WORKING PAPER SERIES

Monetary Aggregation Theory and Statistical Index Numbers

Richard G. Anderson
Barry Jones
Travis Nesmith

Working Paper 1996-007B
http://reseach.stlouisfed.org/wp/1996/96-007.pdf

PUBLISHED: Federal Reserve Bank of St. Louis Review, 79(1), January/February 1997.

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
411 Locust Street
St. Louis, MO 63102

MONETARY AGGREGATION THEORY AND STATISTICAL INDEX NUMBERS

October 1996

ABSTRACT

This paper is the first of two from the Monetary Services Indices (MSI) Project at the Federal Reserve Bank of St. Louis. The second paper, Working Paper 96-008B, summarizes the methodology, construction and data sources for the an extensive new database of monetary services indices, often referred to as Divisia monetary aggregates, for the United States. This paper surveys the microeconomic theory of the aggregation of monetary assets, bringing together results that are not otherwise readily available in a single source. In addition to indices of the flow of monetary services, the Project's database contains dual user cost indices, measures of potential aggregation error in the monetary services indices, and measures of the stock of monetary wealth. An overview of the Project and the concept of monetary aggregation is included here as a preface.

JEL CLASSIFICATION: C43, C82, E51

KEYWORDS: monetary aggregates, aggregation theory, Divisia monetary aggregates

Richard G. Anderson
Assistant Vice President
Research Department
Federal Reserve Bank of St. Louis
411 Locust Street
St. Louis, MO 63102
anderson@stls.frb.org

Barry Jones
Visiting Scholar-Federal Reserve Bank
Washington University
Campus Box 1208
One Brookings Drive
St. Louis, MO 63130
jonesb@wuecon.wustl.edu

Travis Nesmith
Visiting Scholar-Federal Reserve Bank
Washington University
Campus Box 1208
One Brookings Drive
St. Louis, MO 63130
nesmith@wuecon.wustl.edu

Richard G. Anderson, Barry E. Jones, and Travis D. Nesmith*

Richard Anderson is an assistant vice president and economist at the Federal Reserve Bank of St. Louis. Barry Jones and Travis Nesmith are visiting scholars at the Federal Reserve Bank of St. Louis and Ph.D. candidates at Washington University in St. Louis.

Introduction to the St. Louis Monetary Services Index (MSI) Project

October 2, 1996

(For Publication in the January/February 1997 *Review*)

Economists have long recognized that the equilibrium between the demand and the supply of money is the primary long-run determinant of an economy's price level. There is far less agreement, however, on how to *measure* the aggregate quantity of money in the economy. The Federal Reserve Bank of St. Louis' monetary services index project seeks to provide researchers and policy makers with an extended database of new monetary services indices and related data.[1]

Measurement of the MSI differs considerably from that of the monetary aggregates that have been published by the Federal Reserve Board for more than 35 years, even though both begin with the same basic observation: households choose to hold monetary assets, in equilibrium, because the assets provide valuable services to the household. In other words, the household's level of utility is higher when they choose to hold positive, rather than zero,

---

* The authors thank the referees William A. Barnett and Adrian Fleissig for their careful comments on this this research. Any remaining errors are, of course, the responsibility of the authors.

[1] The monetary services indices have sometimes been referred to as Divisia monetary aggregates because their construction uses a discrete approximation to Divisia's (1925) continuous time index. The label MSI emphasizes the fact that the indices measure the flow of monetary services received, rather than the outstanding stock of monetary assets (which is the discounted value of that flow).

quantities of monetary assets, given their budget constraint. The increased utility arises, in part, because some of the assets are medium of exchange: other things equal, a larger quantity of such assets increases utility by reducing shopping time, permitting immediate purchase of bargain priced goods, providing a cushion against unanticipated expenses, and reducing the amount of time spent on cash management. Assets that are not medium of exchange, such as mutual fund shares and savings and time deposits, may also increase utility, in particular, if they are convertible to medium of exchange at relatively low cost.[2] Samuelson (1947, p. 117-8), for example, noted that

> ...it is a fair question as to the relationship between the demand for money and the ordinal preference fields met in utility theory. In this connection, I have reference to none of the tenuous concepts of money, as a numeraire commodity, or as a composite commodity, but to money proper, the distinguishing features of which are its indirect usefulness, not for its own sake but for what it can buy, its conventional acceptability, its not being "used up" by use, etc.
>
> Possession of an average amount of it [money] yields convenience in permitting the consumer to take advantage of offers of sale, in facilitating exchanges, in bridging the gap between receipt of income and expenditure, etc. The average balance is both used and at the same time not used; it revolves but is not depleted; its just being there to meet contingencies is valuable even if the contingencies do not materialize, *ex post*. Possession of this balance then yields a real service, which can be compared with the direct utilities from the consumption of sugar, tobacco, etc. in the sense that there is some margin at which the individual would be indifferent between having more tobacco and less of a cash balance, with all of the inconvenience which the latter condition implies.

