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—Why Aren't They More Effective?

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Evaluating Unconventional Monetary Policies

—Why Aren’t They More Effective?*

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Abstract

We use a general equilibrium finance model that features explicit government purchases of private debts to shed light on some of the principal working mechanisms of the Federal Reserve’s large-scale asset purchases (LSAP) and their macroeconomic effects. Our model predicts that unless private asset purchases are highly persistent and extremely large (on the order of more than 50% of annual GDP), money injections through LSAP cannot effectively boost aggregate output and employment even if inflation is fully anchored and the real interest rate significantly reduced. Our framework also sheds light on some long-standing financial puzzles and monetary policy questions facing central banks around the world, such as (i) the flight to liquidity under a credit crunch and debt crisis, (ii) the liquidity trap, and (iii) the low inflation puzzle under quantitative easing.

Keywords: Liquidity Trap, Yield Curve, Flight to Quality, Large-Scale Asset Purchases, Quantitative Easing, Qualitative Easing, Optimal Exit Strategies.

JEL codes: E3, E4, E5.

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1 Introduction

Our approach—which could be described as "credit easing"—resembles quantitative easing in one respect: It involves an expansion of the central bank's balance sheet. However, in a pure QE regime, the focus of policy is the quantity of bank reserves, which are liabilities of the central bank; the composition of loans and securities on the asset side of the central bank's balance sheet is incidental....In contrast, the Federal Reserve's credit easing approach focuses on the mix of loans and securities that it holds and on how this composition of assets affects credit conditions for households and businesses.

Federal Reserve Chairman Ben S. Bernanke (January 13, 2009)

The main focus of this paper is on credit easing (CE) and its macroeconomic impact on employment and output. CE involves increasing the money supply by the purchase not of government bonds, but of private assets such as corporate bonds and residential mortgage-backed securities. When undertaking CE, the Federal Reserve increases the money supply not by buying government debt, but instead by buying private debt.¹

Many believe that CE has had significant effects on lowering the real interest rate of private credits and raising asset prices,² but its impact on the real economy is far from obvious. Figure 1 shows the behavior of corporate interest rates (top panel), industrial bond yields (middle panel), and headline inflation rates (lower panel) along with CE announcement dates. Figure 2 shows the logarithm of aggregate output along with CE announcement dates (top panel) and the postwar long-run trends in real GDP, total real consumption and business investment (lower panel). From Figure 1, it appears that given the well-anchored inflation rate (bottom panel), CE has lowered the real interest rates of corporate debt by at least 2 to 3 percentage points on average, even though it is debatable whether this decline is an endogenous response of interest rates to financial shocks or a result of CE. However, Figure 2 indicates that aggregate output, consumption, and investment still remain about 10% below their respective long-run trends, as if the U.S. economy has suffered permanent losses from the 2007 financial crisis and CE has had zero effect on closing the gap. By Okun's law, a permanent 10% drop in gross domestic product

¹For our purpose, this paper treats CE, quantitative easing (QE), and large-scale asset purchases (LSAP) as synonymous.

²See, e.g., Gagnon, Raskin, Remache, and Sack (2010), Brian Krishnamurthy and Vissing-Jorgensen (2011); and Chung, Laforte, Reifschneider, and Williams (2011)

(GDP) implies that the unemployment rate must have remained 3 to 4 percentage points above its natural rate. Therefore, despite the seemingly large impact of LSAP on financial variables, the picture on the real side of the economy looks gloomy, making it difficult to claim victory for CE despite 5 years in the making.³

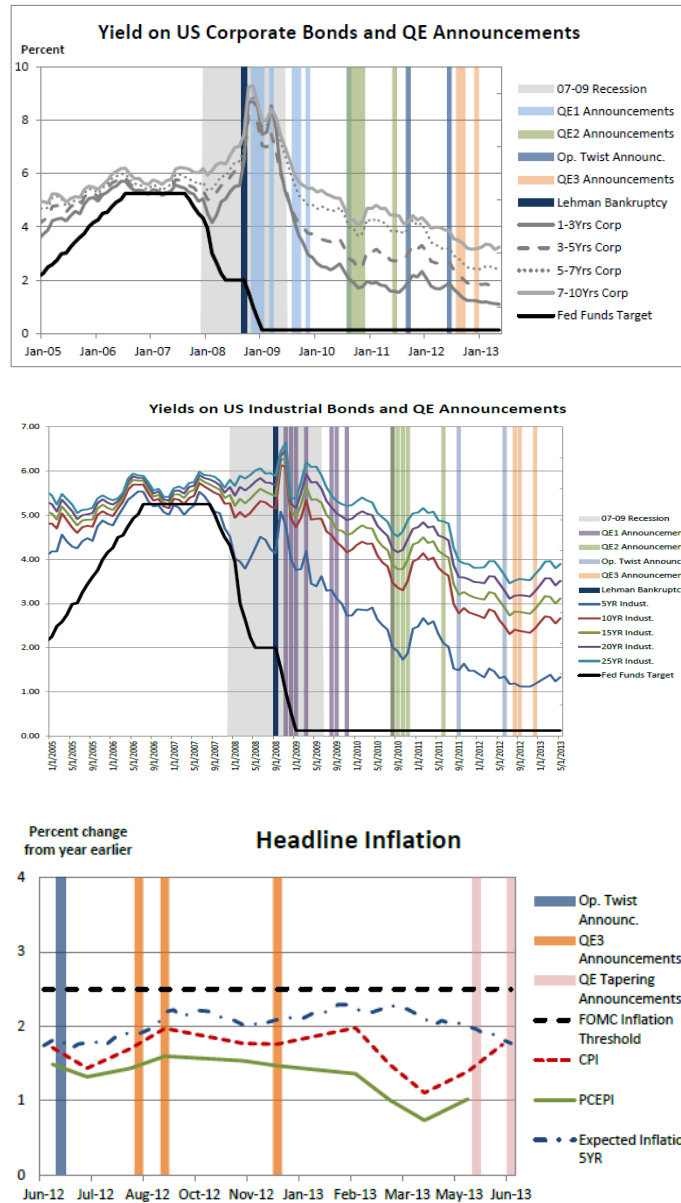


Figure 1. U.S. Interest Rates and Inflation Rates.

The sharp contrast between the impact of LSAP on financial variables and the impact on real

³The situation in Europe is similar: QE has had significant effects on lowering the real interest rates, but its impact on eliminating the output gap is far from obvious.

variables is striking and demands serious explanations. It is no wonder that questions concerning the effects of CE are still generating heated debates among economists and policymakers 5 years after its inception. Would the U.S. economy have suffered even greater losses had CE never been implemented? Will the Fed ever be able to bring the economy back to its full-employment level by continuing the current pace of asset purchases (or even more aggressively)? How long should CE continue, and when is it optimal to exit? Would tapering and unwinding of CE adversely affect the economy and undo the gains (if any) so far?

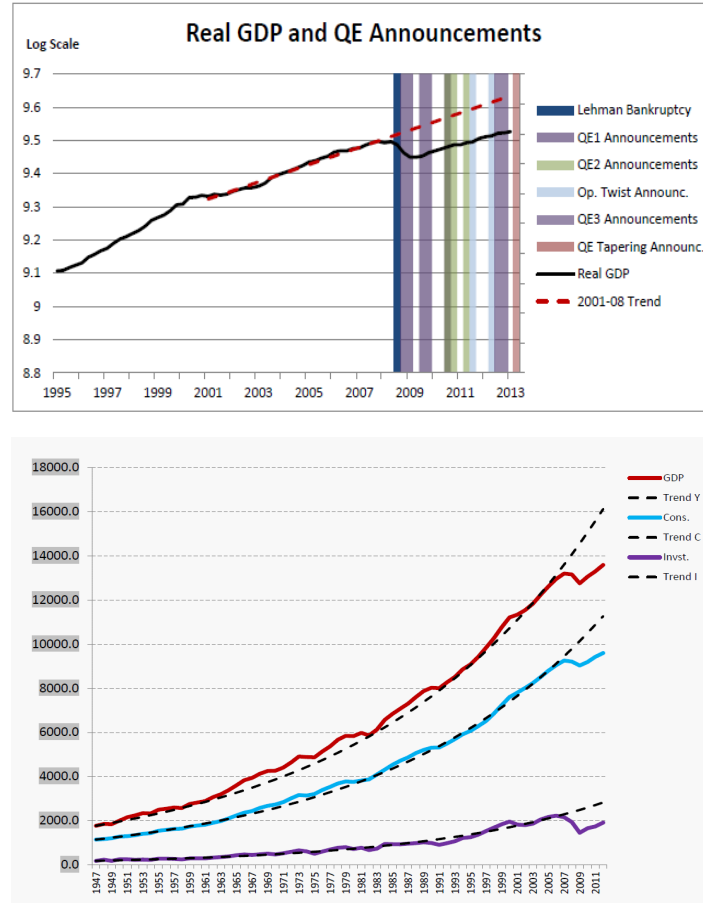


Figure 2. U.S. Real Output, Consumption, and Investment
along with their long-run trends (dashed lines).

To answer these questions and more importantly, to explain the apparent significant impact of CE on financial variables but insignificant impact on aggregate output, a model that can explicitly mimic the Federal Reserve's LSAP in a setting calibrated to the U.S. economy and the financial crisis is needed. We offer such a model and use it to illustrate how CE works in reducing real interest rates and raising asset prices, and how such effects in the financial

markets get transmitted into the real economy.

To anticipate our major findings, we find that given the best scenario with fully anchored inflation, CE can lower the real interest rate and raise asset prices significantly, which in turn can relax borrowing constraints. However, CE is unable to stimulate aggregate spending and increase employment unless its scale is (i) extraordinarily large, (ii) highly persistent, and (iii) operating in a relatively high inflation-targeting environment—due to the liquidity-trap risk in a low-inflation environment. In particular, when calibrated to match the broad features of the U.S. economy and the magnitude of the recent financial crisis, our model predicts that total private asset purchases should exceed at least 50% of annual GDP (or \$7 trillion) and persist for much longer to have a positive and significant impact on aggregate output and employment—thanks to the highly nonlinear nature of the general equilibrium effects of CE. For example, suppose the Federal Reserve’s target inflation rate is 12% per year (Table 1), then the steady-state output level can be increased by 0.04% if the steady-state asset purchase-to-GDP ratio reaches 50%. However, the increase in output would be 2.4% if that ratio reaches 100%, and the increase becomes 9.2% if the ratio is close to 140%. The macroeconomic effects are essentially zero if asset purchases are either below 50% of GDP or highly transitory. In addition, if the target inflation rate is 2% per year, then even with permanent asset purchases at the scale of nearly 140% of GDP, steady-state output can be raised by at most 6.7% instead of 9.2% because of the liquidity trap.

Table 1. Predicted Effects of LSAP.

$\frac{B}{Y}$	50%	100%	137%
$\Delta Y (\pi^* = 12\%)$	0.04%	2.4%	9.2%
$\Delta Y (\pi = 2\%)$	0.04%	2.3%	6.7%

Currently the U.S. real GDP is about 10% below its long-run trend (see Figure 2) and total asset purchases stand at \$3.7 trillion (or less than 25% of GDP). Our model predicts that this level of asset purchases (even if permanent) would have little effect on aggregate output and employment even though it could reduce the real interest rate significantly by 2 to 3 percentage points. These predictions are consistent with the empirical evidence. Thus, based on our model the Federal Reserve’s total asset purchases must be more than quadrupled and remain active for several more years if the Fed intends to eliminate the 10% output gap caused by the financial crisis.

Our model can also shed light on questions regarding optimal exit strategies. In a companion paper (Wen, 2013), the model is used to show that fully unwinding LSAP does not undo the positive effects of LSAP if (i) the timing of exit is sufficiently postponed, (ii) the event is

fully unexpected, and (iii) the exit is sufficiently gradual, provided that inflation remains fully anchored.

The model is a fairly standard off-the-shelf model based on the recent macro-finance literature.⁴ A key feature of this class of models is that an endogenously determined distribution of heterogeneous creditors/debtors (instead of households' time preference per se) pins down the real interest rate and asset prices in each asset market through the demand and supply of public/private debt. LSAP affect the real economy through their impact on the allocations (distributions) of credits/debts in the asset markets. Depending on which asset market LSAP intervene directly and how asset markets are connected to each other and the rest of the economy, the quantitative effects of LSAP may differ, but the main findings emerging from the model remain valid. Two key assumptions in the model dictate our findings:

- Debtors are relatively more productive than creditors—in other words, more productive agents opt to issue debt and less productive agents opt to lend.
- Financial markets are incomplete—that is, agents face uninsurable idiosyncratic shocks, are borrowing constrained, and have unequal access (costs of entry) to all available financial markets.

Under these fairly standard assumptions, the following properties emerge naturally from the model: (i) The demand for liquid assets (such as money and government bonds) is strong despite the low returns of these instruments relative to capital investment. (ii) Assets with different degrees of liquidity command different premiums, and more-liquid assets pay a lower equilibrium rate of return than less-liquid assets;⁵ (iii) When asset scarcity (e.g., caused by the Federal Reserve's open market operations or QE) drives down yields, "flight to liquidity" causes funds to be reallocated away from scarcer assets toward relatively more abundant assets (such as money or government bonds), which implies lower interest rates and higher asset prices across segmented financial markets and potentially lower aggregate price level or short-run inflation rate. (iv) When the cost of borrowing is reduced, marginal creditors in each financial market self-select to become debtors—raising the quantity aggregate debt but lowering the average quality (efficiency) of loans.

With these core properties, it is clear that CE affects aggregate output mainly through the impact on the distribution of credit/debt in each financial market. That is, CE works by

⁴See, e.g., Azariadis and Kaas (2013), Bernanke and Gertler (1989), Buera and Moll (2013), Eggertson and Krugman (2012), Guerrieri and Lorenzoni (2010), Kiyotaki and Moore (2008, 2012), and Wang and Wen (2009, 2012, 2013).

⁵The liquidity of an asset is defined as the scope of its acceptance (or re-salability) as a store of value.

pushing more creditors to become debtors, which in turn increases the total quantity of loans but decreases the average efficiency of loans.⁶ When economic activities depend not only on the extent and scope of credit/debt but also on the quality of loans, such a trade-off between quantity and quality implies that (i) aggregate output and employment are *insensitive* to small-scale asset purchases even with relatively large changes in the real interest rate and asset prices, and (ii) the positive *quantitative* effect of LSAP on credit/debt and aggregate investment may become dominating their adverse *qualitative* effect *only if* asset purchases are sufficiently large and highly persistent relative to the magnitudes of financial shocks—so that the long-term productivity of labor can increase sufficiently to induce labor demand and attract labor supply (in the absence of technology changes).⁷ This lack of increase in labor productivity explains the weak demand and supply of labor in the model (as well as in the data) under CE.

To highlight these general equilibrium effects of CE, we start with a real model in which long-run inflation is fully anchored and money serves purely as a store of value for portfolio investors (asset holders). In the real model, all transactions and payments are conducted by goods. The advantage of using a real model is that the mechanisms behind the real effects of LSAP on credit allocation can be seen in their bare bones without being masked by money or specific monetary models. The framework can be easily extended to specific monetary models in which money plays a more active role in allocating resources (such as the New Monetarism model or the New Keynesian model). In the real model we do not need to distinguish monetary authority from fiscal authority—that is, we assume there is a consolidated government (as in Williamson, 2012) that can purchase private assets using revenues raised from lump-sum taxes or through sales of public debt. As noted by Sims (2013), all monetary policies must involve fiscal policies to be effective.⁸ We also abstain from maturity and risk considerations involving longer-term debt.

Our approach also sheds light on some monetary-policy-related puzzles and their intimate relationships, such as (i) the flight to liquidity under credit crunch and debt crisis, (ii) the liquidity trap, and (iii) the paradox of inflation. Our model also clearly explains (i) why the inflation rate can be positive in a liquidity trap, (ii) how the inflation rate can remain low despite QE—the paradox of inflation, (iii) when there are differential effects of *qualitative easing* (—

⁶In general equilibrium, agents are either creditors or debtors and they sum to a constant population. Hence, more debtors naturally implies fewer creditors unless there are unmatched creditors and debtors in the credit markets. Such "unemployed" credit resources are not modeled in this paper since this approach requires a radically different framework from standard macro-finance models.

⁷This means that the aggregate capital stock must increase significantly to raise the marginal product of labor.

⁸The original quote reads as follows: "Monetary policy actions, to be effective, must induce a fiscal policy response" (Sims, 2013, p. 564).

changing the central bank's composition of balance sheet without expanding it) and *quantitative easing* (—expanding the central bank's balance sheet through purchases of private debt), and (iv) what the appropriate exit strategies are.

Despite the wide practice of QE around the world, how exactly unconventional monetary policies work and what their ultimate impact is on the real economy still remain largely unclear and highly controversial among policymakers and academics.⁹ To the best of our knowledge, the literature that tempts to provide a general equilibrium framework for the explicit analysis and evaluation of QE remains surprisingly thin. Eggertson and Krugman (2012) use a model similar to ours to study the debt crisis and the liquidity trap, but they do not focus on unconventional monetary policies. Farmer (2012) studies the effectiveness of qualitative easing in an overlapping-generations model with sunspots, and shows that a change in the asset composition of the central bank's balance sheet will change equilibrium asset prices when unborn agents are unable to participate in the asset market. Chen, Curdia, and Ferrero (2012) study the impact of LSAP in a New Keynesian model with segmented financial markets. Their focus is on the Federal Reserve's purchases of long-term government debt (instead of private debt) and the likely impact on households' (instead of firms') saving behavior. They reach the conclusion that the effects of LSAP on macroeconomic variables are likely to be moderate. Gertler and Karadi (2010) study LSAP in a DSGE model with agency costs. They argue that LSAP are effective in boosting aggregate output and employment under the assumption that the central bank can act as a financial intermediary that is better able than private banks to channel credit to firms during a crisis. In other words, Gertler and Karadi interpret CE as expanding central bank credit intermediation to offset a disruption of private financial intermediation, and argue that the primary advantage of the central bank over private intermediaries is its ability to elastically obtain funds by issuing riskless government debt.

