ as follows:

$$\widehat{a}_{s+1}^v\left(\theta^s\right) \pi_{s+1}\left(\theta^{s+1}\right) = \begin{cases} \widehat{a}_{s+1}\left(\theta^s\right) \pi_{s+1}\left(\theta^{s+1}\right) & \text{if } \theta^s \neq \hat{\theta}^t, \\ \widehat{a}_{s+1}\left(\theta^s\right) \pi_{s+1}\left(\theta^{s+1}\right) + \Omega\left(\theta^s\right) & \text{if } \theta^s = \hat{\theta}^t, \end{cases} \tag{39}$$

where

$$\Omega\left(\hat{\theta}^t\right) \equiv \beta \sum_{\hat{\theta}_{t+1}} \pi(\hat{\theta}_{t+1}|\hat{\theta}_t) \left\{ \begin{array}{c} \left(\frac{C_{t+1}}{C_t}\right)^{1-\alpha} \left(\frac{H_{t+1}}{H_t}\right)^{\alpha-1} \left(\zeta_{t+1}^v(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{-1}{\alpha}} - \zeta_{t+1}(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{-1}{\alpha}}\right) \\ -\frac{L_{t+1}^{\gamma} J_{t+1}^{-\gamma}}{C_t^{-\alpha} H_t^{\alpha}} \hat{\theta}_{t+1}^{\frac{\gamma}{\gamma-1}} \left(\zeta_{t+1}^v(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{1}{\gamma-1}} - \zeta_{t+1}(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{1}{\gamma-1}}\right) \end{array} \right\}.$$

According to (36), both $\left(\zeta_{t+1}^v(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{-1}{\alpha}} - \zeta_{t+1}(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{-1}{\alpha}}\right)$ and $\left(\zeta_{t+1}^v(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{1}{\gamma-1}} - \zeta_{t+1}(\hat{\theta}^t, \hat{\theta}_{t+1})^{\frac{1}{\gamma-1}}\right)$ in $\Omega\left(\hat{\theta}^t\right)$ are functions of ϵ_2, and the value of $\Omega\left(\hat{\theta}^t\right)$ is negative.

Note that there are two degrees-of-freedom for (ϵ_1, ϵ_2). As such, it is feasible for us to pick (ϵ_1, ϵ_2) such that the chosen $\{\zeta_s^v\left(\theta^s\right)\}$ according to (36) satisfies the following condition:

$$\left[\zeta_t^v(\hat{\theta}^t)^{\frac{-1}{\alpha}} - \zeta_t(\hat{\theta}^t)^{\frac{-1}{\alpha}} - \frac{L_t^{\gamma} J_t^{-\gamma}}{C_t^{1-\alpha} H_t^{\alpha-1}} \hat{\theta}_t^{\frac{\gamma}{\gamma-1}} \left(\zeta_t^v(\hat{\theta}^t)^{\frac{1}{\gamma-1}} - \zeta_t(\hat{\theta}^t)^{\frac{1}{\gamma-1}}\right)\right] + \Omega\left(\hat{\theta}^t\right) = 0, \tag{40}$$

where the term in the square brackets is positive and a function of ϵ_1.

Part 2: Verify that the v allocation satisfies Conditions 1 to 5 of Proposition 1 and (35).

First, Condition 1 is satisfied according to (38). Condition 2, the implementability condition, is satisfied given equation (40). Indeed, if we multiply (40) by $\beta^t C_t^{1-\alpha} H_t^{\alpha-1}$, then the resulting sum of net savings between state $\hat{\theta}^t$ and all subsequent states $\hat{\theta}_{t+1}$ remains unchanged in terms of present value. Condition 3, the measurability conditions, holds for all time periods and all states by the construction of the asset holdings according to (39). The asset holdings, $\widehat{a}^v_{s+1}(\theta^s)$, do not change if $\theta^s \neq \hat{\theta}^t$. The measurability condition for $\widehat{a}^v_{t+1}(\hat{\theta}^t)$ if $\theta^s = \hat{\theta}^t$ is also satisfied according to (39) since the variation term, $\Omega\left(\hat{\theta}^t\right)$, does not depend on $\hat{\theta}_{t+1}$.

Next, we verify whether Condition 4, the borrowing constraints, holds. The borrowing constraints for θ^s, $s \leq t$, are satisfied owing to (40). For θ^s, $s > t+1$, and $\theta^{t+1} \neq \hat{\theta}^{t+1}$, the borrowing constraints still hold as the associated allocations are irrelevant to the variation. For $\hat{\theta}^{t+1}$, its corresponding borrowing constraint becomes tighter. This is because $\Omega\left(\hat{\theta}^t\right) < 0$ leads to $\widehat{a}^v_{t+1}(\hat{\theta}^t) < \widehat{a}_{t+1}(\hat{\theta}^t)$, which implies less wealth inherited from time t to $t+1$. However, since for $\left\{\hat{\theta}^t, \hat{\theta}_{t+1}\right\}$ the borrowing constraints are not binding at time $t+1$, this variation does not violate the borrowing constraint at $\hat{\theta}^{t+1}$ provided that the (ϵ_1, ϵ_2)-variation is sufficiently small.

Finally, Condition 5 is satisfied by our construction of the v allocation.