Monetary aggregates published by the Federal Reserve Board are constructed by simply summing the total dollar values of the included assets.[3] Summation implicitly assumes that the

---

[2] Although most money market mutual funds allow customers to write "checks", shares in the fund are not medium of exchange. The checks themselves are drawn against a bank demand deposit owned by the mutual fund firm, an account that is replenished by the liquidation of the customer's shares.

[3] The first monetary aggregate published by the Federal Reserve, M1, was constructed in 1960 at the Federal Reserve Bank of St. Louis (Abbott, 1960). In April 1971, the Federal Reserve Board introduced two additional monetary aggregates, M2 and M3. The monetary aggregates currently published by the Federal Reserve Board differ only slightly from the revised definitions introduced in 1980 (see Anderson

monetary assets that are included in the aggregate are regarded as perfect substitutes by their owners. Microeconomic theory demonstrates that when rational decision makers are allocating resources over perfect substitutes they choose corner solutions. Thus, simple sum monetary aggregation is only consistent with microeconomic theory in the case where economic decision makers hold only one monetary asset.

In contrast, the monetary services indices (MSI) are based on explicit models of microeconomic decision making that do not make strong prior assumptions about the elasticities of substitution between monetary assets. For example, household demand for monetary assets can be modeled as the decision of a representative household which maximizes a utility function, $U(m_1,\ldots,m_n,q_1,\ldots,q_m)$, that includes both real stocks of monetary assets $m=(m_1,\ldots,m_n)$ and quantities of non-monetary goods and services $q=(q_1,\ldots,q_m)$.[4] In this model, monetary assets are treated as durable goods in the utility function, furnishing a flow of monetary services to the household. Stocks of monetary assets are assumed to depreciate, but to not fully depreciate within one period.[5] Expressions for the rental prices, or user costs, of monetary assets were derived by Barnett (1978).[6] In real terms, the user cost of a particular monetary asset is the discounted spread between a rate of return on an asset that does not furnish monetary services (called the benchmark asset) and the own rate of that particular monetary asset. The spread is

---

and Kavajecz, 1994; Kavajecz, 1994; and Whitesell and Collins, 1996). Current data are published in the Board's H.6 release and the *Federal Reserve Bulletin*.

[4] For exposition, we restrict this discussion to a simple household model. Anderson, Jones, and Nesmith (1997a) discuss an intertemporal version of the household model, as well as extensions of the household model to other decision makers, such as profit maximizing firms.

[5] Treating money as a consumer durable in household utility functions dates (at least) from Walras (1896, 1954). Non-interest bearing money (such as cash) is assumed to depreciate at the inflation rate. For a precise statement of the depreciation rate of interest bearing monetary assets see Anderson, Jones, and Nesmith (1997a).

[6] Donovan (1978) provides a definition for the current period user costs of monetary assets that are the same as Barnett's (1978) general definition in the current period. Barnett (1978) also derived the user costs of monetary assets in future periods. In addition, Barnett (1987) extends the definition of user costs to the case of manufacturing firms and financial intermediaries.

discounted to account for the payment of interest at the end of the period. Thus, the user cost of a monetary asset is the (discounted) interest foregone by the household as a result of choosing to hold the asset.

More precisely, assume that the household maximizes the utility function $U(m_1,\ldots,m_n, q_1,\ldots,q_m)$ subject to the budget constraint

$$\sum_{i=1}^{n} \pi_i m_i + \sum_{j=1}^{m} p_j q_j = Y,$$

where $\pi = (\pi_1,\ldots,\pi_n)$ is the vector of user costs of monetary assets m, Y is the household's total expenditure on non-monetary goods and services and on the services of monetary assets, and $p = (p_1,\ldots,p_m)$ denotes the vector of prices of q. Solving the household's constrained utility maximization problem yields demand functions for real monetary assets and for quantities of non-monetary goods and services

$$m_i^* = f_i(\pi, p, Y), \quad i = 1,\ldots,n$$

$$q_j^* = g_j(\pi, p, Y), \quad j = 1,\ldots,m$$

The optimization problem is discussed in detail in Anderson, Jones and Nesmith (1997a).[7]