Williamson (2012) provides a micro-founded monetary model (a la Lagos and Wright, 2005) with both public and private liquidity (debt) to study the effects of open-market operations and QE. His model features an explicit treatment of the Federal Reserve's asset purchases that mimic the key entries on real-world central bank balance sheets. In the model, agents face uncertainty in the demand for money and public/private debt. Government injection of money through asset purchases may lower the nominal interest rate and change the distribution of debt

⁹The president and CEO of the Federal Reserve Bank of Minneapolis, Narayana Kocherlakota, advocated more aggressive and longer-duration QE to reduce the U.S. unemployment rate (consistent with our model's recommendations), but admitted in an interview that "[it is] actually much more sophisticated to see how [QE] works, in terms of the economic mechanisms involved....The empirical work that I mentioned has validated that there does seem to be an impact on yields. What that means in terms of the impact on economic activity, I'm still sorting through, to be honest. As of now, I would say that I think quantitative easing works in the right direction, but gauging the actual magnitude of its impact remains challenging." (Kocherlakota, 2013).

through financial intermediation. However, Williamson shows that private asset purchases by the central bank are either irrelevant or they reallocate credit and redistribute wealth, with no obvious net benefits.

The physical structure of our model is directly based on recent works by Wang and Wen (2009, 2012, 2013), which in turn follow Kiyotaki and Moore (2008, 2012). However, our approach in spirit (methodology) is most closely related to Williamson (2012). The value added of our work to the existing literature is the provision of an explicit and straightforward general equilibrium corporate-finance approach for a quantitative evaluation of the real impact of CE, alternative to the New Keynesian household-based model of Chen, Curdia, and Ferrero (2012), the financial-accelerator model of Gertler and Karadi (2010) and the money-search model of Williamson (2012). Since our heterogeneous-agent finance model nests the standard representative-agent real business cycle model as a special (limiting) case and yet remains analytically tractable, it can be readily applied to calibration or econometric analyses under various aggregate shocks. It can also be easily embedded into the other frameworks for alternative types of policy analysis and business-cycle studies.

2 The Benchmark Model

2.1 Outline of the Model

The key actors in our model are firms that make production and investment decisions in an uncertain world with an infinite horizon. There are two types of firms—"large" and "small" firms—indexed by $\ell = \{L, S\}$ respectively. Large firms can participate in a private credit/debt market to lend and borrow from each other. Small firms cannot borrow at all and must rely on retained earnings to self-finance investment projects. Because of default risks and limited contract enforceability, private debt issued by large firms is collateralized by the market value of firms' capital stock. All firms, regardless of size, face idiosyncratic uncertainty in the rate of return on investment projects, modeled specifically as an idiosyncratic shock to the marginal efficiency of firm-level investment (to be specified below).¹⁰ When investment is irreversible or reselling used capital is costly, inaction (not investing) in some periods is optimal because waiting has a positive option value.

Three types of financial assets serve as a store of value: money with a real rate of return $\frac{1}{1+\pi}$ (where π is the inflation rate), public debt (government bonds) with a real rate of return $1 + r_t^g$, and private debt (corporate bonds) with a real rate of return $1 + r_t^c$. For simplicity, both

¹⁰The idiosyncratic shock is meant to capture the scarcity of good investment opportunities as in Kiyotaki and Moore (2008) and the lumpiness of firm-level investment documented by Cooper and Haltiwanger (2006).

government and corporate bonds are modeled as one-period debt.¹¹

With borrowing constraints, idiosyncratic uncertainty, irreversible investment, and the availability of financial assets as a store of value, firms that draw low-return projects opt to be inactive—they do not undertake fixed-capital investment in the current period. Instead, they opt to save by investing in financial assets. This generates the fundamental demand for liquidity (public and private debt, including money) in our model.

While small firms cannot borrow at all, large firms can borrow from the financial market by issuing private debt. In particular, large firms with good investment opportunities opt to issue (supply) debt, whereas those temporarily without good investment opportunities opt to purchase debt as a store of value if its rate of return dominates other assets and can be liquidated with low costs when better investment opportunities arrive. Public and private debt may be subject to systemic default risks, which are modeled as aggregate shocks to the probability of default. The real interest rates of private and public debt are determined by the demand and supply in each asset market—thus can also be influenced by monetary policies through the Federal Reserve’s asset purchase/sales programs. Hence, as in Bernanke and Gertler (1989) and Kiyotaki and Moore (1997, 2008), our model captures a general flavor of the macro-finance literature that private debtors are the productive (or low-cost) agents and private creditors are the less productive (or high-cost) agents. This endogenous self-selection into creditors/debtors and the changes in their distribution caused by monetary policies are key to understanding our results.

2.2 Arbitrage Conditions

For simplicity, we assume a representative household that faces no borrowing constraints. Because firms’ borrowing constraints may be binding in both the present and the future, all financial assets in our model command a positive liquidity premium in equilibrium. Thus, the equilibrium interest rates of private and public debt are all bounded above by the time preference of households. Since by assumption large firms can hold both public and private debt and small firms cannot participate in the corporate bonds market, private debt are not as liquid (or widely accepted as a store of value) as public debt. This asymmetry in liquidity (caused by market segmentation) gives rise to a wedge between the liquidity premium of public and private debt, leading to a spread between the interest rates on more-liquid and less-liquid financial assets.

Since by default money can always serve as a store of value in the model, all interest rates are bounded below by the real rate of return on money—the inverse of the inflation rate. So

¹¹Introducing longer-term debts does not change our basic insights and quantitative results.

our setup immediately implies that the equilibrium rates of return on different financial assets must satisfy the following arbitrage conditions (as in the New Monetarism model of Williamson, 2012):¹²

$$\frac{1}{1+\pi} \leq (1+r^g) \leq (1+r^c) \leq \frac{1}{\beta}. \quad (1)$$

Equation (1) consists of a chain of inequalities. Depending on whether or not a particular link in the chain holds with strict equality, different equilibrium regimes arise and the effect of CE differs across them. We use these arbitrage conditions and equilibrium regimes to explain the puzzling phenomena of (i) the flight to liquidity during a financial crisis, (ii) the liquidity trap, (iii) the limitations of conventional monetary policies on the interest rates of private debt, and (iv) the detachment of inflation and money supply in low-inflation economies, among others. Specifically, we first study five equilibrium regimes and how monetary policies (i.e., government purchases/sales of private/public debt) can determine which equilibrium regime prevails and how the equilibrium conditions affect the effectiveness of CE on the aggregate economy (as in Williamson, 2012). To that end, below we detail the supply and demand sides in each asset market.

2.3 The Government

The consolidated government supplies public liquidity (including money and public debt) and balances its budget in each period. Denote the total money supply by M_t , the aggregate price level by P_t , and the inflation rate by $1+\pi_t = \frac{P_{t+1}}{P_t}$; then the government budget constraint in each period is given by

$$G_t + \frac{1}{1+r_t^c} B_{t+1}^c + B_t^g = B_t^c + \frac{B_{t+1}^g}{1+r_t^g} + \frac{(M_{t+1} - M_t)}{P_t} + T_t, \quad (2)$$

where the left-hand side (LHS) is total government expenditures and the right-hand side (RHS) is total government revenues. Government outlays include government spending G_t , new purchases of private debt B_{t+1}^c at price $\frac{1}{1+r_t^c}$, and repayment of public debt B_t^g at price 1 when the debt mature. Total government revenues include debt repayment B_t^c from the private sector, new issues of public debt B_{t+1}^g at price $\frac{1}{1+r_t^g}$, real seigniorage income $\frac{(M_{t+1}-M_t)}{P_t}$, and lump-sum

¹²As will become clear shortly, once we introduce default risks, the upper bound on the real interest rates can be larger than the inverse of time preference $\frac{1}{\beta}$. But, allowing for a larger upper bound under default risks has no effect on our analysis in this section, we therefore defer the analysis of default risk to the next section.

taxes T_t .

Qualitative easing (in contrast to quantitative easing) in our model can then be defined as a simultaneous change in B_{t+1}^c and B_{t+1}^g , holding the other components $\left\{G_t, \frac{M_t}{P_t}, T_t\right\}$ in equation (2) constant. That is, qualitative easing can be specified as changes in only the composition of public and private debt on the government balance sheet (holding total asset purchases and long-run inflation constant) such that

$$\frac{B_{t+1}^c}{1+r_t^c} - B_t^c = \frac{B_{t+1}^g}{1+r_t^g} - B_t^g. \quad (3)$$

2.4 Large Firms' Problem

Large firms affect both the supply side and the demand side of the private debt market; they may also affect the demand side of the public debt and money markets when the real rates of return in these assets are high enough. A large firm i 's objective is to maximize the present value of discounted future dividends,

$$V_t(i) = \max E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} d_{t+\tau}(i), \quad (4)$$

where $d_t(i)$ is firm i 's dividend in period t and Λ_t is the representative household's marginal utility, which firms take as given. The production technology of all firms in this paper is given by the constant returns to scale (CRS) function

$$y_t(i) = A_t k_t(i)^{\alpha} n_t(i)^{1-\alpha}, \quad (5)$$

where A represents aggregate technology level, and n and k are firm-level employment and capital, respectively. Firms accumulate their own capital stock through the law of motion,

$$k_{t+1}(i) = (1 - \delta)k_t(i) + \varepsilon_t(i) i_t(i), \quad (6)$$

where investment is irreversible:

$$i_t(i) \geq 0, \quad (7)$$

and $\varepsilon_t(i)$ denotes an i.i.d. idiosyncratic shock to the marginal efficiency of investment. The idiosyncratic shock $\varepsilon_t(i)$ has the cumulative distribution function $F(\varepsilon)$.¹³ In each period t , a firm needs to pay wages $W_t n_t(i)$ and decides whether to invest in fixed capital or distribute dividends $d(i)$ to households. Firms' investment is financed by internal cash flow and external funds. Firms raise external funds by issuing one-period debt, $b_{t+1}(i)$, which pays the competitive market interest rate r_t^c .¹⁴ Note that $b_{t+1}(i)$ can be negative— $b_{t+1}(i) < 0$ when a firm opts to hold bonds issued by other firms.

A large firm's dividend in period t is then given by

$$\begin{aligned} d_t(i) = & y_t(i) - i_t(i) - W_t n_t(i) \\ & + \left[\frac{b_{t+1}^c(i)}{1 + r_t^c} - b_t^c(i) \right] - \left[\frac{b_{t+1}^g(i)}{1 + r_t^g} - b_t^g(i) \right] - \left[\frac{M_{t+1}(i)}{P_t} - \frac{M_t(i)}{P_t} \right]. \end{aligned} \quad (8)$$

Firms cannot short-sell public debt and money,

$$b_{t+1}^g(i) \geq 0 \quad (9)$$

$$M_{t+1}(i) \geq 0, \quad (10)$$

nor can they pay negative dividends:

$$d_t(i) \geq 0. \quad (11)$$

Constraint (11) is the same as saying that fixed investment and financial investment must be financed entirely by internal cash flow ($y_t(i) - W_t n_t(i)$) and external debt net of debt repayment ($\frac{b_{t+1}^c(i)}{1 + r_t^c} - b_t^c(i)$).

Private debt are subject to collateral constraints, as in the models of Kiyotaki and Moore (1997) and Wang and Wen (2009). That is, firm i is allowed to pledge a fraction $\theta \in (0, 1]$ of its fixed capital stock $k_t(i)$ at the beginning of period t as collateral. In general, the parameter θ represents the extent of financial market imperfections, or the tightness of the financial market. At the end of period t , the pledged collateral is priced by the market value of newly installed capital, so the market value of collateral is simply Tobin's q , denoted by q_t , which is equivalent

¹³As in Wang and Wen (2013), the model is tractable with closed-form solutions because (i) the idiosyncratic shock is i.i.d. and (ii) the production technology is CRS.

¹⁴We focus on debt financing because it accounts for 75% to 100% of the total external funds of corporations. See Wang and Wen (2009).

to the expected value of a firm that owns collateralizable capital stock $\theta k_t(i)$. The borrowing constraint is thus given by

$$b_{t+1}(i) \leq \theta q_t k_t(i), \quad (12)$$

which specifies that any new debt issued cannot exceed the collateral value (q_t) of a firm with the pledged capital stock $\theta k_t(i)$. The parameter $\theta \geq 0$ measures the changes in the tightness of the financial market. When $\theta = 0$ for all t , the model is identical to one that prohibits external financing.¹⁵

2.5 Small Firms' Problem

Small firms are also owned by the household. For simplicity, we assume that small firms are not publicly traded in the equity market. Since small firms cannot borrow or issue debt, their actions influence only the demand side of the public debt markets and the money market. Each small firm j maximizes the present value of future profits $d_t(j)$ by solving

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} d_t(j) \quad (13)$$

subject to

$$d_t(j) \equiv R_t k_t(j) - I_t(j) - \left[\frac{b_{t+1}^g(j)}{1 + r_t^g} - b_t^g(j) \right] - \left[\frac{M_{t+1}(j)}{P_t} - \frac{M_t(j)}{P_t} \right] \geq 0 \quad (14)$$

$$k_{t+1}(j) \leq (1 - \delta) k_t(j) + \xi_t(j) I_t(j) \quad (15)$$

$$I_t(j) \geq 0 \quad (16)$$

$$b_{t+1}^g(j) \geq 0 \quad (17)$$

$$M_{t+1}(j) \geq 0. \quad (18)$$

The first constraint implies that small firms can finance investment projects only through internal cash flows $R_t k_t(j)$ and past precautionary savings $b_t^g(j) + \frac{M_t(j)}{P_t}$ on public liquidity or money. The other constraints are identical to those facing large firms. The crucial difference between a small firm and a large firm is that the latter can borrow from the financial market

¹⁵If firms cannot issue bonds, then the corporate bond market would not exist. Hence, large firms would hold government bonds as a store of value. In this case, large firms are identical to small firms.

by issuing private debt whereas the former cannot.

2.6 The Household Problem

There is a representative household composed of two workers; one works for large firms and another works for small firms. Outputs produced by small firms and large firms are not perfect substitutes in the household utility. We assume that the household is subject to the cash-in-advance (CIA) constraint for consumption purchases, $C_t = C_t^L + C_t^S \leq \frac{M_t^h}{P_t}$, where C^ℓ denotes consumption of goods produced by ℓ -type firms with $\ell = \{\text{large, small}\}$. Since money may also be held by firms as a store of value, the CIA constraint implies that the household is a residual money holder in the economy. The representative household chooses nominal money demand M_{t+1} , consumption plans C_t^L and C_t^S , labor supply schedules N_t^L and N_t^S , and share holdings $s_{t+1}(i)$ of large firm i to solve

$$\max \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t^L + \log C_t^S - \frac{(N_t^L)^{1+\gamma}}{1+\gamma} - \frac{(N_t^S)^{1+\gamma}}{1+\gamma} \right\} \quad (19)$$

subject to the constraints,

$$C_t^L + C_t^S \leq \frac{M_t^h}{P_t} \quad (20)$$

$$\begin{aligned} & C_t^L + C_t^S + \frac{M_{t+1}^h}{P_t} + \int_{i=0}^1 s_{t+1}(i) [V_t(i) - d_t(i)] di \\ & \leq \frac{M_t^h}{P_t} + W_t^L N_t^L + W_t^S N_t^S + \int_{i=0}^1 s_t(i) V_t(i) di + \int_{j=0}^1 d_t(j) dj - T_t, \end{aligned} \quad (21)$$

where T_t denotes lump-sum income taxes, $d_t(j)$ is small firm j 's profit income, $s_t(i) \in [0, 1]$ is the household's holding of large firm i 's equity shares, and $V_t(i)$ is the value (stock price) of the firm i .¹⁶ Denoting Λ_t as the Lagrangian multiplier of budget constraint (21), the first-order condition for $s_{t+1}(i)$ is given by

$$V_t(i) = d_t(i) + E_t \beta \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(i). \quad (22)$$

¹⁶We will show that the household has no incentive to buy bonds issued by firms in equilibrium.

Equation (22) implies that the stock price $V_t(i)$ of a large firm i is determined by the present value of this firm's discounted future dividends, as in equation (4).

3 Competitive Equilibrium

Given initial real money balance M_0^h held by the household and the initial distribution of capital stocks $\{k_0(i), k_0(j)\}_{i,j \in [0,1]}$ and asset holdings $\{b_0^c(i), b_0^g(i), m_0(i), b_0^g(j), M_0(j)\}_{i,j \in [0,1]}$ for all firms, a competitive equilibrium consists of the sequences and distributions of quantities $\{C_t, N_t, M_t^h\}_{t=0}^\infty$, $\{i_t(\ell), n_t(\ell), y_t(\ell), k_{t+1}(\ell), b_{t+1}^c(\ell), b_{t+1}^g(\ell), M_{t+1}(\ell)\}_{t \geq 0}$ for $\ell = [i, j] \in [0, 1]^2$, and the sequences of prices $\{W_t, P_t, V_t(i), r_t^g, r_t^c\}_{t=0}^\infty$ such that

(i) Given prices $\{W_t, P_t, r_t^c, r_t^g\}_{t \geq 0}$, the sequences $\{i_t(i), n_t(i), y_t(i), k_{t+1}(i), b_{t+1}^c(i), b_{t+1}^g(i), M_{t+1}(i)\}_{t \geq 0}$ solve all large firm i 's problem (4) subject to constraints (5)-(11), and the sequences $\{i_t(j), n_t(j), y_t(j), k_{t+1}(j), b_{t+1}^g(j), M_{t+1}(j)\}_{t \geq 0}$ solve all small firm j 's problem (13) subject to constraints (14)-(18).

(ii) Given prices $\{W_t, P_t, V_t(i)\}_{t \geq 0}$, the sequences $\{C_t, N_t, M_t^h, s_{t+1}(i)\}_{t \geq 0}$ maximize the household's lifetime utility (19) subject to its budget constraint (21) and the CIA constraint (20).

(iii) The arbitrage conditions (1) hold.

(iv) All markets clear:

$$\int b_{t+1}^c(i) di = B_{t+1}^c \quad (23)$$

$$\int_{j \in S} b_{t+1}^g(j) dj + \int_{i \in L} b_{t+1}^g(i) di = B_{t+1}^g \quad (24)$$

$$s_{t+1}(i) = 1 \quad \text{for all } i \in [0, 1] \quad (25)$$

$$N_t = \int_{i \in L} n_t(i) di + \int_{j \in S} n_t(j) dj \quad (26)$$

$$C_t + \int_{i \in L} i_t(i) di + \int_{j \in S} i_t(j) dj + G_t = \int_{i \in L} y_t(i) di + \int_{j \in S} y_t(j) dj \quad (27)$$

$$M_t^h + \int_{i \in L} M_t(i) di + \int_{j \in S} M_t(j) dj = \bar{M}_t. \quad (28)$$

Equation (23) states that the net supply of private bonds issued by all firms equals the total purchase of private bonds by the government. Note that if $B_{t+1}^c = 0$, then the government's

holding of private debt is zero and all bonds issued by large firms are circulated only among themselves with zero net supply. Equation (24) states that the net supply of public bonds issued by the government is held by both large firms and small firms. Note that when $r_t^g < r_t^c$, only small firms hold public debt and $\int_{i \in L} b_{t+1}^g(i) di = 0$. For simplicity, we assume that the government does not hold its own debt on its balance sheet. Any public debt not held by firms are remitted to the household as negative lump-sum taxes. Equation (25) is the market-clearing condition for equities, equation (26) is that for labor, equation (27) is that for goods, and equation (28) is that for money.