Part 3: Compare the lifetime utility U under the original allocation and the v allocation.

Note that the lifetime utility U is concave in $\zeta^v_t(\theta^t)^{\frac{-1}{\alpha}}\frac{C_t}{H_t}$ (individual consumption) and convex in $\theta_t^{\frac{1}{\gamma-1}}\zeta^v_t(\theta^t)^{\frac{1}{\gamma-1}}\frac{L_t}{J_t}$ (individual labor) and, therefore, the (ϵ_1, ϵ_2)-variation in the v allocation enhances U.

A.3 Proof of Lemma 1

Using (22), we have from (19):

$$\sum_{\theta_{t+1}} \eta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) = \eta_t(\theta^t) + \sum_{\theta_{t+1}} \varphi^P_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \geq \eta_t(\theta^t),$$

which implies that $\eta_t(\theta^t)$ is non-decreasing on the average, and this average becomes strictly increasing if $\varphi^P_t(\theta^t) > 0$ for some θ^t. Since $\eta_0 = \chi^P > 0$, the value of $\sum_{\theta^t} \eta_t(\theta^t)\pi_t(\theta^t)$ is clearly positive.

A.4 Proof of Proposition 2

Suppose there is a Ramsey steady state. By the FOCs of the Ramsey problem, namely, (23) through (25), there are two possible cases for the existence of the Ramsey steady state: (a) μ_t itself converges to a finite positive limit, and (b) μ_t diverges but the growth rate of μ_t, $\frac{\mu_{t+1}}{\mu_t}$, converges to a finite positive limit.

First, let us consider the case (b). There are three subcases, depending on the value of α.

1. $\alpha = 1$. $W_C(t)$ is reduced to C_t^{-1}, which is a finite constant in the steady state. This implies that the FOC (23) cannot hold since μ_t diverges in the limit. Thus, we have a contradiction with the existence of the Ramsey steady state.

2. $\alpha > 1$. From (24) with $F_L(K_t, L_t)$ being a finite constant in the steady state, we see that the divergence of μ_t implies the divergence of $-W_L(t)$ as well. According to $-W_L(t)$ given by (27), the divergence of $-W_L(t)$ cannot be due to the terms $\widehat{w}_t = w_t(1-\tau_{l,t})$, $C_t^{-\alpha}$, or Part 1 of $-W_L(t)$, because all of them are finite constants in the steady state. Its occurrence must be due to Part 2 of $-W_L(t)$ because $\lim_{t\to\infty} H_t = \infty$ or the weighted sum of $\eta_t(\theta^t)$ stochastically diverges to infinity in the limit. The same divergent force with respect to H_t or the weighted sum of $\eta_t(\theta^t)$ also causes $W_C(t)$ in (26) to explode. However, $W_C(t)$ diverges to negative infinity because $(1-\alpha) < 0$. Since μ_t diverges to positive rather than negative infinity, this leads to the violation of the FOC (23) that needs to be satisfied at the maximization of the Ramsey problem. Again, we have a contradiction with the existence of the Ramsey steady state.

3. $\alpha < 1$. Both $W_C(t)$ in (26) and $-W_L(t)$ in (27) are increasing sequences that are driven by the common force that $\lim_{t\to\infty} H_t = \infty$ or the weighted sum of $\eta_t(\theta^t)$ stochastically diverges to infinity in the limit. It is thus possible to have a steady state in which the FOCs (23)-(25) are all satisfied and μ_{t+1}/μ_t converges to a finite limit. However, depending on the value of μ_{t+1}/μ_t in the limit, we see from (25) that there is no guarantee that the Ramsey planner will implement the MGR in the steady state.

We conclude (i) if $\alpha \geq 1$, there is no possibility for the existence of a Ramsey steady state in which the growth rate of μ_t converges to a finite limit, and (ii) if $\alpha < 1$, there is the possibility for the existence of a Ramsey steady state in which the growth rate of μ_t converges to a finite limit, but the planner may not implement the MGR.

Next, we address the case (a) where μ_t itself converges to a finite limit. Because μ_t itself converges to a finite limit, the MGR holds in this Ramsey steady state by the FOC (25). A finite limit of μ_t also implies that $W_L(t)$ must converge in the limit according to the FOC (24). As noted in Section 4.3, the term $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t)$ in $-W_L(t)$ of (27) represents the shadow price of distorting net aggregate savings via changing labor income. Given that the government policy tools are distortive and hence that the first-best aggregate allocation is infeasible to the Ramsey planner, the constraints associated with $\eta_t(\theta^t)$ of $W(t)$ defined in (18) will bite in the Ramsey steady state. Consequently, analogous to $\chi^P > 0$ at the optimum in the RA model, we will have $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} > 0$ in the Ramsey steady state in the HAIM economy. With this positive shadow price, by (27), one of the necessary conditions for the convergence of $W_L(t)$ is that H_t must be finite in the limit. A finite limit of H_t then leads to $R = 1/\beta$ by equation (21) in this Ramsey steady state. Finally, from (16) and (4), we obtain the steady-state Euler equation in competitive equilibrium:

$$1 = \beta \left[(1 - \tau_{k,t+1})(F_K - \delta) + 1 \right],$$

which together with the MGR leads to a zero capital tax in this Ramsey steady state.