In macroeconomics, the problem of creating a smaller number of monetary aggregates from the individual monetary assets $m_1,\ldots,m_n$ naturally arises. In general, constructing a monetary aggregate by simply summing the dollar values of the individual assets is not consistent with economic theory unless economic agents (households or firms) regard all of the monetary assets as perfect substitutes. A method of aggregation that is consistent with economic theory

[7] Equivalently, a manufacturing firm can be viewed as maximizing profit subject to a production function which contains monetary assets, as in Barnett (1987). This model produces factor demand functions for monetary assets and other inputs to production which are functions of the factor prices of non-monetary inputs and monetary asset user costs (which are the same as the user costs in the household case).

was suggested by Barnett (1980).[8] In his formulation, the household's utility function is assumed to be weakly separable in monetary assets, and may be written $F(u(m_1,\ldots,m_n), q_1,\ldots,q_m)$, where the function u is called a category subutility function.[9] In this case, the marginal rate of substitution between monetary assets m_i and m_j is $\frac{\partial u(m_1,\ldots,m_n)}{\partial m_i} / \frac{\partial u(m_1,\ldots,m_n)}{\partial m_j}$, which is independent of the quantities of all other goods $q_1,\ldots,q_m$.[10] In this form, the household can solve its utility maximization problem in two stages. In the first stage, the household chooses the shares of total household expenditure that it wishes to spend on real monetary services and on quantities of individual non-monetary goods and services. In the second stage, conditional on not exceeding the expenditure on monetary services selected in the first stage, the household selects the real stocks of monetary assets m_i that will provide the largest possible quantities of monetary services.

This two-stage budgeting model of household behavior implies that there exists an aggregator function, u, that measures the total amount of monetary services that the household receives from its holdings of monetary assets $m_1,\ldots,m_n$; the function defines a monetary aggregate as $M = u(m)$.[11] Even with this result, however, a difficulty remains: the specific functional form of the monetary aggregate depends on the household's utility function, which is unknown. Following the theoretical advances of Diewert (1976) and Barnett (1980), the monetary aggregate may be approximated by a statistical index number. The MSI developed in

---

[8] See also Barnett (1981). Additional references to Barnett's work are included in the following article.

[9] The equivalent condition for the case of a manufacturing firm is weak separability of the production function in monetary assets, see Barnett (1987).

[10] For a formal discussion of weak separability and its implications, see Goldman and Uzawa (1964). This statement of the separability assumption includes only current period monetary assets and goods. A more complete statement is that the household's choice over current period monetary assets be weakly separable from its choice over all future period monetary assets and all current and future period quantities of non-monetary goods and services (see Anderson, Jones, and Nesmith, 1997a).

the St. Louis project are based on a high quality statistical index number; details of their construction are discussed in Anderson, Jones and Nesmith (1997b).

The methodology outlined above for construction of the MSI lies solidly in the mainstream of current macroeconomic research. The theory and methods are similar to those now being used by the Department of Commerce to produce improved economic aggregates such as GDP and the GDP deflator (see Triplett, 1992, and Young, 1992, 1993).[12] An advantage of the MSI approach is that it produces an internally consistent "dual" opportunity cost, which relates to the MSI in the same way that the GDP deflator, produced by the Commerce Department, relates to GDP. In addition, the methods are similar to those of modern general-equilibrium business cycle models which often begin with the hypothesis of an optimizing microeconomic representative agent (Cooley and Hansen, 1995). To the extent that such complementary developments in measurement and modeling improve our understanding of economic fluctuations, the MSI may prove particularly valuable.

Recent research also suggests that empirical conclusions regarding issues such as the interest and income elasticities of money demand and the long-run neutrality of money may be sensitive to the choice of monetary aggregate. In other words, empirical conclusions may differ when "money" is measured by the flow of monetary services rather than by simple summation of the dollar amounts of monetary assets, see Barnett, Offenbacher, and Spindt (1984), Barnett, Fisher, and Serletis (1992), Chrystal and MacDonald (1994), and Belongia (1996). Such findings have spurred the construction of MSI data for many countries. Academic studies include: la Cour (1996) for Denmark; Janssen and Kool (1994) for the Netherlands; and Lim and

---

[11] See Green (1964) for more discussion of two stage budgeting and aggregation theory.

[12] The recent revisions in the Department of Commerce aggregates reflect two improvements. The old aggregates were fixed base Laspeyres index numbers. These have been improved to reflect advances in index number theory. The new aggregates are chained superlative indices. The monetary indices in

Martin (1994) for Australia. Central bank studies include: Herrmann, Reimers and Toedter (1994) for Germany; Ishida and Nakamura (1994) for Japan; Longworth and Atta-Mensah (1995) for Canada; and Fisher, Hudson and Pradham (1993) for the United Kingdom. Unique among central banks, the Bank of England publishes monetary services indices alongside other monetary aggregates.