Although our definition of the competitive equilibrium holds for all possible cases (monetary regimes) specified in the arbitrage conditions (1), below we consider firms' decision rules under the equilibrium conditions with strict inequalities: $\frac{1}{1+\pi} < 1 + r^g < 1 + r^c$. We defer discussions for other cases with strict equalities to Sections 3.4 and 4.2.2.¹⁷

3.1 Large Firms' Decision Rules

Under CRS a firm's labor demand is proportional to its capital stock. Hence, a firm's net cash flow (revenue minus wage costs) is also a linear function of its capital stock,

$$y_t(i) - W_t^L n_t(i) = \alpha A_t^L \left(\frac{(1-\alpha) A_t^L}{w_t} \right)^{\frac{1-\alpha}{\alpha}} k_t(i) \equiv R(W_t^L, A_t^L) k_t(i), \quad (29)$$

where R_t depends only on the aggregate state. With this notation for R_t , we have:

Proposition 1 *The decision rule for investment is characterized by an optimal cutoff ε_t^* such that the firm undertakes fixed investment if and only if $\varepsilon_t(i) \geq \varepsilon_t^*$ and holds a positive amount of private debt only if $\varepsilon_t(j) < \varepsilon_t^*$:*

$$i_t(i) = \begin{cases} \left[R_t^L + \frac{\theta_t q_t}{1+r_t^c} \right] k_t(i) - b_t^c(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \quad (30)$$

$$b_{t+1}^c(i) = \begin{cases} \theta_t q_t k_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ - & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \quad (31)$$

¹⁷With strict inequalities, large firms hold only private debts and small firms hold only government debts.

where the aggregate equity price $q_t = \frac{1}{\varepsilon_t^*}$, and the cutoff ε_t^* is independent of any individual firm's history and is a sufficient statistic for characterizing the distribution of large firms' capital investment and asset holdings.¹⁸

Proof. See Appendix 1. ■

Proposition 2 *The equilibrium interest rate of private debt satisfies the following relation,*

$$\frac{1}{1 + r_t^c} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q(\varepsilon_{t+1}^*), \quad (32)$$

where $Q(\varepsilon_t^*) \equiv \int_{\varepsilon(i) < \varepsilon_t^*} d\mathbf{F}(\varepsilon) + \int_{\varepsilon(i) \geq \varepsilon_t^*} \frac{\varepsilon_t(i)}{\varepsilon_t^*} d\mathbf{F}(\varepsilon) \geq 1$ is the liquidity premium of corporate bonds.

Proof. See Appendix 2. ■

Because of borrowing constraints and irreversible investment, cash flows have positive option values. The function $Q(\varepsilon_t^*)$ measures the option value (or liquidity premium) of one unit of cash flow or goods. Given one dollar in hand, if it is not invested (because $\varepsilon_t(i) < \varepsilon_t^*$), its value is still one dollar next period. This case occurs with probability $\mathbf{F}(\varepsilon_t^*)$. If $\varepsilon_t(i) \geq \varepsilon_t^*$, one unit of cash flow can produce $\varepsilon_t(i)$ units of new capital and the cash return is $\frac{\varepsilon_t(i)}{\varepsilon_t^*}$ dollars—where $\frac{1}{\varepsilon_t^*}$ is the market value of Tobin's q (see Wang and Wen, 2009). This case occurs with probability $1 - \mathbf{F}(\varepsilon_t^*)$. Therefore, the expected value of one dollar is $Q(\varepsilon_t^*) = \mathbf{F}(\varepsilon_t^*) + \int_{\varepsilon \geq \varepsilon_t^*} \frac{\varepsilon(i)}{\varepsilon_t^*} d\mathbf{F} = 1 + \int_{\varepsilon(i) \geq \varepsilon_t^*} \frac{\varepsilon(i) - \varepsilon_t^*}{\varepsilon_t^*} d\mathbf{F}(\varepsilon) \geq 1$.

Equation (32) holds because for an inactive firm that decides to lend (purchase corporate bonds), the bond price today is $\frac{1}{1+r_t^c}$ dollars and the return tomorrow is one dollar with an option value of Q_{t+1} . So the present value of returns is $\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q_{t+1}$, which equals βQ in the steady state. Notice that $1 + r^c = \frac{1}{\beta Q} \leq \frac{1}{\beta}$ because $Q(\varepsilon^*) \geq 1$. This also implies that the representative household will not hold private bonds because (unlike firms) the household does not benefit from the liquidity premium of bonds (as in Kehoe and Levine, 2001).¹⁹

Notice that if $\varepsilon^* = \varepsilon_{\max}$, then $Q = 1$. Consequently equation (32) is reduced to a standard asset demand equation in a representative-firm model with complete financial markets. In this

¹⁸Under the nonnegative constraint on dividends, $d_t(i) \geq 0$, the level of the dividend is indeterminate at the firm level when $d_t(i) > 0$. Consequently, the level of debt purchases is determinate only at the aggregate level, but not at the firm level when $\varepsilon_t(i) < \varepsilon_t^*$.

¹⁹The situation changes if we also allow idiosyncratic shocks on the household side. In other words, with idiosyncratic income risks households may find it optimal to hold the privately issued bonds.

limiting case, only the most productive firm (with the draw $\varepsilon(i) = \varepsilon_{\max}$) undertakes fixed investment (being active) in each period and the rest opt to lend (being inactive). This would be the case if there were no borrowing constraints in our model. That is, the option value $Q > 1$ is a consequence of borrowing constraints under uninsurable idiosyncratic shocks, which generate a demand for low-yield liquid bonds (or money) as self-insurance, as well as a well-defined distribution of creditors and debtors. Notice that the distribution of creditors/debtors is fully characterized by the cutoff ε_t^* , which endogenously responds to monetary (credit) policies.

Remark 1 Since $\frac{\partial Q}{\partial \varepsilon^*} < 0$, equation (32) implies that a lower interest rate r^c in the private debt market is associated with a lower cutoff ε^* and a larger number of debtors (active firms). In addition, the market value of capital is given by $q_t = \frac{1}{\varepsilon^*}$. Hence, an increase in government purchases (demand) of private debt will reduce the real interest rate on private debt, raise the collateral value of capital q , and induce more creditors to become debtors, thus potentially increasing aggregate investment.

3.2 Small Firms' Decision Rules

Proposition 3 *A small firm's investment decision rule is also characterized by an optimal cutoff ξ_t^* such that the firm undertakes fixed investment if and only if $\xi_t(j) \geq \xi_t^*$ and holds a positive amount of financial assets as a store of value only if $\xi_t(j) < \xi_t^*$:*

$$i_t(j) = \begin{cases} R_t^S k_t(j) + b_t^g(j) & \text{if } \xi_t(j) \geq \xi_t^* \\ 0 & \text{if } \xi_t(j) < \xi_t^* \end{cases} \quad (33)$$

$$b_{t+1}^g(j) = \begin{cases} 0 & \text{if } \xi_t(j) \geq \xi_t^* \\ + & \text{if } \xi_t(j) < \xi_t^* \end{cases} \quad (34)$$

where the cutoff ξ_t^* is independent of any individual firm's history and is a sufficient statistic for characterizing the distribution of small firms' capital investment and asset holdings.²⁰

Proof. See Appendix 3. ■

²⁰Since the dividend is indeterminate at the firm level in this class of models, debt purchases can be determined only at the aggregate level, not at the firm level.

Proposition 4 *The equilibrium interest rate on public debt satisfies the following relation,*

$$\frac{1}{1+r_t^g} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \tilde{Q}(\xi_{t+1}^*), \quad (35)$$

where $\tilde{Q}(\xi_t^*) \equiv \int_{\xi(j) < \xi_t^*} d\mathbf{F}(\xi) + \int_{\xi(j) \geq \xi_t^*} \frac{\xi_t(i)}{\xi_t^*} d\mathbf{F}(\xi) \geq 1$ is the liquidity premium of government bonds.

Proof. See Appendix 4. ■

Remark 2 Since $\frac{\partial \tilde{Q}}{\partial \xi^*} < 0$, a lower interest rate r^g in the public debt market is associated with a lower cutoff ξ^* and a larger number of active small firms. Similar to the case of large firms, if $\xi^* = \xi_{\max}$, then the option value $\tilde{Q}(\xi^*) = 1$. Consequently, only the most productive small firm (with $\xi(j) = \xi_{\max}$) invests in each period and the rest opt to save by holding government bonds. This would be the case if small firms could borrow without limits by short-selling government bonds. That is, the option value (liquidity premium) $\tilde{Q} > 1$ is also the consequence of the borrowing constraint $b_{t+1}(j) \geq 0$ on small firms.

Remark 3 An increase (decrease) in the supply of government bonds will raise (lower) the real interest rate on government debt and decrease (increase) the number of active firms in the public debt market, thus potentially decreasing (increasing) aggregate investment.

3.3 Aggregation

Proposition 5 *Using our indexation $\ell = \{L, S\}$ for large (L) and small (S) firms, we can define aggregate capital stock as $K_t = K_t^L + K_t^S = \int k_t(i) di + \int k_t(j) dj$, equilibrium aggregate labor demand as $N_t = N_t^L + N_t^S = \int n_t(i) di + \int n_t(j) dj$, aggregate output as $Y_t = Y_t^L + Y_t^S = \int y_t(i) di + \int y_t(j) dj$, and aggregate investment expenditure as $I_t = I_t^L + I_t^S = \int i_t(i) di + \int i_t(j) dj$. Since the cutoffs $\{\varepsilon_t^*, \xi_t^*\}$ are sufficient statistics for characterizing the distributions of firms, for any given sequences of aggregate debt demand/supply $\{B_t^c, B_t^g\}_{t \geq 0}$ and money supply $\{\bar{M}_t\}_{t \geq 0}$, the model's equilibrium can be fully characterized as the sequences of aggregate variables $\{C_t, K_{t+1}^\ell, I_t^\ell, Y_t^\ell, N_t^\ell, R_t^\ell, \varepsilon_t^*, \xi_t^*, r_t^c, r_t^g, W_t, P_t\}_{t=0}^\infty$, which can be solved uniquely by the following system of nonlinear equations (given the path of any aggregate shocks and the initial distribution of assets):*

$$C_t^L + C_t^S = \frac{M_t^h}{P_t} \quad (36)$$

$$\Lambda_t W_t^\ell = (N_t^\ell)^\gamma \quad (37)$$

$$\frac{1}{C_t^\ell} = \Lambda_t + \Theta_t \quad (38)$$

$$\Lambda_t = \frac{\beta}{1 + \pi} E_t (\Lambda_{t+1} + \Theta_{t+1}) \quad (39)$$

$$\sum_\ell (C_t^\ell + I_t^\ell) + G_t = \sum_\ell Y_t^\ell \quad (40)$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1}^L Q(\varepsilon_{t+1}^*) + \frac{\theta_{t+1}}{\varepsilon_{t+1}^*} \frac{[Q(\varepsilon_{t+1}^*) - 1]}{1 + r_{t+1}^c} + \frac{(1 - \delta)}{\varepsilon_{t+1}^*} \right\} \quad (41)$$

$$\frac{1}{1 + r_t^c} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q(\varepsilon_{t+1}^*) \quad (42)$$

$$I_t^L = \left\{ \left(R_t^L + \frac{\theta_t}{(1 + r_t^c) \varepsilon_t^*} \right) K_t^L - B_t^L \right\} [1 - \mathbf{F}(\varepsilon_t^*)] \quad (43)$$

$$\frac{1}{\xi_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1}^S \tilde{Q}(\xi_{t+1}^*) + \frac{(1 - \delta)}{\xi_{t+1}^*} \right\} \quad (44)$$

$$\frac{1}{1 + r_t^g} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \tilde{Q}(\xi_{t+1}^*) \quad (45)$$

$$I_t^S = \{ R_t^S K_t^S + B_t^g \} [1 - \mathbf{F}(\xi_t^*)] \quad (46)$$

$$K_{t+1}^\ell = (1 - \delta) K_t^\ell + P_t^\ell I_t^\ell \quad (47)$$

$$R_t^\ell = \alpha \frac{Y_t^\ell}{K_t^\ell} \quad (48)$$

$$W_t = (1 - \alpha) \frac{Y_t^\ell}{N_t^\ell} \quad (49)$$

$$Y_t^\ell = A_t (K_t^\ell)^\alpha (N_t^\ell)^{1-\alpha}, \quad (50)$$

where $Q(z_t^*) \equiv \int \max \left\{ \frac{z}{z_t^*}, 1 \right\} d\mathbf{F}^\ell(z)$, $P^\ell(z_t^*) \equiv \left[\int_{z \geq z_t^*} z d\mathbf{F}^\ell(z) \right] [1 - \mathbf{F}^\ell(z_t^*)]^{-1}$, and Θ_t denotes the Lagrangian multiplier for the household's CIA constraint.

Proof. Equations (36)-(39) are the household's first-order conditions, equation (40) is the aggregate resource identity derived from the household's budget constraint, the definitions for firms' dividends, and the government budget constraint, equations (41)-(43) are derived from large firm's decision rules based on the law of large numbers, equations (44)-(46) are analogous equations for small firms, equation (47) is the law of motion for aggregate capital stocks, equations (48) and (49) relate firms' marginal products to factor prices, and equation (50) is the aggregate production function (see Appendix 5 for details). ■

Remark 4 The function P in equation (47) can be rewritten as $P(z_t^*) = \frac{\left[\int_{z \geq z_t^*} z d\mathbf{F}^\ell(z) \right]}{\left[\int_{z \geq z_t^*} d\mathbf{F}^\ell(z) \right]}$, which

reflects the average efficiency of firm-level investment. Since $\frac{\partial P}{\partial z^*} > 0$, a lower cutoff z^* (or larger number of active firms) implies lower average (aggregate) investment efficiency. Hence, even if CE can raise aggregate investment, it does not imply a higher aggregate capital stock and output.

Remark 5 Suppose LSAP are such that government spending G_t remains constant; then CE would have no direct effect on the aggregate resource constraint (40). The intuition is that when CE is financed entirely by increases in the lump-sum taxes, there would be an equal decrease in dividends—namely, $T_t = B_{t+1} - (1 + r_t) B_t$ and $D_t = R_t K_t - I_t + B_{t+1} - (1 + r_t) B_t = R_t K_t - I_t + T_t$. Thus, the lump-sum tax on households is transferred completely to firms as dividends, which are remitted to the households. Hence, nothing changes on the household budget constraint. However, such a resource-invariant transfer has a real effect because CE reduces the real interest rate, which may stimulate investment. On the household side, since equity return r_t^E depends on dividend D and stock price Q , $r_t^E = \frac{Q_{t+1} + D_{t+1}}{Q_t}$, both are positively affected by a lower interest cost; thus, the return to equity decreases when Q and D increase. Given income, this can imply a decrease in consumption growth—either a higher current consumption or a lower future consumption.

3.4 Equilibrium Regimes

Similar to the model of Williamson (2012), there exist five monetary-equilibrium regimes in our model, depending on the quantity of supply/demand of public/private debt. The effects of both conventional and unconventional monetary policies differ in different equilibrium regimes. These five regimes include (i) a "liquidity trap" regime with scarcity (shortage) of public/private debt, (ii) a regime with a plentiful supply of public/private debt, (iii) an "engaged interest rate"

regime, (iv) a "disengaged interest rate" regime, and (v) the "Friedman rule" regime with the inflation rate $\pi = \beta - 1 < 0$. The steady-state properties of these different equilibrium regimes are characterized below.

3.4.1 The Liquidity Trap Regime

The real interest rate of public/private debt is bounded below by the real rate of return on money $\frac{1}{1+\pi}$. At this lower bound, money is a perfect substitute for public/private debt since both forms of debt yield the same rate of return in real terms. Therefore, the liquidity trap regime is characterized by the conditions $\frac{1}{1+\pi} = 1 + r^g \leq 1 + r^c < \frac{1}{\beta}$. The liquidity trap can arise, for example, if the supply of public/private debt is sufficiently low, which drives down the real interest rate to $\frac{1}{1+\pi}$. Since the nominal interest rate of government bonds is given by $(1 + i^g) = (1 + \pi)(1 + r^g)$, at the liquidity trap the nominal interest rate is automatically at its zero lower bound $i^g = 0$. At the liquidity trap, further decreases in the supply of public debt or further increases in the demand for public debt have no effect on the real interest rate r^g , given the inflation rate π .

As noted by Williamson (2012), the liquidity trap can be associated with either negative or positive inflation away from the Friedman rule, so the real interest rate can be either positive or negative at the liquidity trap. This is in sharp contrast to the conventional wisdom that the liquidity trap can happen only with negative inflation (deflation) at the Friedman rule $\pi = \beta - 1$. Empirical data also support the model's prediction that the zero nominal interest rate can be associated with either deflation or inflation. For example, currently the U.S. nominal interest rate on short-term government bonds has been essentially zero since the financial crisis, yet the inflation rate has been strictly positive at 1% to 2% per year. In fact, a positive inflation rate provides more room for LASP to lower the real interest rate r^c , so LSAP would be more effective in reducing the real interest rate on private debt in an economy with a positive inflation rate since the gap between $\frac{1}{1+\pi}$ and $1 + r^c$ is potentially larger.²¹

A special case of the liquidity trap occurs when $\frac{1}{1+\pi} = 1 + r^g = 1 + r^c < \frac{1}{\beta}$. In this case, reducing the supply or increasing the demand for private debt are no longer effective in reducing

²¹However, because interest rates are bounded above by the time preference, the liquidity trap may not exist if the inflation rate approaches infinity or is above a finite upper bound π_{\max} . The case depends on the distribution of the idiosyncratic shocks. For example, suppose the lower support of the distribution of ε_t is strictly positive, then the liquidity premium Q is bounded above by $Q_{\max} < \infty$. Hence, given the time preference rate, π is bounded by the relationship $1 + \pi \leq \beta Q_{\max}$. Therefore, any distribution of $\varepsilon \in [\varepsilon_{\min}, \varepsilon_{\max}]$ with the lower support $\varepsilon_{\min} > 0$ would put a finite upper bound on $\pi_{\max} = \beta Q_{\max} - 1 < \infty$. The intuition for this result is that when the inflation rate is too high, or the real rate of return to financial assets is too negative, firms are better off by not saving or holding the asset as a store of value. Therefore, the liquidity trap cannot exist in hyper-inflation economies where $\varepsilon_{\min} > 0$. For recent literature studying the conventional concept of the liquidity trap caused by deflation, see Gavin, Keen, Richter and Throckmorton (2013).

the interest rate on either public or private debt.