Monetary services indices for the United States have been produced previously: by Barnett (1980), Barnett and Spindt (1982), Farr and Johnson (1985), and Thornton and Yue (1992). While this project is a continuation of previous research, it is not an extension of any previous series. The assumptions and methodology used in the construction of the MSI were examined for sustainability and credibility, resulting in a new series of indices which are detailed in Anderson, Jones and Nesmith (1997a,1997b).[13] The first article surveys the literature on the aggregation of monetary assets, seeking to synthesize theoretical results not readily available elsewhere in a single source. Because the analysis is based on the dynamic theory of utility maximization, some aspects are necessarily technical. Readers primarily interested in understanding the construction of the MSI and related data might prefer to move directly to the second article which provides a detailed road map to the MSI database. In addition to the MSI and their dual indices, the data include own-rates of return for some of the monetary assets in the MSI, and the user cost and asset stock data for all the monetary assets included in the MSI. This will allow researchers to use the MSI database to study the demand functions for individual monetary assets, as well as the aggregate monetary service flow. The database also includes other heretofore unpublished indices, such as the second moments of the MSI which were suggested by Barnett and Serletis (1990) as useful measures of the amount of (statistical)

Anderson, Jones, and Nesmith (1997b) are also chained superlative indices. Thus, the monetary services indices (MSI) have the same statistical properties as the Department of Commerce aggregates.

aggregation error contained in the MSI, the CE index which was suggested by Poterba, Rotemberg, and Driscoll (1995), and total expenditures on monetary assets.

The St. Louis' MSI database is maintained by the staff of the Federal Reserve Bank of St. Louis as a part of the Bank's Federal Reserve Economic Database (FRED).[14] To facilitate comparison with monetary aggregates published by the Federal Reserve Board, indices in the database are provided for the same groupings of monetary assets -- M1, M2, M3, and L -- as well as for other widely-used aggregates such as M1A (currency plus non-interest-bearing checkable deposits) and MZM (M2 less small time deposits). The indices, which will be provided at monthly, quarterly, and annual frequencies, will be updated and revised as data become available.

In addition to providing the MSI and related data, the St. Louis MSI project seeks to stimulate research on the role of monetary and financial variables in the conduct of monetary policy. In support of this goal, the MSI database also contains all underlying nonconfidential source data and the computer programs used to construct the indices.

---

[13] In addition, many of the underlying series were previously taken from undocumented outside sources. In these cases, analogous series were constructed from documented sources. These constructions are detailed in Anderson, Jones and nesmith (1997b).

[14] FRED can be reached on the world wide web at www.stls.frb.org and by modem at (314) 444-1824.

REFERENCES

Anderson, Richard G., Barry E. Jones, and Travis D. Nesmith. "Monetary Aggregation Theory and Statistical index Numbers." (1997a) Federal Reserve Bank of St. Louis *Review*

Anderson, Richard G., Barry E. Jones, and Travis D. Nesmith. "Building New Monetary Services Indices: Concepts, Methodology, and Data." (1997b) Federal Reserve Bank of St. Louis *Review*

Anderson, Richard G., and Kenneth A. Kavajecz.(1994) "A Historical Perspective on the Federal Reserve's Monetary Aggregates: Definition, Construction, and Targeting." (March/April 1994). Federal Reserve Bank of St. Louis *Review* v76 n2. 1-31.

Barnett, William A. (1978). "The User Cost of Money," *Economic Letters*, pp. 145-149.

_______________ (1980) "Economic Monetary Aggregates: An Application of Index Number and Aggregation Theory," *Journal of Econometrics* 14(1), pp. 11-48.

_______________ (1981) *Consumer Demand and Labor Supply: Goods, Monetary Assets, and Time* (Amsterdam: North Holland).

_____________. "The Microeconomic Theory of Monetary Aggregation," in William A. Barnett and Kenneth J. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge, MA: Cambridge University Press, 1987.

_____________ and Paul A. Spindt. "Divisia Monetary Aggregates: Compilation, Data, and Historical Behavior," Staff Studies 116, Board of Governors of the Federal Reserve System, May 1982.

Barnett, William A., Douglas Fisher, and Apostolos Serletis, "Consumer Theory and the Demand for Money," *Journal of Economic Literature*, (December 1992), vol.XXX, 2086-2119.

Barnett, William A., Edward K. Offenbacher, and Paul A. Spindt. "The New Divisia Monetary Aggregates." (December 1984). *Journal of Political Economy* v92 n6. 1049-85.

Barnett, William A. and Apostolos Serletis (1990). "A Dispersion Dependency Diagnostic Test for Aggregation Error: With Applications to Monetary Economics and Income Distribution." *Journal of Econometrics* 43, pp. 5-43.

Belongia, Michael T. "Weighted Monetary Aggregates: A Historical Survey." (1995) *Journal of Comparative Economics* 4, 87-114, 1995.

Chrystal, K. Alec, and Ronald MacDonald. "Empirical Evidence on the Recent Behavior and Usefulness of the Simple Sum and Weighted Measures of the Money Stock." (March/April 1994). Federal Reserve Bank of St. Louis *Review*. 73-109.

Cooley, Thomas F. and Gary D. Hansen (1995). "Money and the Business Cycle," in *Frontiers of Business Cycle Research,* Thomas F. Cooley ed. (Princeton NJ: Princeton University Press)

Diewert, Erwin. "Exact and Superlative Index Numbers." (1976), *Journal of Econometrics* 4, 115-145.

Divisia, Francois (1925). "L'indice Monetaire et la Theorie de la Monnaie," *Revue d'Economie Politique*, pp. 883-900.

Donovan, Donald J. "Modeling the Demand for Liquid Assets: An Application to Canada," International Monetary Fund Staff Paper, (December 1978), 25(4), 676-704.

Farr, Helen T. and Deborah Johnson. "Revisions in the Monetary Services (Divisia) Indexes of the Monetary Aggregates," Staff Studies 147, Board of Governors of the Federal Reserve System (December 1985).

Fisher, Paul, Suzanne Hudson, and Mahmood Pradhan. "Divisia Indices for Money: An Appraisal of Theory and Practice," (April 1993), Bank of England Working Paper Series no.9.

Goldman S.M. and H. Uzawa. "A Note on Separability in Demand Analysis," (1964), *Econometrica* vol 32, 387-398.

Herrmann, Heinz, Hans-Eggert Reimers and Karl-Heinz Toedter. "Weighted Monetary Aggregates for Germany. (1994) working paper.

Ishida, Kazuhiko and Koji Nakamura. "Broad and Narrow Divisia Monetary Aggregates for Japan." (1994). working paper.

Janssen, Norbert G.J. and Clemens J.M. Kool. "The Measurement and Relevance of Weighted Monetary Aggregates in the Netherlands." (1994). working paper.

Kavajecz, Kenneth A. (1994). "The Evolution of the Federal Reserve's Monetary Aggregates: A Timeline," Federal Reserve Bank of St. Louis *Review*, March/April 1994.

la Cour, Lisbeth F. "On the Measurement of 'Money': Results From the Experience with Divisia Monetary Aggregates for Denmark and Some Methodological Considerations of the Comparison of Money Demand Relations Based on Alternative Monetary Aggregates." (1996) working paper.

Lim, G.C. and Vance Martin. "Weighted Monetary Aggregates: Empirical Evidence for Austrailia." (1994). working paper.

Longworth, David and Joseph Atta-Mensah. "The Canadian Experience with Weighted Monetary Aggregates." (1994) working paper.

Rotemberg, Julio J., John C. Driscoll, and James M. Poterba. "Money, Output, and Prices: Evidence From a New Monetary Aggregate," *Journal of Business and Economic Statistics*. vol 13 n1 (January 1995), pp. 67-83.

Samuelson, Paul A. (1947). *Foundations of Economic Analysis* (Cambridge, MA: Harvard University Press)

Thornton, Daniel L., and Piyu Yue. "An Extended Series of Divisia Monetary Aggregates," Federal Reserve Bank of St. Louis, *Review*. vol 74 n6 (November/December 1992), pp. 35-46.

Triplett, Jack E. (1992) "Economic Theory and BEA's Alternative Quantity and Price Indexes," *Survey of Current Business*, April, vol 72 no 4, pp. 49-54.

Young, Allan H. (1992) "Alternative Measures of Change in Real Output and Prices," *Survey of Current Business*, April, vol 72 no 4, pp. 32-48.

________. (1993) "Alternative Measures of Change in Real Output and Prices, Quarterly Estimates for 1959-92." *Survey of Current Business*, March, vol 73 no 3, 31-41.

Walras, L. (1896) *Elements of Pure Economics*, W. Jaffe, trans. (Homewood IL: Richard D. Irwin, 1954)

Whitesell, William and Sean Collins (1996). "A Minor Redefinition of M2," Finance and Economics Discussion Series paper number 96-7, Board of Governors of the Federal Reserve System, February 1996.