A liquidity trap can be the consequence of either (i) an insufficient supply of public/private debt or (ii) an excessive demand for public/private debt. Thus, at least two scenarios can lead to the liquidity trap: large-scale open-market operations by the central bank to purchase government bonds and excessive demand for government bonds by the private sector, such as during financial crisis with a flight to liquidity or a savings glut in developing countries. For example, if the debt limit (or leverage ratio) θ is reduced during a financial crisis, then the supply (issuing) of private debt goes down, causing a fall in the real interest rate on private assets. If initially $1 + r^g = 1 + r^c$, a fall in r^c will push large firms to take flight to public debt as an alternative store of value, driving r^g down toward the liquidity trap regime.

3.4.2 Plentiful Public/Private Debt Regime

When the economy is saturated with public debt, the interest rates of both public and private debt will reach their upper bound $\frac{1}{\beta}$ with $\frac{1}{1+\pi} < 1 + r^g = 1 + r^c = \frac{1}{\beta}$ —as the interest rate on public debt approaches $\frac{1}{\beta}$, arbitrage by large firms will also drive up the interest rate on private debt toward the same upper bound. At the limit, all liquidity premiums disappear and the cutoffs are at their maximum values, $\varepsilon^* = \varepsilon_{\max}$ and $\xi^* = \xi_{\max}$. This regime corresponds to the constrained optimum where only the most productive firm (large or small) undertakes investment and the rest opt to lend. This suggests that high interest rate policies are welfare improving, consistent with Williamson’s (2012) finding in the new monetarism model.

3.4.3 Engaged Interest Rate Regime

When $\frac{1}{1+\pi} < 1 + r^g = 1 + r^c < \frac{1}{\beta}$, the interest rate on public debt and that on private debt are equal but both rates are away from their lower and upper bounds. With this regime, (i) increases in r^g can drive up r^c and (ii) decreases in r^c can push down r^g . However, the reverse is not true: increases in r^c do not necessarily drive up r^g and decreases in r^g do not necessarily reduce r^c , except at the margin.

First, given r^c , if r^g rises above r^c (say because the supply of public debt increases), by arbitrage large creditors (firms) opt to switch to public debt and reduce the demand for private debt, which will drive up r^c until the two rates are equalized. This equalization process under arbitrage can continue until both interest rates reach their upper bound $\frac{1}{\beta}$. On the other hand, given r^c , if r^g falls below r^c (say because the supply of government bonds decreases), large creditors opt to decrease their demand on public debt by selling government bonds. However,

since large firms cannot hold a negative amount of government bonds, the decrease in their holdings of public debt is limited. Therefore, there exists a lower bound \underline{r}^c on the interest rate of public debt such that if $r^g \leq \underline{r}^c$, a further decrease in r^g can no longer cause a fall in r^c . Note that at or below this lower bound, large firms no longer hold any public debt.

Second, given r^g , if r^c falls below r^g (say because the supply of private debt decreases), under arbitrage large firms opt to increase their demand for public debt, which drives down r^g until the two rates are equalized. This equalization process under arbitrage can continue until both interest rates reach their lower bound $\frac{1}{1+\pi}$ at the liquidity trap. On the other hand, given r^g , if r^c rises above r^g (say because the supply of corporate bonds increases), large firms opt to decrease their demand for public debt. However, since large firms cannot short-sell government bonds, the decrease in their holdings of public debt is limited. Therefore, there exists an upper bound \bar{r}^g on the interest rate of government bonds such that if $\bar{r}^g \leq r^c < \frac{1}{\beta}$, a further rise in r^c can no longer cause an increase in r^g .

Case (i) in the engaged interest rate (EIR) regime shows the limitation of open-market operations. That is, the open-market sale of public debt is effective in driving up the market interest rate on private debt, but open-market purchases of public debt may not be effective in driving down the market interest rate on private debt. Case (ii) shows the power of flight to liquidity on the interest rate of public debt—the interest rate of government bonds decreases whenever firms decide to hold fewer illiquid assets (such as private debt) and more liquid assets (such as public bonds).

3.4.4 Disengaged Interest Rate Regime

The above analyses imply that there exists a disengaged interest rate regime (DEIR) in which $\frac{1}{1+\pi} < 1 + r^g < 1 + r^c < \frac{1}{\beta}$. This equilibrium regime arises either because the supply of government bonds is neither too low nor too high, or the supply of private debt is neither too low nor too high. At this regime, open market operations of the central bank have no effects on the interest rate of private debt.

Remark 6 The effectiveness of LSAP on aggregate investment and private spending hinges on the condition $1 + r^g < 1 + r^c$. That is, the economy must be in the DEIR regime for private asset purchases to be effective in reducing the real interest rate r^c while not allowing open-market sales of public debt to raise r^g to counter the effects of LSAP. Hence, qualitative easing is easiest to implement when $\frac{1}{1+\pi} = 1 + r^g < 1 + r^c$ so that open market sales of public debt does not increase r^g , because at the liquidity trap reducing the supply of public debt has no effects on its interest rate.

3.4.5 The Friedman Rule

If aggregate money stock shrinks at the rate of time preference, $\frac{1}{1+\pi} = \frac{1}{\beta}$, then the economy is at the Friedman rule regime with $\frac{1}{1+\pi} = 1 + r^g = 1 + r^c = \frac{1}{\beta}$. At this regime all assets command the same rate of return: $\frac{1}{1+\pi} = 1 + r^g = 1 + r^c = \frac{1}{\beta}$. At the Friedman rule, only the most productive firm (large and small) invests and the rest opt to lend, as in the regime with plentiful public debt.

3.5 Graphic Representations

The five equilibrium regimes can be visually presented and summarized by Figure 3, where the left panel is the public debt market and the right panel the private debt market. The horizontal axes denote the quantity of debt and the vertical axes denote the real interest rate. The supply curve is vertical in the public debt market and downward sloping in the private debt market (since a higher interest rate or cost of borrowing discourages the issuing of private debt). The demand curve is upward sloping in both markets, because a higher interest rate encourages the purchase (demand) of debt. In the public debt market, given the demand curve D , the position of the supply curve of public debt can determine the equilibrium regimes discussed above. For example, the supply curve S_1 corresponds to the liquidity trap regime where the real interest rate equals $\frac{1}{1+\pi}$; the supply curve S_4 corresponds to the regime with plentiful public liquidity where the real interest rate equals $\frac{1}{\beta}$; the supply curve S_3 corresponds to the EIR regime where $r^g = r^c$; and the supply curve S_2 corresponds to the DEIR regime with $r^g < r^c$.

When the supply curve S_3 shifts to S_2 (say under open market purchases of public debt), the interest rate r^g decreases along the demand curve D in the left panel. Under arbitrage, large firms opt to reduce their holdings of public debt and increase their portfolios toward private debt. Thus, the demand curve in the private debt market shifts out from D_2 to D_1 . Once large firms fully deplete their stocks of public debt, the demand curve D_2 stops shifting, so the equilibrium interest rate on private debt \bar{r}^c exceeds that of public debt r^g in equilibrium and the two interest rates become disengaged. This illustrates the limitation of open market purchases of government bonds on reducing the interest rate on private debt. Hence, monetary policies are most powerful in shifting the market interest rate only in the upward direction (e.g., in tightening) but not in the downward direction (e.g., in expansion). Finally, the Friedman rule regime occurs when the horizontal line $(1 + \pi)^{-1}$ shifts upward until it overlaps with the top line $\frac{1}{\beta}$. In this case, all equilibrium interest rates collapse to a single point on the vertical

axis at $\frac{1}{\beta} = \frac{1}{1+\pi} = 1 + r^g = 1 + r^c$.

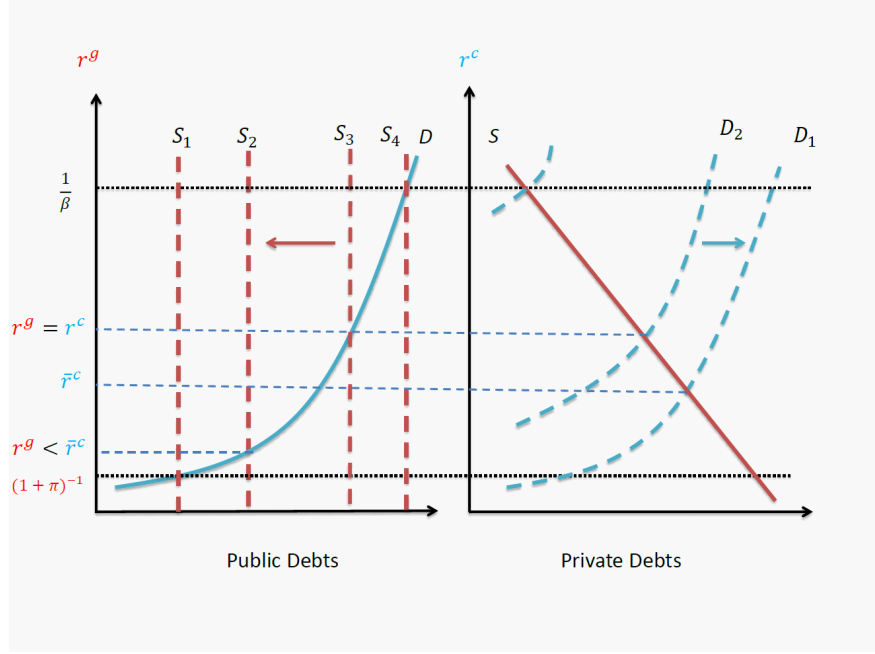


Figure 3. Five Equilibrium Regimes.

3.5.1 Flight to Liquidity

The flight to liquidity during a financial crisis can be illustrated in Figure 4, where the left panel is the market for more-liquid assets (e.g., government bonds) and the right panel is the market for less-liquid assets (e.g., corporate debt). A financial crisis can be modeled in several ways: (i) as a negative shock to the debt limit θ imposed in the private credit market (see, e.g., Eggertson and Krugman 2012), (ii) as a negative shock to TFP, and (iii) as an increase in the default risk of private/public debt. Under the first scenario, suppose the demand curve is D_1 in the public debt market and the supply curve is S_1 in the private debt market, so we are in the DEIR regime with $r^g < r^c$. The financial shock to the debt limit θ will reduce the capacity of large firms to issue debt, thus causing an inward shift of the supply curve in the right panel from S_1 to S_2 . Consequently, the interest rate on less-liquid assets falls from r^c to r_1^c , which now lies below r^g (the interest rate on the liquid assets). Under arbitrage, large firms adjust their portfolios by decreasing the demand for private debt and increasing the demand for public debt. This adjustment is reflected in the private debt market by movement along the demand curve D and in the public debt market by the outward shift of the demand curve from D_1 to D_2 . Because the drop in the interest rate on less-liquid assets is sufficiently large, the demand

curve in the liquid asset market will continue to shift until $r_1^g = r_1^c$, so the economy enters the EIR regime.

Furthermore, if the leftward shift of the supply curve is sufficiently large (say from S_1 to S_3), the demand curve in the public debt market will continue to move out to D_3 and reach the liquidity trap interest rate $\frac{1}{1+\pi}$. This can happen, for example, if the debt limit θ in equation (12) decreases sufficiently. Therefore, a credit crunch and a flight to liquidity during financial crisis can lead to a liquidity trap (as in Eggertson and Krugman, 2012). Notice that to prevent the liquidity trap in a credit crunch, the correct policy response is not to inject money through open market operations by purchasing public debt—which would shift the vertical supply curve S in the left panel leftward and thus exacerbate the problem, but rather to sell public debt to meet the suddenly increased demand for liquid assets—which would shift the supply curve out to the right and thus avoid the liquidity trap. Alternatively, if the government has private assets on its balance sheets, it can also shift the demand curve D in the right panel leftward by selling private debt. However, private asset sales would not avoid the liquidity trap if the interest rate on public debt has already reached the liquidity trap, in which case the two interest rates would become disengaged again when r^c increases, so it would have no effect on r^g .

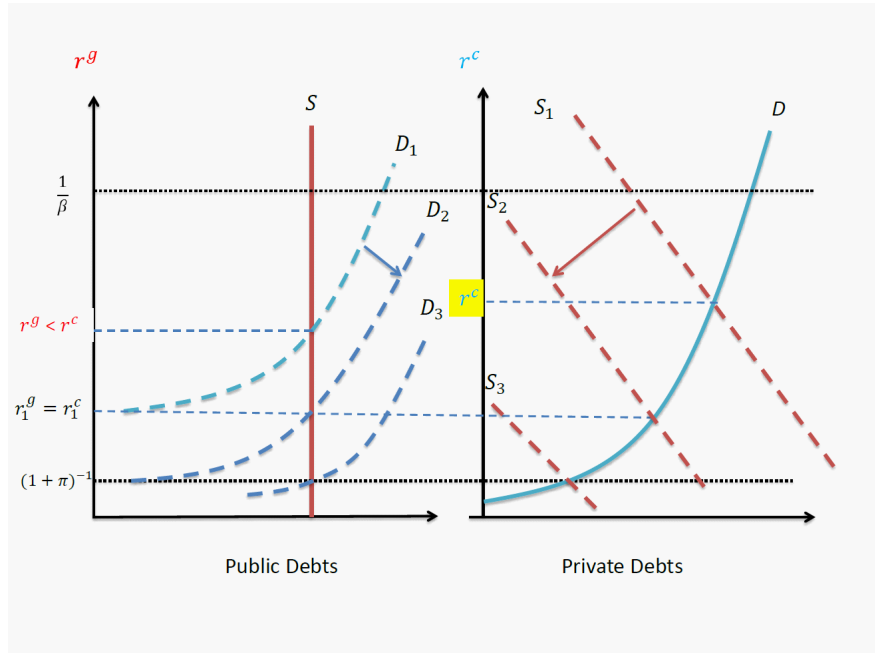


Figure 4. Credit Crunch and Flight to Liquidity.

4 Macroeconomic Effects of LSAP

If we buy assets and hold them for a day, they are not having any impact on the economy. If we buy assets and hold them for three years, yes, they can start to have an impact on the economy.

Narayana Kocherlakota (March 4, 2013)

The previous analysis shows that LSAP can reduce the real interest rate and raise the collateral value of productive assets, thus potentially stimulating firm investment by shifting the distributions of credit/debt in the debt markets. But to gauge the quantitative impact of LSAP on aggregate employment and output is a task that can be carried out only in a calibrated general equilibrium framework. The following remark anticipates the results in this section:

Remark 7 LSAP cannot effectively mitigate the negative impact of a financial crisis on aggregate output and employment unless the extent of purchases is extremely large and highly persistent. The main reason is that LSAP do not improve the allocative efficiency of credits nor the productivity of debt. By lowering the costs of borrowing and the rate of return on savings, LSAP can stimulate aggregate investment by "pushing" creditors—firms without good investment opportunities who opt to save during bad times—to become debtors and undertake investment projects that would otherwise not be taken (even during normal times). Consequently, although the number of debtors and the volume of aggregate investment may increase, the average efficiency of investment (quality of loans) would decline, leaving the "quality-adjusted" aggregate capital stock and labor productivity barely changed. On the household side, although saving becomes less attractive when the rate of return on financial wealth declines, consumption does not necessarily increase if labor income has not. However, when total asset purchases are sufficiently large and persistent, the real interest rate becomes sufficiently negative and sensitive to asset purchases, and the asset price of capital becomes sufficiently high, so that firms' borrowing constraints are sufficiently relaxed—thanks to the extremely low borrowing cost and high collateral value of capital. In such a case the intensive margin of aggregate investment would increase significantly and strongly dominate the efficiency loss of investment along the extensive margin, so that the quality-adjusted aggregate capital stock starts to rise significantly, pushing up labor productivity and the real wage rate. As a result, the elasticity of labor supply would rise significantly in response to large asset purchases because workers eventually find it sufficiently attractive to increase hours worked when the wage is significantly high, leading to a significantly larger elasticity of output and employment toward additional asset purchases.

4.1 Calibration

Assume symmetry in the technology level between small and large firms, $A^L = A^S$. Let the time period be one quarter, the time discount rate $\beta = 0.99$, the rate of capital depreciation $\delta = 0.025$, the capital income share $\alpha = 0.36$, and the inverse labor supply elasticity $\gamma = 0.5$. In the U.S., the total private debt-to-GDP ratio of nonfinancial firms doubled from 23% to 48% over the past half century. The model-implied private debt-to-output ratio is about 25% when $\theta = 0.1$ and about 50% when $\theta = 0.5$. We choose two values for the steady-state ratio of government debt to GDP: $\bar{b}^g = \{0.5, 0.6\}$. When $\bar{b}^g = 0.6$, the model-implied real interest rate of government bonds is greater than the inverse inflation rate: $1 + r^g > \frac{1}{1+\pi}$; and when $\bar{b}^g = 0.5$, the model-implied real interest rate of government bonds is less than the inverse inflation rate: $1 + r^g < \frac{1}{1+\pi}$. By arbitrage, small firms opt to switch from holding government bonds to holding money, so in equilibrium we have $1 + r^g = \frac{1}{1+\pi}$ (a case of the liquidity trap).

Assume that the idiosyncratic shocks $\{\varepsilon, \xi\}$ follow the same power distribution $\mathbf{F}(z) = \left(\frac{z}{z_{\max}}\right)^\eta$ with $z \in [0, z_{\max}]$ and $\eta > 0$. We set the shape parameter $\eta = \frac{z}{z_{\max} - z}$ so that it is easy to control for the mean \bar{z} and conduct mean-preserving experiments on the variance of idiosyncratic shocks by changing the upper bound z_{\max} . The distribution becomes uniform when the mean $\bar{z} = \frac{1}{2}z_{\max}$. These parameter values are summarized in Table 2.

Table 2. Parameter Values

Parameter	β	δ	γ	α	θ	\bar{b}^g	π	ξ_{\max}	$\bar{\xi}$	ε_{\max}	$\bar{\varepsilon}$
Calibration	0.99	0.025	0.5	0.36	0.5	$\{0.5, 0.6\}$	0.03	2.0	1.0	2.0	1.0

4.2 Steady-State Analysis

The steady-state allocation and the elasticity of the economy with respect to CE depend on the monetary regimes in Section 3.4. Below we focus on two of them.