Richard G. Anderson, Barry E. Jones, and Travis D. Nesmith*

Richard Anderson is an assistant vice president and economist at the Federal Reserve Bank of St. Louis. Barry Jones and Travis Nesmith are visiting scholars at the Federal Reserve Bank of St. Louis and Ph.D. candidates at Washington University in St. Louis.

Monetary Aggregation Theory and Statistical Index Numbers

October 2, 1996

(For Publication in the January/February 1997 *Review*)

Introduction

The aggregate quantities of monetary assets held by households, firms, and other economic decision makers play important roles in macroeconomics. Monetary aggregates for the United States published by the Board of Governors of the Federal Reserve System are simple unweighted sums of the total dollar amounts of monetary assets held by the nonbank public, such as currency, checkable deposits, money market mutual fund shares, and savings and time deposits. Implicitly, this method of aggregation assumes that the owners of these assets regard them as perfect substitutes. Although all the included assets are either medium of exchange or convertible at relatively low cost into medium of exchange, there are significant differences in their opportunity costs, suggesting that many, if not most, economic decision makers do not regard them as perfect substitutes.

A method of monetary aggregation has been developed by Barnett (1978, 1980) and others from the insight that economic agents generally choose to consume a flow of monetary services in addition to leisure, nondurable goods and the flow of services provided by durable goods. In this context, monetary assets can be viewed as durable goods in the household's utility

---

*The authors thank the referees William A. Barnett and Adrian Fleissig for their careful comments on this this research. Any remaining errors are, of course, the responsibility of the authors. The authors also thank

function. The treatment of monetary assets as durable goods in a household's utility function dates (at least) from Walras (1896), but the appropriate user cost of monetary assets in this context was derived by Barnett (1978).[1] Solving the household's constrained utility maximization problem yields demand functions for monetary assets, durable goods, nondurable goods, and leisure (see Barnett, 1980, 1981).

The user cost of monetary assets can also be derived in the context of a firm's constrained profit maximization (cost minimization) problem, when money is an argument in the firm's (derived) production function. Solution of the firm's problem produces factor demand functions for monetary assets and for the other inputs to production (see Barnett, 1987, 1990). The user costs derived from the household and firm optimization problems are the same.[2] In addition, supply side user costs of monetary assets can be derived from a model in which financial intermediaries are multi-product firms (see Barnett and Zhou, 1994, Barnett, 1987, Barnett, Hinich and Weber, 1986, and Hancock, 1985, 1986). Under the assumption that the reserves held by financial intermediaries are "sterile", the supply side user costs differ from the previous demand side user costs by a reserve tax.[3] More generally, the user costs can be extended to allow for the taxation of interest (see Barnett 1980).

The appropriate method of aggregating monetary assets is an important question in macroeconomics. Although the microfoundations of money have been widely discussed (see for

---

Kelly Morris and Mary Lohmann for research assistance.

[1] Donovan (1978) provides a definition of the user cost of monetary assets which agrees with Barnett's (1978) general definition in the current period. See also Diewert (1974), who derives a general user cost for household durable goods and applies it to the demand for money assets.

[2] The household's user costs of monetary assets are analogous to user costs for durable consumer goods, and the firm's user costs of monetary assets are analogous to user costs for durable physical capital. If firms face different interest rates than households the user costs will be different. In addition, if firms and households face different non-monetary price deflators then the user costs will also differ. However in both cases, the general form of the user costs will be a discounted interest rate spread between the interest rate of a non-monetary benchmark asset and the own rate of the monetary asset, multiplied by a price deflator. This general formula identifies (discounted) foregone interest as the opportunity cost of holding monetary assets.

example Pesek and Saving, 1967; Fama, 1980; Samuelson, 1968 and Niehans, 1978), prior to Barnett (1980) only a few authors had been concerned with application of aggregation and/or index number methods to monetary assets; see for example Chetty, 1969; Friedman and Schwarz, 1970; and Hutt, 1963, who suggested the index now known as the currency equivalent, or CE, index. Barnett, Fisher and Serletis, 1992 and Belongia, 1995a survey the early literature on the subject. After forming four aggregates by simple summation of monetary assets, Friedman and Schwartz (pp. 151-2) cautioned:

> The restriction of our attention to these four combinations seems a less serious limitation to us than our acceptance of the common procedure of taking the quantity of money as equal to the aggregate value of the assets it is decided to treat as money. This procedure is a very special case of the more general approach ...[which]... consists of regarding each asset as a joint product having different degrees of "moneyness", and defining the quantity of money as the weighted sum of the aggregates value of all assets, the weights for individual assets varying from zero to unity with a weight of unity assigned to that asset or assets regarded as having the largest quantity of "moneyness" per dollar of aggregate value. The procedure we have followed implies that all weights are either zero or unity.
>
> The more general approach has been suggested frequently but experimented with only occasionally. We conjecture that this approach deserves and will get much more attention than it has so far received. The chief problem with it is how to assign the weights and whether the weights assigned by a particular method will be relatively stable for different periods or places or highly erratic.