4.2.1 Disengaged Interest Rate Regime

To study the effects of LSAP in a DEIR regime, we assume $\bar{b}^g = 0.6$, so $\frac{1}{1+\pi} < 1 + r^g < 1 + r^c < \frac{1}{\beta}$. The detailed steps to compute the equilibrium allocations are provided in Appendix 6. Figure 5 shows the effects of CE. Panels [2,2] and [2,3] show that LSAP are effective in reducing the real interest rate and raising asset prices. However, the other panels show that LSAP are not effective in increasing aggregate consumption, investment, employment, and output unless

private asset purchases become so large that they account for more than 75% of annual GDP. In particular, when the asset purchases-to-GDP ratio ($\frac{B^c}{Y}$) increases from 0 to 50%, the real annualized interest rate on private debt drops by 62 basis points (from 3.59% down to 2.97%) but aggregate annual output increases only by 0.04% from the steady state. When the $\frac{B^c}{Y}$ ratio reaches 75%, the real interest rate drops by nearly 150 bases points but aggregate output increases only by 0.56%.

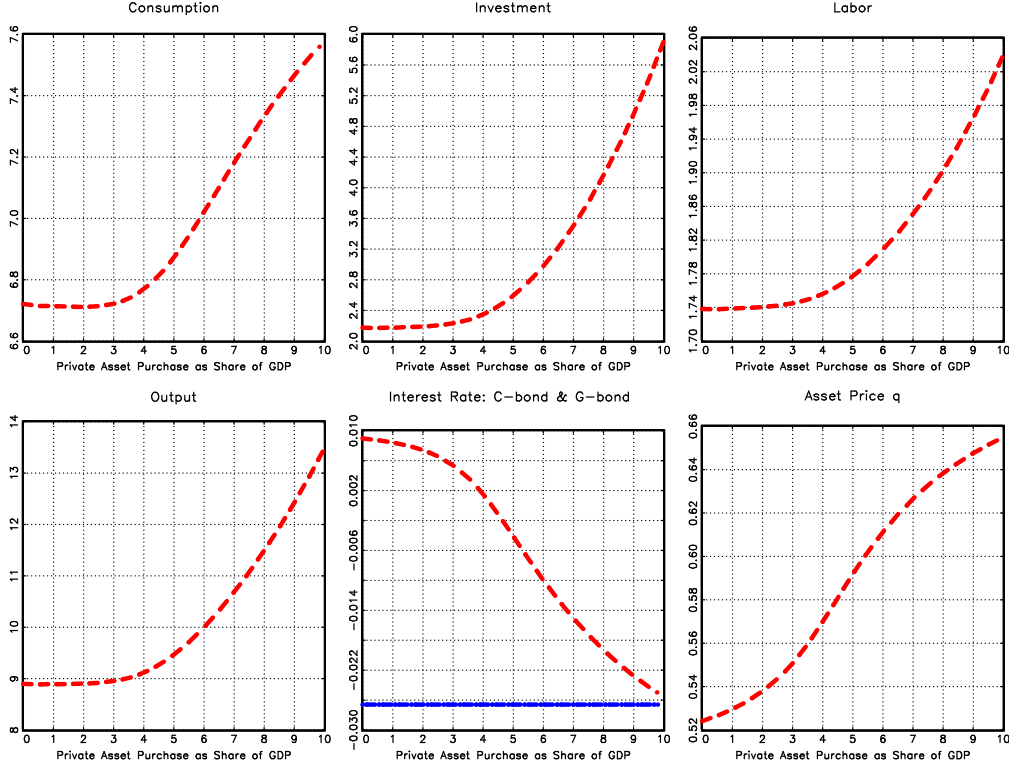


Figure 5. Macroeconomic Effects of LSAP.

Recall that just before the financial crisis the U.S. real annual (average) interest rate on long-term corporate (or industrial) bonds was less than 4% per year (see Figure 1) but the financial crisis has generated a 10% output gap. As of 2013, the total average drop in real interest rate on long-term corporate (or industrial) bonds is about 2 percentage points in the data. Our model predicts that this scale of decrease (a 2-percentage-point drop) in the real interest rate is not enough to significantly close the output gap.

Figure 6 shows the marginal product (real wage) of large firms (left panel) and the aggregate labor demand/supply (right panel) in the model. It indicates that the insensitivity of output to small-scale asset purchases is largely the result of the insensitivity of the real wage to such unconventional policy operations. In the absence of TFP changes, the only way to increase the

marginal product of labor and thus inducing a stronger labor demand/supply is to increase the capital stock. But the accumulation of the capital stock depends not only on the quantity, but also on the quality (efficiency) of investment. CE induces some of the less productive creditors to become debtors, thus achieving more investment at the cost of the average efficiency of investment.²²

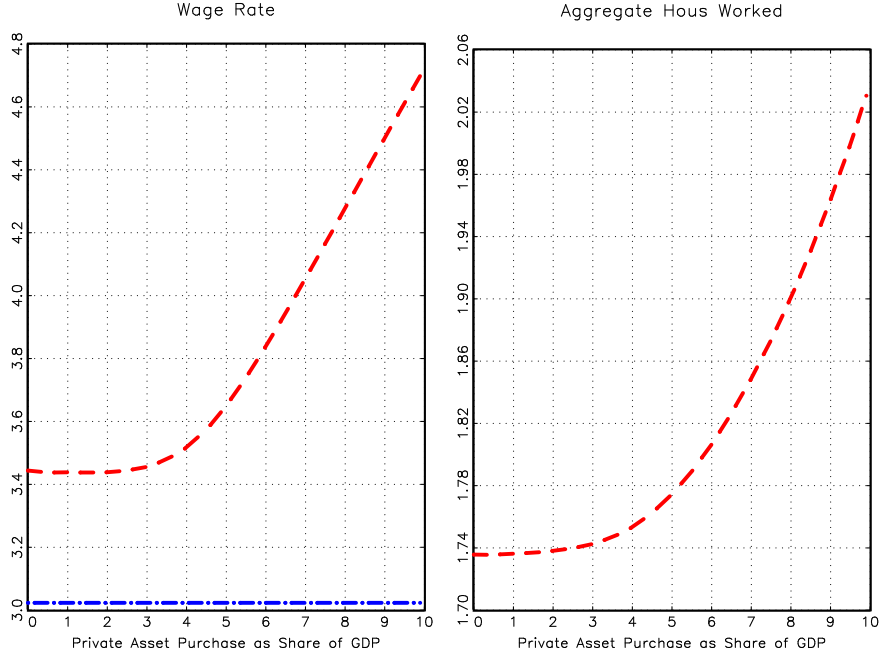


Figure 6. Wage and Hours Worked.

4.2.2 Mixed Regimes

Instead of studying all other regimes individually, we study an important case where the supply of public debt is sufficiently small $\bar{b}^g = 0.5$ so that $1 + r^g = \frac{1}{1+\pi}$ (i.e., the economy is in a liquidity trap judged from the view point of the government bond market). We start from zero private asset purchases and continuously increase CE so that the economy transitions from a "disengaged liquidity trap" regime where $\frac{1}{1+\pi} = 1 + g^g < 1 + r^c < \frac{1}{\beta}$, to an "engaged liquidity trap" regime, where $\frac{1}{1+\pi} = 1 + g^g = 1 + r^c < \frac{1}{\beta}$. This case is more relevant to the current U.S. situation because the American economy has essentially been operating at a zero nominal interest rate with $\frac{1}{1+\pi} = 1 + r^g$ since 2009.

When the model economy is in the disengaged liquidity trap regime with $\frac{1}{1+\pi} = 1 + g^g <$

²²The purchases of mortgage-backed securities have a similar quantity-quality trade-off—by lowering the mortgage interest rate and inducing more low-income households to become homeowners, such policy operations increase the amount of debts by reducing the quality of loans.

$1 + r^c$, small firms are indifferent between holding money and public debt. The problem of a large firm is identical to that studied in the DEIR regime in the previous section. However, if private asset purchases are large enough to push the interest rate on private debt down to the level of $\frac{1}{1+\pi}$, the model economy enters a different regime in which large firms also become indifferent between holding money, public debt, and private debt. We call this regime the engaged liquidity trap regime. In this case, both large and small firms opt to hold money in their portfolios. In the engaged liquidity trap regime, if the central bank's purchases of private debt increase, the real interest rate r^c cannot fall further; instead, the demand for real money balances will rise. Given the total nominal money supply \bar{M}_t , a higher demand for real balances is possible if and only if the price level becomes permanently lower. Therefore, the economy will experience a temporary deflation and then settle at a permanently lower price level. However, this situation does not imply a permanent deflation since it is still the money growth rate that determines the long-run inflation rate and this fact has not changed. Therefore, deflation cannot be a permanent phenomenon of the liquidity trap unless the growth rate of money is negative in the steady state. This explains the positive inflation rate observed in the U.S. economy despite the fact that the nominal interest rate has been at its zero lower bound since early 2009.

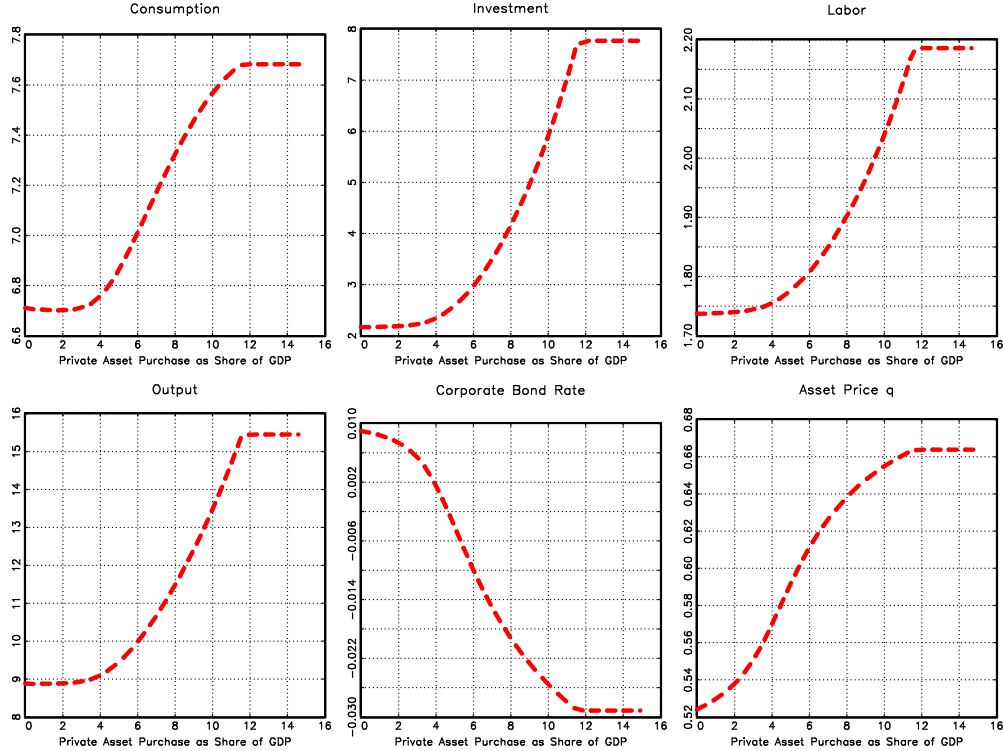


Figure 7. Effects of LSAP at Liquidity Traps.

The detailed steps for computing the mixed liquidity-trap regimes are provided in Appendix 7. Figure 7 shows the effects of LSAP on the economy. Assuming the target inflation rate is

12% per year (or 3% per quarter), the economy enters the engaged liquidity trap when asset purchases are more than 300% of annual GDP. However, if the target inflation rate is 2% per year, then the economy enters the liquidity trap (with a zero nominal interest rate not only in the public debt market but also in the private debt market) when asset purchases reach about 115% of annual GDP, at which point the maximum increase in GDP is 6.7%. Any further increases in asset purchases beyond 115% of GDP would have no additional effect on the economy.

It is worth noting that before the private debt market enters the liquidity trap ($\frac{B^c}{Y} < 115\%$), the elasticity of the economy (e.g., output, consumption, employment, investment, interest rate and asset price) with respect to LSAP in the mixed regime (with $\bar{b}^g = 0.5$ and $\frac{1}{1+\pi} = 1 + r^g$) is almost identical to that in the previous case where $\bar{b}^g = 0.6$ and $\frac{1}{1+\pi} < 1 + r^g$.

This case is just another manifestation of the insensitivity of the economy to monetary policies. When the supply of public debt is 60% of quarterly GDP or 15% of annual GDP, the real (or nominal) interest rate of government bonds is -10.6% (or $+1.4\%$) per year and the GDP level is 8.97 without QE ($\bar{b}^c = 0$). In this case, the economy is away from the liquidity trap in the public debt market. When the supply of public debt decreases to 12.5% of annual GDP (a 20% decrease in public debt supply), the real (or nominal) interest rate in the government bond market is -12% (or 0%) per year, a drop of 2 percentage points and the GDP level is 8.881 without QE ($\bar{b}^c = 0$). In this case, the economy is in a liquidity trap from the viewpoint of the public debt market, so further open-market operations under conventional monetary policies should have no effects on the real economy. However, since the private debt market is still far away from the liquidity trap, the Fed can use unconventional monetary policies to lower the real interest rate on private debt. But the results here show that the effects of such unconventional monetary policies are almost identical regardless of the interest rate in the public debt market. This insensitivity is analogous to the near-zero elasticity of output with respect to small-scale private asset purchases despite significant changes in the real interest rate of corporate bonds. That is, even if we set the public debt level to 80% of quarterly GDP (or 20% of annual GDP) in the mixed regime case, the steady-state output level is only 9.06 without QE (just 1 percentage point higher), even though the real (or nominal) interest rate of government bonds has increased by 6 percentage points from -12% (or 0%) to -6% (or $+6\%$) per year with a 12% annual inflation target.²³

²³Similar to the findings in Williamson (2012), in this class of models a high interest rate on public debt is beneficial for aggregate output because it encourages more productive firms to invest and less productive firms to lend.

4.2.3 The Low Inflation Puzzle

Inflation is the most capricious of economic variables and central banks are cursed with the responsibility for it. It has defied all predictions in the US during the past five years and, once again, inflation's general perversity is complicating life for the Federal Reserve.

Financial Times (July 23, 2013)

It has been feared that large-scale purchases of public/private debt will lead to high inflation because of the enormous amount of money injected under such operations. For this reason, both news media and some Fed officials (most notably Philadelphia Fed president Charles Plosser) predicted back in 2009 that QE would cause high inflation. But the reality has defied such predictions (see Figure 1).

The conventional explanation proposed by Fed vice chair Janet Yellen (then the San Francisco Fed president) was that monetary injection would not cause high inflation when economy-wide resources are highly underutilized during recessions, because there is little pressure for prices and wages to increase.

Our model provides an alternative explanation for the low inflation level. The Federal Reserve's LSAP alone can depress inflation near the liquidity trap: Once the real interest rate of financial assets is low enough, QE induces flight to liquidity because portfolio investors opt to switch from interest-bearing assets to money. Hence, the aggregate price level must fall to accommodate the increased demand for real money balances for any given target level of long-run money growth (or anticipated inflation rate).²⁴ Therefore, monetary injection through LSAP exerts downward pressure on the price level because it increases aggregate money demand for any given inflation rate. This deflationary effect is particularly strong at the liquidity trap, as shown in Figure 8. The U.S. economy has been in a liquidity trap since late 2008/early 2009 when the nominal short-term interest rate became essentially zero. Subsequent QE further reinforced the low inflation by keeping high pressure on real money demand.

Here, the paradox of inflation arises: Money injections can lead to lower rather than higher inflation. In normal situations, since monetary injection does not affect the real demand for money, the aggregate price level rises one-for-one with the money supply. However, near the liquidity trap the real demand for money increases due to portfolio adjustments by asset holders, and the aggregate price level does not rise one-for-one with the money supply. Hence, low inflation or deflation need not be caused by the shrinkage of money supply—in sharp contrast to the conventional wisdom—instead, it can be caused by expansion of the money supply.

²⁴One-time asset purchases alone cannot change the long-run inflation target (or inflation expectations) unless the scale of asset purchases is permanently growing at a positive rate.

One potential risk of low inflation is that it reinforces the liquidity trap by allowing the liquidity trap to occur more easily, thus reducing the chance for the efficacy of unconventional monetary policies. For example, Figure 3 shows that the equilibrium regimes depend crucially on the gap between the rate of time preference $\frac{1}{\beta}$ and the inverse inflation rate $(1 + \pi)^{-1}$. When inflation declines, the horizontal bar $(1 + \pi)^{-1}$ at the bottom rises. Consequently, the chance for the other regimes to exist shrinks. In this regard, central banks should avoid deflation and low inflation if they want monetary policies to remain effective in influencing real interest rates.

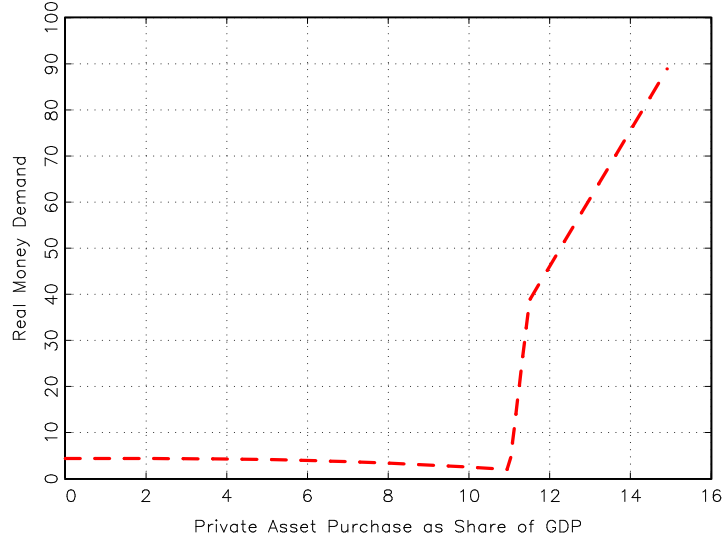


Figure 8. The Paradox of Money Injection.

4.2.4 Qualitative Easing vs. Quantitative Easing

Buiter (2008) proposes a terminology to distinguish quantitative easing (QE^a) and qualitative easing (QE^b). The former is an expansion of a central bank's balance sheet through asset purchases. The latter is the central bank's portfolio adjustment process of adding riskier (or less-liquid) assets to its balance sheet, holding constant the average liquidity and riskiness of its asset portfolio.²⁵

We can study the differential effects of QE^a and QE^b in our model by defining QE^a as increases in B_t^c in the consolidated government's balance sheet, holding B_t^g constant; and QE^b as adjustment in the government's portfolio such that $B_t^c - B_t^g = 0$.

Proposition 6 *The necessary condition for large-scale QE^b to have potential effects on the*

²⁵Because the Fed has financed its balance sheet expansion largely with interest-bearing reserves, which in effect are overnight government debt, LSAP to a large degree can also be interpreted as *qualitative easing* instead of as *quantitative easing*.

economy is $r_t^g < r_t^c$, such as in the DEIR regime. If $r^g = r^c$, then QE^b has zero marginal effects on the economy.

Proof. The intuition behind the above proposition can be seen from dependence of the distributions of creditors/debtors (i.e., the cutoffs $\{\varepsilon^*, \xi^*\}$) on QE^b . When $r^g = r^c$, large firms are indifferent between holding public debt and private debt, so an increase in the supply of government bonds tends to increase r^g and r^c together. However, an equivalent increase in the demand for private debt tends to reduce r^c and r^g together by the same amount on the margin. Therefore, as long as the economy is in the EIR regime, these effects offset each other on the margin, thereby having no effect on the real interest rates across financial markets. ■

4.3 State-Contingent LSAP: Dynamic Analysis

The previous analysis is confined to the steady state where CE is completely exogenous. When CE is endogenous and state dependent, and agents rationally anticipate this, they may react differently. In addition, the economy's elasticity with respect to transitory CE may differ significantly from that to permanent CE. Therefore, it is necessary to study state-dependent policies in a dynamic setting. Our model offers a convenient framework for this task.

We introduce three aggregate shocks into the benchmark model and use the model to evaluate the effects of the central bank's unconventional monetary policy to combat a simulated financial crisis. To this purpose, we assume (i) the debt limit θ is a stochastic process with the law of motion,

$$\log \theta_t = (1 - \rho_\theta) \log \bar{\theta} + \rho_\theta \log \theta_{t-1} + \varepsilon_{\theta t}; \quad (51)$$

(ii) TFP is a stochastic process with the law of motion,

$$\log A_t = (1 - \rho_A) \log \bar{A} + \rho_A \log A_{t-1} + \varepsilon_{At}; \quad (52)$$

and (iii) the default risk \mathbf{P} is a stochastic process with the law of motion,

$$\log \mathbf{P}_t = (1 - \rho_{\mathbf{p}}) \log \bar{\mathbf{P}} + \rho_{\mathbf{p}} \log \mathbf{P}_{t-1} + \varepsilon_{\mathbf{p}t}. \quad (53)$$

In this paper, we introduce default risk only in the private debt market. It is straightforward to study default risk shocks to public debt as in the case of the European debt crisis. Since the model's impulse responses under LSAP are insensitive to the interest rate in the public debt market (as shown in the previous sections), all dynamic analyses conducted in this section are without small firms.

All three shocks—a negative shock to θ_t and A_t , and a positive shock to \mathbf{P}_t —can generate financial-crisis-like effects on output, consumption, investment, and employment: They all decline sharply. The real interest rate, however, decreases under a negative shock to either the credit limit θ_t or TFP (as in the U.S.), but increases under a positive shock to default risk (as in Europe). The asset price increases under the first and the third shocks but decreases under TFP shock.

A state-contingent QE policy is specified as

$$\hat{B}_{t+1}^c = \rho_B \hat{B}_t^c + \sigma_x \hat{\mathbf{X}}_t, \quad (54)$$

where $\rho_B \in [0, 1]$ measures the persistence of QE, $\sigma_x = \begin{bmatrix} \sigma_\theta & \sigma_A & \sigma_P \end{bmatrix}$ is a 1×3 row vector, and $\hat{\mathbf{X}} = \begin{bmatrix} \hat{\theta}_t & \hat{A}_t & \hat{\mathbf{P}}_t \end{bmatrix}'$ is a 3×1 column vector. Under a credit crunch shock (51), we set $\sigma_\theta = -1.0$, $\bar{b}^c = \{0.1 \text{ or } 0.9\}$, and $\rho_b = \{0.5, 0.95, 1.0\}$, respectively. These different parameter values attempt to capture the different aggressiveness of LSAP. For example, $\bar{b}^c = 0.9$ implies that the steady-state private asset purchases are equivalent to 90% of quarterly GDP or 22.5% of annual GDP; and $\sigma_\theta = -1$ implies that for every 1% decrease in credit limit θ_t below its steady-state value, private asset purchases B_{t+1} would increase by 1% above its steady state value. We set the innovation $\varepsilon_{\theta t} = 50$ in the impact period, implying that the credit limit is suddenly tightened by 50%, which generates an initial 1% drop in GDP below its steady state when $\bar{b}^c = 0.1$.

If QE is effective in combating or mitigating the credit crunch, the drop in output will be less than 1%. More importantly, the most desirable outcome from the viewpoint of output stabilization is that the shock is completely mitigated by QE so that output remains at its steady state (or its cumulative change around the steady state is zero).²⁶

Figure 9 plots the impulse responses of GDP (left column), the real interest rate of corporate bonds (middle column), and total asset purchases (right column) to a credit crunch (50% tightening of the credit limit θ) in two parameter settings: (i) $\bar{b}^c = 0.1$ (top panels) and (ii) $\bar{b}^c = 0.9$ (lower panels). In each panel, there are four impulse response functions (the steady state is the straight horizontal line), corresponding to the case without QE (the line with open circles in each panel) and the other three cases with three different values of QE persistence $\rho_B = \{0.5, 0.95, 1.0\}$.

²⁶With sufficiently aggressive QE policies, output response can even be positive, but this does not imply a free lunch because a high output level must be supported by high employment or less leisure.

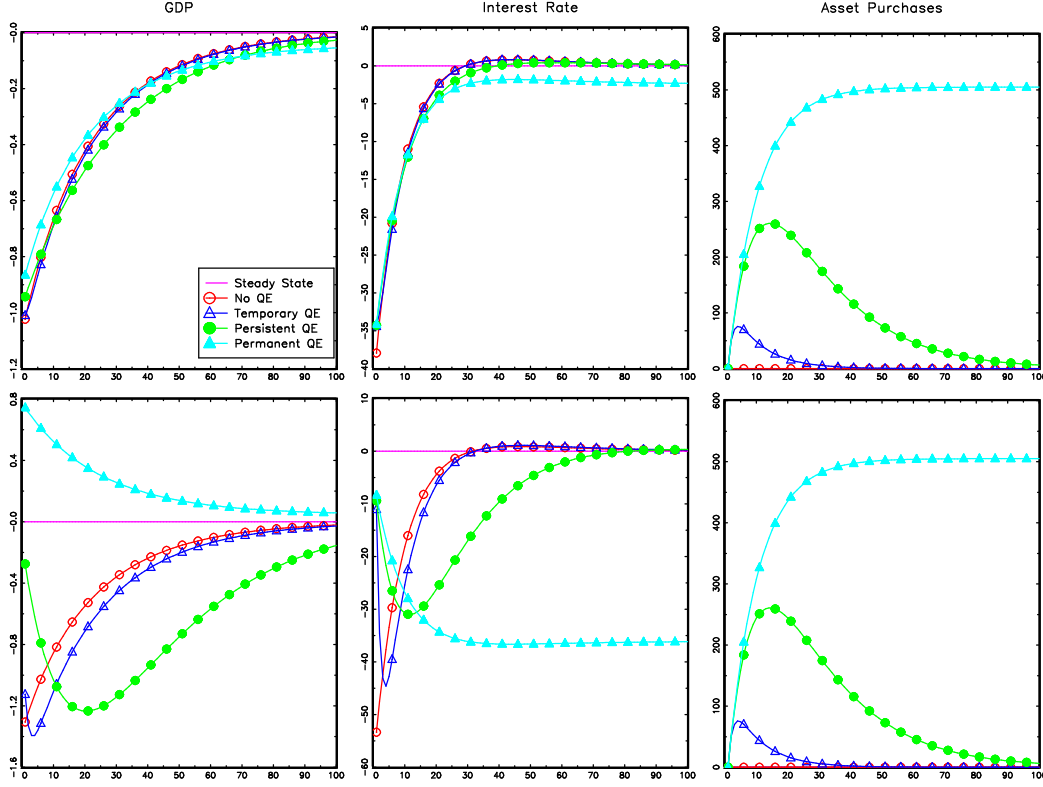


Figure 9. Effect of QE under Credit Crunch.

In the first parameter setting with a low steady-state asset purchase level ($\bar{b}^c = 0.1$, top panels), dynamic state-contingent CE is not effective in mitigating the impact of fundamental shocks on aggregate output regardless of its persistence (despite a permanently lower interest rate under permanent QE)—that is, the impulse responses of output are essentially the same with or without QE. In the second parameter setting with a substantially higher steady-state asset purchase level ($\bar{b}^c = 0.9$, lower panels), dynamic state-contingent CE remains ineffective when it is transitory ($\rho_B = 0.5$) except in the impact period of the shock. However, a highly persistent CE ($\rho_B = 0.95$) can significantly mitigate the shock initially, but the deep recession in output is simply postponed for about 20 quarters (instead of eliminated). State-contingent CE can completely offset the shock only when it is permanent (lower-left panel, $\rho_B = 1.0$). In this latter case, output is significantly positive: It increases by nearly 0.8% on impact, the real interest rate is permanently lowered by nearly 40% below its steady-state value of 3.4% per year, and total asset purchases are 40% permanently above its steady state level.

Now consider adverse TFP shocks. We set $\varepsilon_A = 1$, $\sigma_A = -15$, $\bar{b}^c = 0.1$ or 0.9 , and $\rho_A = \{0.5, 0.95, 1.0\}$, respectively. Notice that $\sigma_A = -15$ implies that for every 1% drop in TFP, the Federal Reserve Bank will increase its asset purchases by 15% above its steady-state purchase level. Figure 10 shows again that if the steady-state private asset purchases are

relatively small ($\bar{b}^c = 0.1$, top-left panel), even extremely responsive state-contingent policy with $\sigma_A = -15$ and $\rho_A = 1.0$ (and permanent QE) is not effective in mitigating the shock. However, if the steady-state asset purchases are relatively large with $\bar{b}^c = 0.9$ (lower-left panel), then with an extremely large state-contingent policy coefficient ($\sigma_A = -15$), permanent QE can significantly mitigate the negative TFP shock: The drop of output changes from -1.4% (without QE) to -0.8% (with QE) and the half-life of the recession is shortened by 50% from 10 quarters to 5 quarters. However, QE is still unable to completely offset the shock. Therefore, unconventional monetary policies—no matter how aggressive—are not an effective tool for combating TFP shocks even when inflation is fully anchored. The intuition is that under adverse TFP shocks, it is extremely hard to motivate high-quality firms to invest and low-quality firms to become debtors even with an extremely easy credit policy (under aggressive asset purchases by the government).

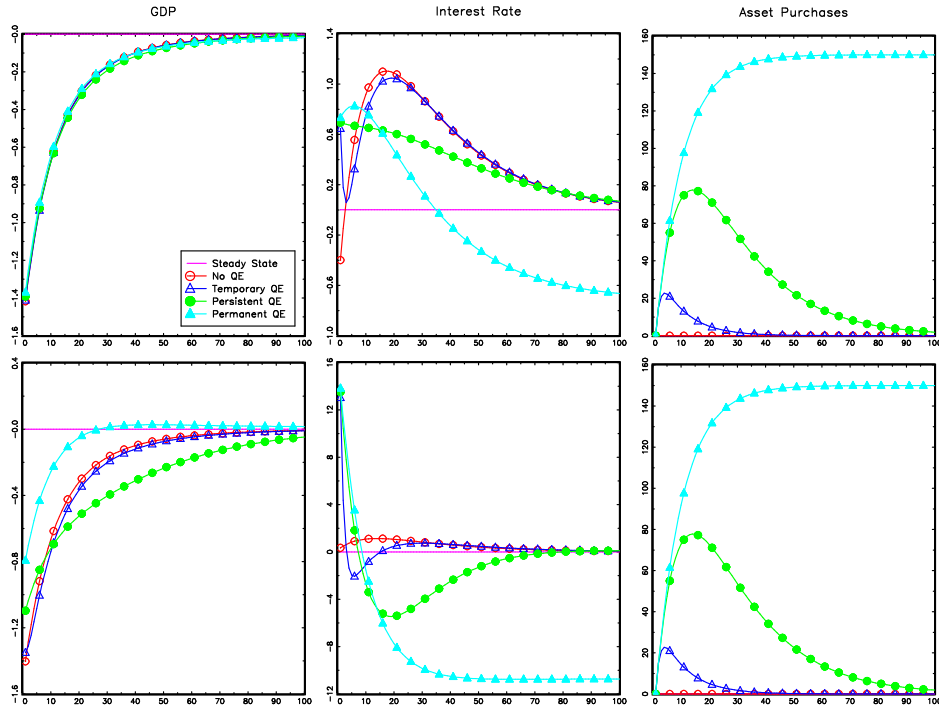


Figure 10. Effect of QE under TFP Shock.

4.3.1 Default Risk Shock

During the recent European debt crisis, the interest rate on government bonds increased to unprecedented levels, far exceeding the time preference of households and the interest rate on private bonds. The arbitrage conditions in equation (1) cannot directly explain these puzzles. We explain and reconcile these puzzles with the arbitrage conditions by introducing aggregate

default risk. When default risk exists, the "effective" interest rate on debt does not equal the actual interest rate, but only the rate adjusted by the default risk. As a result, the interest rate on public/private debt (or more-liquid assets) can far exceed the rate of time preference and the interest rates on less-liquid assets, even though the risk-adjusted interest rates still obey the arbitrage conditions provided by Williamson (2012).

To simplify the analysis, suppose only large firms exist and let $1 + r^c > \frac{1}{1+\pi}$; a large firm i solves

$$V_t(i) = \max E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} d_{t+\tau}(i) \quad (55)$$

subject to

$$d_t(i) \equiv R_t k_t(i) - i_t(i) + \frac{b_{t+1}(i)}{1 + r_t^c} - (1 - \mathbf{P}_t) b_t \geq 0, \quad (56)$$

$$k_{t+1}(i) = (1 - \delta) k_t(i) + \varepsilon_t(i) i_t(i), \quad (57)$$

$$b_{t+1}(i) \leq \theta (1 - \mathbf{P}_t) q_t k_t(i) \quad (58)$$

$$i_t(i) \geq 0, \quad (59)$$

where \mathbf{P}_t denotes systemic default risk (probability). When the probability of default \mathbf{P}_t increases, each firm's expected debt level is reduced from $b_t(i)$ to $(1 - \mathbf{P}_t) b_t$, which also reduces the firm's ability to pledge collateral by the factor of $(1 - \mathbf{P}_t)$. Thus, firms' ability to issue debt is severely hindered when the aggregate default risk rises. In the extreme case of a 100% default probability, firms are no longer able to issue debt, so the asset market shuts down and the real interest rate shoots up to infinity.

The detailed steps for solving the model are provided in Appendix 8. We set the state-contingent policy coefficient $\sigma_{\mathbf{P}} = 1.0$ and the steady-state asset purchases $\bar{b}^c = \{0.1 \text{ or } 0.9\}$. We set the steady-state default risk $\bar{\mathbf{P}} = 0.2$ and the innovation $\varepsilon_{\mathbf{P}} = 35$, implying a 35% increase in the probability of default above its steady state value, so that the drop of output on impact is about 1% below its steady state when $\bar{b}^c = 0.1$. It is worth noting that under default risk shocks, the investment decrease is far more severe than the drop in output. For example, in responding to a 35% increase in the default risk, output drops by 1%, whereas aggregate investment drops by 8%. Therefore, default risk shocks can be a potentially important source of business cycles featuring large investment swings.

Figure 11 shows the impulse responses of output (left column), the real interest rate (middle

column), and total asset purchases (right column) to a 35% percent increase in the default probability \mathbf{P}_t when the steady-state asset purchases $\bar{b}^c = 0.1$ (top panels) or $\bar{b}^c = 0.9$ (bottom panels). Notice that the real interest rate jumps up sharply by 40% under a default risk shock, but its dynamic path is insensitive to QE (middle column). Similar to previous cases with other shocks, however, unless the steady-state asset purchases are large ($\bar{b}^c = 0.9$) and the state-contingent QE is highly persistent ($\rho_B > 0.95$, bottom panels) in response to the default-risk shock, QE is not effective in reviving the economy.

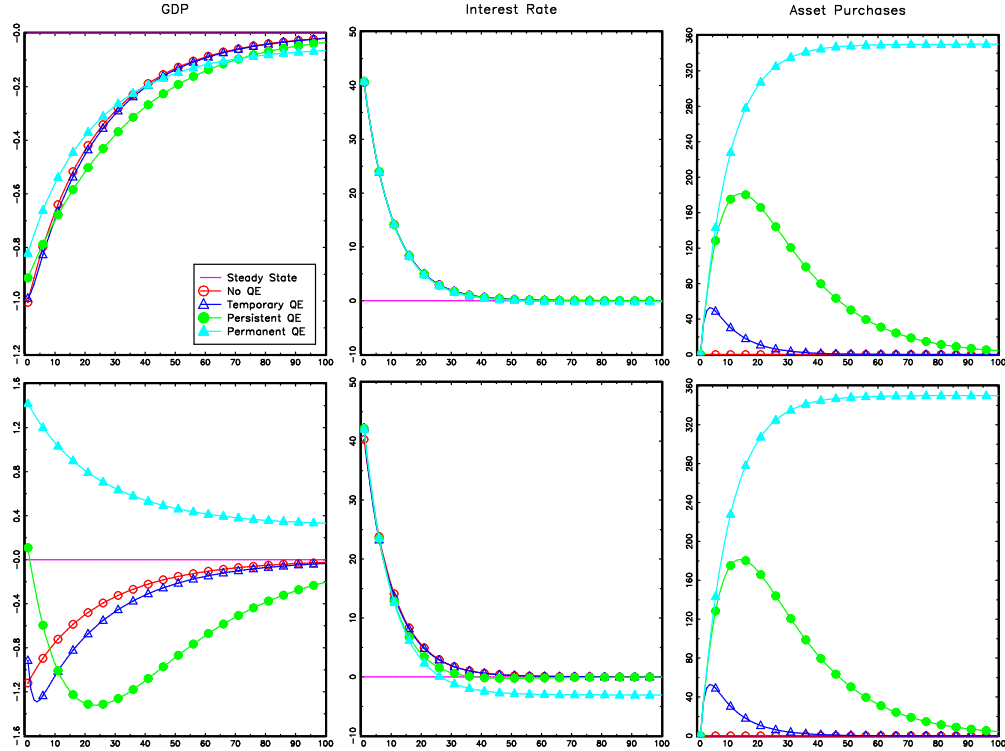


Figure 11. Effect of QE under Increased Default Risk.