In general, monetary aggregation methods should be flexible enough to preserve the information contained in the elasticities of substitution between the various monetary assets, and in particular, aggregation methods should not impose perfect substitutability between monetary assets.[4] A barrier to doing so is that the functional form (and parameters) of the representative

---

[3] Reserves are said to be sterile if they do not earn interest and are not included as factor inputs in the financial firm's production function.

[4] The aggregation and index number theory discussed in this paper is based on the use of flexible functional forms, which have the property that they can attain arbitrary elasticities of substitution at a single point. Hence the aggregation methods described in this paper will satisfy the criteria that the information

agent's utility or production function are unknown. Further, even if a suitable parametric form is assumed for the utility or production function, the implied aggregator function must be estimated. An attractive alternative to direct estimation of the aggregator function is the construction of a statistical index number. Such an index may provide an approximation to an economic aggregator function, which requires no estimation, contains no unknown parameters, and is independent of the specific form of the aggregator function (Barnett, 1980, 1981, 1987).

Monetary services indices, based on statistical index number theory, are consistent with the maintained hypothesis of microeconomic optimization. Thus, the standard tools of economic demand theory may be applied to study the behavior of monetary services indices, including estimates of income and interest elasticities, in the same manner as they are applied to other elementary economic goods and services. Because the economic agent in such models views the monetary aggregate as a single economic good, the behavior of the aggregate "...can be interpreted in terms of a theory analogous to the theory of individual behavior." (Green, 1964, p. 56).

The balance of this paper is as follows. In the next section, we discuss the general conditions under which aggregation of a block of decision variables is valid, and then derive specific results for monetary aggregation in the context of a general microeconomic choice problem for a representative consumer. In the following section, we discuss the use of statistical index number theory to track monetary aggregator functions. In the following section, we discuss the consumer's budget constraint and the implied monetary wealth and stock concepts. The last section of the paper examines the robustness of the theoretical aggregation results to key assumptions and presents a test for the aggregation error that might arise when some of the assumptions are invalid.

---

contained in the elasticities of substitution between the various monetary assets be maintained. This subject is taken up again in the section of this paper titled "Statistical Index Number Theory".

MONETARY AGGREGATION THEORY

There are two distinct aggregation problems in economics: aggregation over the various goods purchased by a single agent (firm or household), and aggregation across heterogeneous agents.[5] Although this paper focuses on aggregation of the monetary assets held by a single representative household, the next paragraph reviews the issues related to aggregation across households and firms.

Consistent aggregation across individual consumers generally requires the highly restrictive assumption, due to Gorman (1953), that all consumers have parallel and linear Engel curves, or in other words, have the same homothetic preferences up to an affine transformation.[6] Even though this condition is false for the entire economy, it often is desirable to apply the economic theory of household behavior and consumer demand to aggregate data. To do so, aggregate models commonly employ the concept of a "representative agent". In such models, decision rules and/or demand functions are developed from models of a single agent, and are estimated or tested with aggregate data. For aggregation across consumers, the existence of representative agent is equivalent to Gorman's conditions. The strong assumptions necessary to aggregate consumers are due to the existence of the familiar consumer budget constraint. Aggregation across firms differs. Because firms maximize profits (not subject to a budget constraint), they are not subject to the distribution effects which produce the need for the strong assumptions necessary to aggregate across consumers. Under perfect competition, Debreu

---

[5] For discussions of aggregation theory, see Green (1964), Samuelson and Swamy (1974), Diewert (1980) and Barnett (1981).

[6] Muellbauer (1976) generalized Gorman's conditions. Gorman's conditions result in linear Engel curves. The extension, due to Muellbauer, comes from the idea of defining the representative consumer "through the representativeness of his or her budget shares rather than the quantities or values purchased. It turns out that this allows the Engel curves to be non-linear." (Muellbauer, 1976, pg. 980) The extended result includes Gorman's (1953) result as a special case. The income variable for the implied representative agent is (mean) average income in Gorman (1953). Muellbauer's (1976) extension allows the income of the representative agent to be a function of individual incomes as well as prices. In general, this income will not be the mean of individual incomes.