5 Conclusion

We provide a general equilibrium finance model featuring explicit government purchases of private debt to evaluate the efficacy of unconventional monetary policies. We identify a particular channel to explain the apparent ineffectiveness of CE on aggregate output and employment (Figure 2) despite a significant drop in the real interest rate and increase in real asset prices (Figure 1). This channel is based on the trade-off between the quantity of loans and the quality of loans in the private debt market. We show that CE can reduce borrowing costs (the real interest rate), increase the collateral value of fixed assets, and hence relax borrowing constraints. However, since the most productive agents are always willing and able to get the loans they

want up to a borrowing limit, any additional loans generated by the government's asset purchase programs under CE can mainly go to the less productive firms, leading to the trade-off between quantity and quality of loans. Hence, for CE to have a significant impact on real economic activities at the aggregate level, the scope of asset purchases must be extraordinarily large and the extent highly persistent, so that the collateral value of fixed assets can increase significantly and, as a result, the positive quantitative effect on aggregate investment (along the intensive margin) can dominate the adverse qualitative effect on investment efficiency (along the extensive margin).

The main reason is that CE by itself does not improve the allocative efficiency of credits or the productivity of debtors. By lowering the costs of borrowing and the rate of return on savings, CE can stimulate aggregate investment mainly by "pushing" creditors—such as firms without good investment opportunities who opt to save at bad times—to become debtors and undertake investment projects that would otherwise not be undertaken (even during normal times). Consequently, although the number of debtors and the volume of aggregate investment may increase, the average efficiency of investment (quality of loans) would decline, leaving the quality-adjusted aggregate capital stock and labor productivity barely changed. On the household side, although saving becomes less attractive when the rate of return on financial wealth declines, consumption would not increase if labor income has not. Therefore, only when the extent of asset purchases becomes sufficiently large and persistent that the collateral value of capital becomes sufficiently high and the real interest rate becomes sufficiently negative, can the positive quantity effect dominate the negative quality effect, raising aggregate capital stock and the demand for labor, hence leading to significantly higher output.²⁷

We believe that our results are quite general despite the specific features of our model. Namely, we believe that any model with endogenous credit/debt markets featuring more productive agents as debtors and less productive agents as creditors would generate similar results because of the quantity-quality trade-off, which would prevent real wages from rising when the scale of asset purchases was small (even though the drop in the real interest rate might be large). In addition, our results would continue to hold even if we replace government purchases of private debts by government purchases of public debts. The intuition is that when large firms in our model can hold only public debt as a store of value, in equilibrium it would still be the case that only the productive firms undertake investment while the less productive firms opt to save by holding government bonds. Thus, government purchases of public debt would push the less productive firms to become active borrowers, thus decreasing the average efficiency of

²⁷However, unconventional policies are always welfare reducing (if leisure is included in welfare) regardless of their impact on output. This welfare implication is similar to that in Azariadis et al. (2013) and Williamson (2012).

investment across firms.

However, our dynamic state-contingent policy analysis raises an important question: Suppose LSAP are large enough to raise aggregate output; would tapering or unwinding LSAP completely undo these positive effects? As far as we know, little existing work to date has attempted to answer this important question. Our model allows us to study the optimal timing and pace of exiting LSAP. In a companion paper (Wen, 2013), we apply our model to study exit strategies and show that the result is surprising: Even though the dynamics of our model are symmetric around the steady state, tapering or unwinding of LSAP does not necessarily undo the gains (if any) under LSAP, depending on the exit strategies pursued. In particular, CE would be more effective if the exit is (i) not anticipated and (ii) gradual.

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Appendices (Not for Publication)

Appendix 1. Proof of Proposition 1

Applying the definition in equation (29), the firm's problem can be rewritten as

$$\max_{\{i_t(i), b_{t+1}(i), k_{t+1}(i)\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left(R_t k_t(i) - i_t(i) + \frac{b_{t+1}^c(i)}{1 + r_t^c} - b_t^c(i) \right) \quad (60)$$

subject to

$$k_{t+1}(i) = (1 - \delta)k_t(i) + \varepsilon_t(i)i_t(i) \quad (61)$$

$$i_t(i) \geq 0 \quad (62)$$

$$i_t(i) \leq R_t k_t(i) + \frac{b_{t+1}^c(i)}{1 + r_t^c} - b_t(i) \quad (63)$$

$$b_{t+1}^c(i) \leq \theta_t q_t k_t(i). \quad (64)$$

Notice that if $r_t^c > r_t^g \geq \frac{1}{1+\pi_t} - 1$, large firms do not hold any public debt or money. On the other hand, if $r_t^c = r_t^g = \frac{1}{1+\pi_t} - 1$, firms are indifferent between holding private debt and public debt (or money). Which case prevails depends on the steady-state supply of public debt and inflation rate. We proceed by first assuming $\frac{1}{1+\pi} < 1 + r^g < 1 + r^c < \frac{1}{\beta}$ in equilibrium and defer proofs for the other cases to Appendix 7.

Denoting $\{\lambda_t(i), \pi_t(i), \mu_t(i), \phi_t(i)\}$ as the Lagrangian multipliers of constraints (61)-(64), respectively, the firm's first-order conditions for $\{i_t(i), k_{t+1}(i), b_{t+1}(i)\}$ are given, respectively, by

$$1 + \mu_t(i) = \varepsilon_t(i)\lambda_t(i) + \pi_t(i), \quad (65)$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \mu_{t+1}(i)]R_{t+1} + (1 - \delta)\lambda_{t+1}(i) + \theta_{t+1}q_{t+1}\phi_{t+1}(i) \right\}, \quad (66)$$

$$\frac{1 + \mu_t(i)}{1 + r_t^c} = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} [1 + \mu_{t+1}(i)] \right\} + \phi_t(i). \quad (67)$$

The complementarity slackness conditions are $\pi_t(i)i_t(i) = 0$, $[R_t k_t(i) - i_t(i) + b_{t+1}(i)/(1 + r_t^c) - b_t(i)]\mu_t(i) = 0$, and $\phi_t(i)[\theta_t q_t k_t(i) - b_{t+1}(i)] = 0$.

Proof. Consider two possible cases for the efficiency shock $\varepsilon_t(i)$.

Case A: $\varepsilon_t(i) \geq \varepsilon_t^*$. In this case, firm i receives a favorable shock. Suppose this induces

the firm to invest, we then have $i_t(i) > 0$ and $\pi_t(i) = 0$. By the law of iterated expectations, equations (65) and (66) then become

$$\frac{1 + \mu_t(i)}{\varepsilon_t(i)} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{ [1 + \bar{\mu}_{t+1}] R_{t+1} + (1 - \delta) \bar{\lambda}_{t+1} + \theta_{t+1} q_{t+1} \bar{\phi}_{t+1} \}. \quad (68)$$

Since the multiplier $\mu_t(i) \geq 0$, this equation implies

$$\varepsilon_t(i) \geq \left[\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{ [1 + \bar{\mu}_{t+1}] R_{t+1} + (1 - \delta) \bar{\lambda}_{t+1} + \theta_{t+1} q_{t+1} \bar{\phi}_{t+1} \} \right]^{-1} \equiv \varepsilon_t^*. \quad (69)$$

So equation (66) implies $\lambda_t(i) = \frac{1}{\varepsilon_t^*}$. Since $\pi(i) = 0$, equation (65) then becomes

$$\frac{1 + \mu_t(i)}{\varepsilon_t(i)} = \frac{1}{\varepsilon_t^*}. \quad (70)$$

Hence, $\mu_t(i) > 0$ if and only if $\varepsilon_t(i) > \varepsilon_t^*$. It follows that under Case A firm i opts to invest at full capacity,

$$i_t(i) = R_t k_t(i) + \frac{b_{t+1}(i)}{1 + r_t^c} - b_t(i), \quad (71)$$

and pays no dividend. Also, since $\mu_t(i) \geq 0$, equation (67) implies

$$\phi_t(i) \geq \frac{1}{1 + r_t^c} - \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{ [1 + \bar{\mu}_{t+1}] \} \equiv \phi_t^*, \quad (72)$$

where the right-hand side defines the cutoff ϕ_t^* , which is independent of i . Note that $\phi_t^* \geq 0$ because it is the value of the Lagrangian multiplier when $\mu_t(i) = 0$. Hence, equation (67) can also be written as

$$\phi_t(i) = \frac{\varepsilon_t(i) - \varepsilon_t^*}{\varepsilon_t^*} \frac{1}{1 + r_t^c} + \phi_t^*. \quad (73)$$

Because $\phi_t^* \geq 0$, we have $\phi_t(i) > 0$ when $\varepsilon_t(i) > \varepsilon_t^*$, which means that under Case A firms are willing to borrow up to the borrowing limit $b_{t+1}(i) = \theta q_t k_t(i)$ to finance investment. Therefore, the optimal investment equation (71) can be rewritten as

$$i_t(i) = \left[R_t + \frac{\theta q_t}{1 + r_t^c} \right] k_t(i) - b_t(i). \quad (74)$$

Case B: $\varepsilon_t(i) < \varepsilon_t^*$. In this case, firm i receives an unfavorable shock, so the firm opts to underinvest, $i_t(i) < R_t k_t(i) + \frac{b_{t+1}}{1+r_t^c} - b_t$, then the multiplier $\mu_t(i) = 0$. Equation (65) implies $\pi_t(i) = \frac{1}{\varepsilon_t(i)} - \frac{1}{\varepsilon_t^*} > 0$. Thus, the firm opts not to invest at all, $i_t(i) = 0$. Since $\int_0^1 b_{t+1}(i) di = 0$, and $b_{t+1}(i) = \theta q_t k_t(i) > 0$ when $\varepsilon_t(i) > \varepsilon_t^*(i)$, there must exist firms indexed by j such that $b_{t+1}(j) < 0$ if $\varepsilon_t(j) < \varepsilon_t^*$. It then follows that $\phi_t(j) = \phi_t^* = 0$ under Case B. That is, firms receiving unfavorable shocks will not invest in fixed capital but will instead opt to invest in financial assets in the bond market by lending a portion of their cash flows to other (more productive) firms.

A firm's optimal investment policy is thus given by the decision rules in Proposition 1, and the Lagrangian multipliers must satisfy:

$$\pi_t(i) = \begin{cases} 0 & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ \frac{1}{\varepsilon_t(i)} - \frac{1}{\varepsilon_t^*} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \quad (75)$$

$$\mu_t(i) = \begin{cases} \frac{\varepsilon_t(i) - \varepsilon_t^*}{\varepsilon_t^*} & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \quad (76)$$

$$\phi_t(i) = \begin{cases} \frac{\varepsilon_t(i) - \varepsilon_t^*}{\varepsilon_t^*} \frac{1}{1+r_t^c} & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} = \frac{\mu_t(i)}{1+r_t^c}. \quad (77)$$

Using equations (75) to (77) and equations $\lambda_t(i) = \frac{1}{\varepsilon_t^*}$ and (66), we can express the cutoff ε_t^* as a recursive equation

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1}^L Q(\varepsilon_{t+1}^*) + \frac{\theta_{t+1} [Q(\varepsilon_{t+1}^*) - 1]}{\varepsilon_{t+1}^* (1+r_{t+1}^c)} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} \right\}, \quad (78)$$

which determines the cutoff as a function of aggregate states only. Finally, equations (75) to (77) also imply that all the Lagrangian multipliers $\{\lambda_t(i), \pi_t(i), \mu_t(i), \phi_t(i)\}$ depend only on aggregate states and the current idiosyncratic shock $\varepsilon_t(i)$. Hence, their expected values $\{\bar{\lambda}_t, \bar{\pi}_t, \bar{\mu}_t, \bar{\phi}_t\}$ are independent of individual history and i . ■

Appendix 2. Proof of Proposition 2

Proof. Using equation (76), we can rewrite equation (67) as

$$\frac{[1 + \mu_t(i)]}{r_t^c} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q(\varepsilon_{t+1}^*) + \phi_t(i). \quad (79)$$

Evaluating this equation for firms with $\varepsilon_t(i) < \varepsilon_t^*$ yields equation (32). ■

Appendix 3. Proof of Proposition 3

Proof. The proof is analogous to that in Appendix 1. ■

Appendix 4. Proof of Proposition 4

Proof. The proof is analogous to that in Appendix 2. ■

Appendix 5. Proof of Proposition 5

Proof. Equation (41) is identical to equation (78) in Appendix 1, and equation (44) can be derived analogously. For the rest of the equations in Proposition 5, it suffices to illustrate the derivations of equations (46), (47), and (40) because of the similarity between large and small firms' problems. By definition, the aggregate investment of small firms is $I_t \equiv \int i_t(j) dj$. Integrating equation (33) gives

$$I_t = R_t \int_{\xi_t(i) \geq \xi_t^*} k_t(j) dj + \int_{\xi_t(i) \geq \xi_t^*} b_t^g(j) dj. \quad (80)$$

Because $\int b_t^g(j) dj = B_t^g$ and $\xi_t(j)$ is independent of the predetermined variables $\{b_t^g(j), k_t(j)\}$ and any aggregate shocks, by the law of large numbers the aggregate investment becomes

$$I_t = (R_t K_t + B_t^g) [1 - \mathbf{F}(\varepsilon_t^*)], \quad (81)$$

which is equation (46). The aggregate capital stock evolves according to

$$K_{t+1} = (1 - \delta) K_t + \int_{\xi_t(j) \geq \xi_t^*} i_t(j) \xi_t(j) dj, \quad (82)$$

which by the small firm's investment decision rule implies

$$\begin{aligned}
K_{t+1} &= (1 - \delta)K_t + (R_t K_t + B_t^g) \int_{\xi_t(j) \geq \xi_t^*} \xi_t(j) dj \\
&= (1 - \delta)K_t + I_t [1 - \mathbf{F}(\varepsilon_t^*)]^{-1} \int_{\xi_t(j) \geq \xi_t^*} \xi_t(j) dj.
\end{aligned} \tag{83}$$

Define $P(\xi_t^*) \equiv \left[\int_{\xi \geq \xi_t^*} \xi d\mathbf{F}(\xi) \right] [1 - \mathbf{F}(\xi_t^*)]^{-1}$ as the measure of aggregate (or average) investment efficiency, we obtain equation (47). Equation (29) implies $(1 - \alpha) \left[\frac{Y_t}{N_t} \right]^{1-\sigma} A_t^\sigma = w_t$. Since the capital-labor ratio is identical across firms, it must be true that $\frac{k(i)}{n(i)} = \frac{K}{N}$. It follows that the aggregate production function is given by $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$. By the property of constant returns to scale, the defined function $R(w_t, A_t)$ in equation (29) is then the capital share, $R_t = \alpha \left(\frac{Y_t}{K_t} \right)^{1-\sigma}$, which equals the marginal product of aggregate capital. Because $\int_0^1 b_t^c(i) di = B_t^c$, $\int_0^1 b_t^g(j) dj = B_t^g$ and the equity share $s_{t+1}(i) = 1$ in equilibrium, the aggregate dividend and profit income are given by $D_t + \Pi_t = \sum_{\ell=L,S} [Y_t^\ell - I_t^\ell - w_t^\ell N_t^\ell] + \left[\frac{B_{t+1}^c}{1+r_t^c} - B_t^c \right] - \left[\frac{B_{t+1}^g}{1+r_t^g} - B_t^g \right]$. Hence, given the government budget constraint, the household resource constraint becomes $\sum_{\ell=L,S} [C_t^\ell + I_t^\ell] + G_t = \sum_{\ell=L,S} Y_t^\ell$, as in equation (40). ■

Appendix 6.

We solve the steady state of the model in the DEIR regime in three steps: (i) solving the aggregate quantities associated with large firms, including the cutoff ε^* and MPK R^L as well as the real interest rate r^c ; (ii) solving the aggregate quantities associated with small firms, including the cutoff ξ^* and MPK R^S as well as the interest rate r^g ; and (iii) solving the contributions (equilibrium weights) of large firms and small firms in aggregate output, consumption, labor, and investment, as well as the aggregate portfolio allocations of public/private debt and money across large firms and small firms.

Large Firms. For large firms, we use Proposition 5 to obtain

$$\frac{1}{\varepsilon^*} = \beta \left\{ R^L Q + \frac{\theta}{\varepsilon^*} \frac{[Q - 1]}{(1 + r^c)} + \frac{(1 - \delta)}{\varepsilon^*} \right\} \tag{84}$$

$$1 = \beta (1 + r^c) Q \tag{85}$$

$$I^L = \left\{ \left(R + \frac{\theta}{\varepsilon^* (1 + r^c)} \right) K^L - B^c \right\} (1 - \mathbf{F}) \quad (86)$$

$$\delta K^L = \tilde{P}(\varepsilon^*) I^L \quad (87)$$

which imply

$$I^L = \left(R^L + \frac{\theta}{\varepsilon^* (1 + r^c)} \right) (1 - \mathbf{F}) \frac{\tilde{P}}{\delta} I^L - B^c (1 - \mathbf{F}) \quad (88)$$

or

$$\left(R + \frac{\theta \beta Q}{\varepsilon^*} \right) (1 - \mathbf{F}) \frac{P}{\delta} - 1 = R \frac{P}{\alpha \delta} (1 - \mathbf{F}) \bar{b}. \quad (89)$$

Rearranging gives

$$R^L = \frac{\alpha}{(\alpha - b^c)} \left[\frac{\delta}{\tilde{P}(\varepsilon^*) (1 - \mathbf{F})} - \frac{\theta \beta Q}{\varepsilon^*} \right]. \quad (90)$$

Substituting R^L into equation (84) gives

$$1 = \beta \left\{ \frac{\alpha}{(\alpha - b^c)} \left[\frac{\delta \varepsilon^*}{\tilde{P}(1 - \mathbf{F}^L)} - \theta \beta Q^L \right] Q^L + \theta \beta Q^L [Q^L - 1] + (1 - \delta) \right\} \quad (91)$$

which determines the equilibrium cutoff ε^* . Given the cutoff, equation (90) determines the marginal product of capital R^L for large firms. Since

$$\alpha \frac{Y^L}{K^L} = R^L(\varepsilon^*), \quad (92)$$

the large firms' aggregate investment-to-output ratio is given by

$$s_i^L \equiv \frac{I^L}{Y^L} = \frac{I^L}{K^L} \frac{K^L}{Y^L} = \frac{\delta}{P^L(\varepsilon^*)} \frac{\alpha}{R^L(\varepsilon^*)}, \quad (93)$$

and the consumption-to-output ratio given by

$$s_c^L = 1 - s_i^L = 1 - \frac{\delta}{P^L(\varepsilon^*)} \frac{\alpha}{R^L(\varepsilon^*)}. \quad (94)$$

Small Firms. For small firms, Proposition 5 implies

$$\frac{1}{\xi^*} = \beta \left\{ R^S Q^S + \frac{(1-\delta)}{\xi^*} \right\} \quad (95)$$

$$1 = \beta (1 + r^g) Q^S \quad (96)$$

$$I = \{ R^S K^S + B^g \} (1 - \mathbf{F})$$

$$\delta K^S = \tilde{P}^S I^S \quad (97)$$

which imply

$$I^S = R^S (1 - \mathbf{F}) \frac{\tilde{P}^S}{\delta} I^S + B^g (1 - \mathbf{F}) \quad (98)$$

or

$$R^S = \frac{\alpha}{[b^g + \alpha]} \frac{\delta}{\tilde{P}^S (1 - \mathbf{F}^S)} \quad (99)$$

Substituting R^S into equation (95) gives

$$1 = \beta \left\{ \frac{\alpha}{(\alpha + b^g)} \frac{\delta \xi^*}{\tilde{P}^S (1 - \mathbf{F}^S)} Q^S + (1 - \delta) \right\}. \quad (100)$$

This equation can be rewritten as

$$1 - \beta (1 - \delta) = \frac{\beta \alpha \delta}{(\alpha + b^g)} \frac{\xi^* Q^S}{\tilde{P}^S (1 - \mathbf{F}^S)} = \frac{\beta \alpha \delta}{(\alpha + b^g)} \frac{\int \max \{ \xi, \xi^* \} d\mathbf{F}(\xi)}{\int_{\xi \geq \xi^*} \xi d\mathbf{F}(\xi)} \quad (101)$$

which uniquely determines the equilibrium cutoff ξ^* . Given the cutoff, equation (99) determines the marginal product of capital R^S . Therefore, we can solve the following ratios:

$$\alpha \frac{Y^S}{K^S} = R^S (\xi^*) = \alpha A^S \left(\frac{N^S}{K^S} \right)^{1-\alpha} \quad (102)$$

$$s_i^S \equiv \frac{I^S}{Y^S} = \frac{I^S}{K^S} \frac{K^S}{Y^S} = \frac{\delta}{P^S (\xi^*)} \frac{\alpha}{R^S (\xi^*)} \quad (103)$$

Relative size of the two sectors. The household's FOCs $\frac{1}{C^\ell} = \Lambda + \Theta$ and $\Lambda = \frac{\beta}{1+\pi} (\Lambda + \Theta)$

imply $\frac{1}{C^\ell} = \frac{1+\pi}{\beta} \Lambda$ for $\ell = \{L, S\}$. Hence,

$$C^L = C^S = \frac{1}{2}C. \quad (104)$$

The household's FOC for labor supply becomes

$$(N^\ell)^{1+\gamma} = \Lambda W^\ell N^\ell = \frac{\beta}{1+\pi} (1-\alpha) \frac{Y^\ell}{C^\ell} = \frac{2\beta(1-\alpha)}{1+\pi} \frac{Y^\ell}{C}, \quad (105)$$

which implies

$$(N^\ell)^* = \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}}. \quad (106)$$

So output in each sector is given by

$$Y^\ell = A^\ell (K^\ell)^\alpha (N^\ell)^{1-\alpha} = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{K^\ell}{Y^\ell} \right)^{\frac{\alpha}{1-\alpha}} (N^\ell)^* = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}} \quad (107)$$

$$(Y^\ell)^{1-\frac{1}{1+\gamma}} = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{1}{C} \right)^{\frac{1}{1+\gamma}} \quad (108)$$

$$Y^\ell = (A^\ell)^{\frac{1}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} \quad (109)$$

Hence, the aggregate output satisfies

$$Y = Y^L + Y^S = C + I^L + I^S + G \quad (110)$$

$$(1 - \delta s_i^L) Y^L + (1 - \delta s_i^S) Y^S = C + G \quad (111)$$

Let $A^\ell = 1$, we have

$$\left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right] \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} = (G + C) \quad (112)$$

Let $G = 0$ without loss of generality, we have

$$\left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right] \left(\frac{2\beta(1-\alpha)}{1+\pi} \right)^{\frac{1}{\gamma}} = C^{\frac{1+\gamma}{\gamma}} \quad (113)$$

$$C = \left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right]^{\frac{\gamma}{1+\gamma}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \right)^{\frac{1}{1+\gamma}} \quad (114)$$

which determines aggregate consumption C . Hence, the rest of aggregate variables are given by

$$Y^\ell = \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} \quad (115)$$

$$(N^\ell)^* = \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}} \quad (116)$$

$$I^\ell = s_i^\ell Y^\ell \quad (117)$$

$$P_t = \frac{\bar{M}_t}{C}. \quad (118)$$

Appendix 7.

Similar to Appendix 6, we solve the steady state of the model in the mixed regimes in three steps: (i) solving the aggregate quantities associated with large firms, including the cutoff ε^* , large firms' MPK R^L , and the interest rate r^c ; (ii) solving the aggregate quantities associated with small firms, including the cutoff ξ^* and MPK R^S as well as r^g ; and (iii) solving the contributions (equilibrium weights) of large firms and small firms in aggregate output, consumption, labor, and investment, as well as the aggregate portfolio allocations of public/private debt and money across large firms and small firms, as well as the aggregate price level P_t .

Large Firms' Problem. Whenever $r^g < r^c$, the large firm's problem is the same as before. However, once LSAP pushes down the real interest rate r^c to the level $1 + r^c = 1 + r^g = \frac{1}{1+\pi}$, large firms' problem changes because they opt to hold a portfolio of public/private debt and money. Let $1 + r_t^c = 1 + r_t^g \equiv 1 + \tilde{r}_t = \frac{1}{1+\pi_t}$, a large firm i solves

$$V_t(i) = \max E_t \sum_{\tau=0}^{\infty} \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} d_{t+\tau}(i) \quad (119)$$

subject to

$$d_t(i) \equiv R_t^L k_t(i) - i_t(i) + \frac{[b_{t+1}^c(i) - b_{t+1}^g(i)]}{1 + \tilde{r}_t} - [b_t^c(i) - b_t^g(i)] - \left[\frac{m_{t+1}(i) - m_t(i)}{P_t} \right] \geq 0, \quad (120)$$

$$k_{t+1}(i) = (1 - \delta)k_t(i) + \varepsilon_t(i) i_t(i), \quad (121)$$

$$b_{t+1}(i) \leq \theta q_t k_t(i) \quad (122)$$

$$i_t(i) \geq 0 \quad (123)$$

$$b_{t+1}^g \geq 0 \quad (124)$$

$$m_{t+1}(i) \geq 0, \quad (125)$$

where $b_{t+1}^c - b_{t+1}^g$ denotes the sum of newly issued private debt and newly purchased public debt.

Following similar steps in Appendix 1, the decision rules are given by

$$i_t(i) = \begin{cases} \left[R_t^L + \frac{\theta_t q_t}{1 + \tilde{r}_t} \right] k_t(i) - \left[b_t^c(i) - b_t^g(i) - \frac{m_t(i)}{P_t} \right] & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \quad (126)$$

$$b_{t+1}^c(i) = \begin{cases} \theta q_t k_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ \pm & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \quad (127)$$

$$b_{t+1}^g(i) = \begin{cases} 0 & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ + & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \quad (128)$$

$$\frac{m_{t+1}(i)}{P_t} = \begin{cases} 0 & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ + & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \quad (129)$$

Denoting M^L as total money demand by large firms, aggregation of large firms' decision rules

leads to

$$\frac{1}{\varepsilon^*} = \beta \left\{ R^L Q^L + \frac{\theta}{\varepsilon^*} \frac{[Q^L - 1]}{(1 + \tilde{r})} + \frac{(1 - \delta)}{\varepsilon^*} \right\} \quad (130)$$

$$1 = \beta (1 + \tilde{r}) Q^L \quad (131)$$

$$I^L = \left\{ \left(R + \frac{\theta}{\varepsilon^* (1 + \tilde{r})} \right) K^L - B^c + B_L^g + \frac{M_t^L}{P_t} \right\} (1 - \mathbf{F}^L) \quad (132)$$

$$\delta K^L = P^L (\varepsilon^*) I^L \quad (133)$$

In a liquidity trap, the real interest rate r^c is pinned down by the inflation rate π . Hence, large firms' allocation is completely determined by inflation in the liquidity trap. Specifically, given π , the following equation determines the cutoff ε^* for large firms:

$$1 + \pi = \beta Q (\varepsilon^*). \quad (134)$$

Given the cutoff, the following equation determines large firms' marginal product of capital R^L :

$$\frac{1}{\varepsilon^*} = \beta \left\{ R^L Q^L + \frac{\theta}{\varepsilon^*} (1 + \pi) [Q^L - 1] + \frac{(1 - \delta)}{\varepsilon^*} \right\} \quad (135)$$

$$\frac{1 - \beta (1 - \delta)}{\varepsilon^*} - \beta \frac{\theta}{\varepsilon^*} (1 + \pi) [Q^L - 1] = \beta R^L Q^L \quad (136)$$

$$R^L = \frac{1 - \beta (1 - \delta) - \beta \theta (1 + \pi) [Q^L - 1]}{\varepsilon^* \beta Q^L (\varepsilon^*)}. \quad (137)$$

To solve the ratios, we have

$$\alpha \frac{Y^L}{K^L} = R^L (\varepsilon^*) \quad (138)$$

$$s_i^L = \frac{I^L}{Y^L} = \frac{I^L}{K^L} \frac{K^L}{Y^L} = \frac{\delta}{\tilde{P}(\varepsilon^*)} \frac{\alpha}{R^L(\varepsilon^*)} \quad (139)$$

which imply

$$I^L = \left(R^L + \frac{\theta}{\varepsilon^* (1 + r^c)} \right) (1 - \mathbf{F}) \frac{\tilde{P}}{\delta} I^L - \left(B^c - B_L^g - \frac{M_t^L}{P_t} \right) (1 - \mathbf{F}) \quad (140)$$

Small Firms' Problem. Since $1 + r^g = \frac{1}{1 + \pi}$, small firms' problem also changes since they

opt to hold both government bonds and money in a liquidity trap. Now the total demand for real money balances by households and all firms (small and large) jointly determine the aggregate price level P_t . Suppose small firms' aggregate asset demand is denoted by $Z_t = B_t^g + \frac{M_t^S}{P_t}$ where M_t^S is total money demand by small firms, and the aggregate money supply is \bar{M}_t . Then the CIA constraint on the household and money market clearing imply $\frac{M_t^S}{P_t} = \frac{\bar{M}_t}{P_t} - \frac{M_t^L}{P_t} - C_t$. So in a liquidity trap, equation (46) needs to be modified to

$$I^S = \left\{ R^S K^S + B^g + \frac{\bar{M}_t}{P_t} - \frac{M_t^L}{P_t} - C \right\} (1 - \mathbf{F}(\xi^*)) \quad (141)$$

where $\frac{M_t^L}{P_t} \geq 0$ is large firms' total real money demand and $\left(\frac{\bar{M}_t}{P_t} - \frac{M_t^L}{P_t} - C \right)$ equals aggregate demand for real money balances by small firms.

In a liquidity trap, the real interest rate r^g is pin down by the inflation rate π . Hence, small firms' allocation is completely determined by inflation in the liquidity trap. Specifically, given π , the following equation determines the cutoff ξ^* for small firms' asset demand:

$$1 + \pi = \beta Q(\xi^*). \quad (142)$$

Given the cutoff, the following equation determines small firms' marginal product of capital R^S :

$$\frac{1}{\xi^*} = \beta \left\{ R^S Q(\xi^*) + \frac{(1 - \delta)}{\xi^*} \right\} \quad (143)$$

$$R^S = \frac{1 - \beta(1 - \delta)}{\xi^* \beta Q(\xi^*)}. \quad (144)$$

To solve the level, we have

$$\alpha \frac{Y^S}{K^S} = R^S(\xi^*) \quad (145)$$

$$s_i^S = \frac{I^S}{Y^S} = \frac{I^S}{K^S} \frac{K^S}{Y^S} = \frac{\delta}{\tilde{P}(\xi^*)} \frac{\alpha}{R^S(\xi^*)} \quad (146)$$

Relative Size of the Two Sectors. The household's FOCs $\frac{1}{C^\ell} = \Lambda + \Theta$ and $\Lambda =$

$\frac{\beta}{1+\pi} (\Lambda + \Theta)$ imply $\frac{1}{C^\ell} = \frac{1+\pi}{\beta} \Lambda$ for $\ell = \{L, S\}$. Hence,

$$C^L = C^S = \frac{1}{2}C. \quad (147)$$

$$(N^\ell)^{1+\gamma} = \Lambda W^\ell N^\ell = \Lambda (1 - \alpha) Y^\ell = \frac{\beta}{1 + \pi} (1 - \alpha) \frac{Y^\ell}{C^\ell} = \frac{2\beta (1 - \alpha)}{1 + \pi} \frac{Y^\ell}{C} \quad (148)$$

$$(N^\ell)^* = \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}} \quad (149)$$

$$Y^\ell = A^\ell (K^\ell)^\alpha (N^\ell)^{1-\alpha} = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{K^\ell}{Y^\ell} \right)^{\frac{\alpha}{1-\alpha}} (N^\ell)^* = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}} \quad (150)$$

$$(Y^\ell)^{1-\frac{1}{1+\gamma}} = (A^\ell)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \frac{1}{C} \right)^{\frac{1}{1+\gamma}} \quad (151)$$

$$Y^\ell = (A^\ell)^{\frac{1}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} \quad (152)$$

$$Y = Y^L + Y^S = C + I^L + I^S + G \quad (153)$$

$$(1 - \delta s_i^L) Y^L + (1 - \delta s_i^S) Y^S = C + G \quad (154)$$

Let $A^\ell = 1$, we have

$$\left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right] \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} = (G + C) \quad (155)$$

Let $G = 0$, we have

$$\left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right] \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \right)^{\frac{1}{\gamma}} = C^{\frac{1+\gamma}{\gamma}} \quad (156)$$

$$C = \left[(1 - s_i^L) \left(\frac{\alpha}{R^L} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} + (1 - s_i^S) \left(\frac{\alpha}{R^S} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \right]^{\frac{\gamma}{1+\gamma}} \left(\frac{2\beta (1 - \alpha)}{1 + \pi} \right)^{\frac{1}{1+\gamma}} \quad (157)$$

which determines aggregate consumption C . Hence, the activity levels of each sector ℓ are given

by

$$Y^\ell = \left(\frac{\alpha}{R^\ell} \right)^{\frac{\alpha}{1-\alpha} \frac{1+\gamma}{\gamma}} \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{1}{C} \right)^{\frac{1}{\gamma}} \quad (158)$$

$$(N^\ell)^* = \left(\frac{2\beta(1-\alpha)}{1+\pi} \frac{Y^\ell}{C} \right)^{\frac{1}{1+\gamma}} \quad (159)$$

$$I^\ell = s_i^\ell Y^\ell \quad (160)$$

The following equations determine money demand for large and small firms, as well as the level of the aggregate price P^M . First,

$$I^L = \left(R^L + \frac{\theta}{\varepsilon^*(1+r^c)} \right) (1-\mathbf{F}) \frac{\tilde{P}}{\delta} I^L - \left(B^c - B_L^g - \frac{M_t^L}{P_t} \right) (1-\mathbf{F}), \quad (161)$$

so we have

$$\left(B^c - B_L^g - \frac{M_t^L}{P_t} \right) = \left[\left(R^L + \frac{\theta(1+\pi)}{\varepsilon^*} \right) \frac{\tilde{P}(\varepsilon^*)}{\delta} - \frac{1}{1-\mathbf{F}(\varepsilon^*)} \right] I^L \quad (162)$$

$$\frac{M_t^L}{P_t} = B^c - B_L^g - \left[\left(R^L + \frac{\theta(1+\pi)}{\varepsilon^*} \right) \frac{\tilde{P}(\varepsilon^*)}{\delta} - \frac{1}{1-\mathbf{F}(\varepsilon^*)} \right] I^L. \quad (163)$$

Second,

$$I^S = \left\{ R^S K^S + B_S^g + \frac{\bar{M}_t - M_t^L}{P_t} - C \right\} (1-\mathbf{F}), \quad (164)$$

which in conjunction with (163) imply

$$\begin{aligned} \frac{\bar{M}_t}{P_t} &= \left[\frac{1}{1-\mathbf{F}(\xi^*)} - R^S(\xi^*) \frac{\tilde{P}(\xi^*)}{\delta} \right] I^S + \left[\frac{1}{1-\mathbf{F}(\varepsilon^*)} - \left(R^L + \frac{\theta(1+\pi)}{\varepsilon^*} \right) \frac{\tilde{P}(\varepsilon^*)}{\delta} \right] I^L \\ &\quad + C + (\bar{b}^c - \bar{b}^g) Y \end{aligned} \quad (165)$$

Notice that in a liquidity trap aggregate money demand is influenced by three components: aggregate investment, aggregate consumption, and aggregate asset purchases of private/public debt $(\bar{b}^c - \bar{b}^g)$.

Appendix 8.

Denoting $\{\mu_t(i), \lambda_t(i), \phi_t(i), \pi_t(i)\}$ as the Lagrangian multipliers of constraints (56)-(58), respectively, the firm's first-order conditions for $\{i_t(i), k_{t+1}(i), b_{t+1}(i)\}$ are given, respectively, by

$$1 + \mu_t(i) = \varepsilon_t(i)\lambda_t(i) + \pi_t(i), \quad (166)$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \{ [1 + \mu_{t+1}(i)] R_{t+1} + (1 - \delta)\lambda_{t+1}(i) + \theta_{t+1} (1 - \mathbf{P}_{t+1}) q_{t+1} \phi_{t+1}(i) \}, \quad (167)$$

$$\frac{1 + \mu_t(i)}{1 + r_t} = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} [1 + \mu_{t+1}(i)] (1 - \mathbf{P}_{t+1}) \right\} + \phi_t(i). \quad (168)$$

The complementarity slackness conditions are $\pi_t(i)i_t(i) = 0$, $[R_t k_t(i) - i_t(i) + b_{t+1}(i) - (1 + r_t) b_t(i)]\mu_t(i) = 0$, and $\phi_t(i)[\theta q_t k_t(i) - b_{t+1}(i)] = 0$. The model can then be solved the same way as in Appendix 1.