(1959) showed that a group of optimizing firms can be treated as a single profit maximizing (representative) firm, subject to the sum of the production sets of the individual firms. Barnett (1987) discusses aggregation across firms in greater detail. In empirical research, testable empirical propositions implied by representative agent models are often rejected. These rejections either suggest rejection of neoclassical demand theory or of the maintained hypothesis that the data generating process for the aggregate is the same as the (unobservable) process for an individual economic agent.[7] Some implications of violations of this assumption are discussed below, in the final section of this paper, titled "Difficulties and Extensions of Aggregation and Statistical Index Number Theory".

General Conditions for Use of Aggregation and Statistical Index Number Theory

Monetary aggregates based on aggregation and statistical index number theory are based on the same set of assumptions as other commonly used macroeconomic aggregates such as Personal Consumption Expenditures (PCE), Gross Domestic Product (GDP), and their dual prices, the PCE Deflator and the GDP Deflator, all of which are produced by the Department of Commerce. The construction of any macroeconomic aggregate (monetary, consumption, output, or otherwise) can be justified only under certain assumptions, which are the same regardless of the type of data being aggregated. Monetary aggregates that are based on this theory can therefore be interpreted in exactly the same way as (for example) the Department of Commerce aggregates.[8]

---

[7] The assertion that macroeconomic models should embed decision rules obtained from the solution of representative agent optimzation problems has been controversial. See for example the exchange between Lucas and Sargent (1978) and Friedman (1978), or Ando (1981).

[8] The recent revisions in the Department of Commerce aggregates reflect two improvements. First, the index number formula has been improved by switching from a non-superlative (Laspeyres) index number to a superlative (Fisher Ideal) index number. Second, the index has been switched from a fixed base to a chain formula. The monetary indices in Anderson, Jones, and Nesmith (1997) are chained superlative (Törnqvist-Theil formula) indices. Thus, as will be rigorously shown in the remaining sections of this paper, the monetary services indices (MSI) have the same statistical properties as the Department of Commerce aggregates. For comments on the revision of the Department of Commerce aggregates, see Triplett (1992).

The general assumptions necessary for the aggregation of a block (group) of economic decision variables are as follows: (1) existence of a theoretical aggregator function defined over the block of decision variables, or in other words the existence of a subfunction over the block which can be factored out of the economic agent's decision; (2) efficient allocation of resources over the components of the factorable block and; (3) no quantity rationing within the factorable block. If the underlying data being aggregated have been previously aggregated across agents, an additional assumption, (4) the existence of a representative agent, is required.[9]

Although these are the minimal conditions for theoretically rigorous aggregation, they are not sufficient to apply the major tools and results of microeconomic demand theory (such as Slutsky equations or elasticities) to the analysis of the aggregates. In order to use microeconomic theory to study the behavior of the aggregates, we need to make additional assumptions about the structure of the model from the which the aggregator functions are derived.[10] For example, aggregator functions are often derived from neoclassical models of utility (proft) maximization, or expenditure (cost) minimization. In these models, weak separability of the objective function (distance, utility, production, expenditure, or cost function) will be the main requirement for aggregation over the weakly separable block of decision variables.

The theoretical assumptions in this paper are significantly less general than the above assumptions. The less general assumptions both facilitate exposition and provide the reader with the strongest and most elegant linkages between monetary aggregation and microeconomic theory. We discuss the use of aggregation theory in monetary economics in the context of a general neoclassical model of consumer demand. Specifically, a price taking representative

---

For the precise definition of a chained superlative index number, see the section of this paper titled "Statistical Index Numbers".

[9] If a representative agent does not exist over all individuals, one may nevertheless exist over subgroups of individuals, in which case there may need to be separate aggregates for different groups of individuals.

[10] These assumptions are required because condition (1), that a factorable subfunction exist, does not restrict the agent's decision to a rational optimizing microeconomic decision.

consumer is assumed to maximize an intertemporal utility function in which current period monetary assets are weakly separable from other goods and leisure, subject to a set of multi-period budget constraints. This less general but more familiar model is sufficient to allow aggregation of current period monetary assets: the weak separability assumption implies the existence of a theoretical aggregator function defined over current period monetary assets; utility maximization implies that the allocation of resources over the weakly separable block will be efficient; and quantity rationing has been ruled out.

Under the assumptions of this model, statistical index number theory can be used to track the implied monetary aggregates. The assumption that households are price takers is sufficient to allow statistical index numbers to be constructed from the observable user costs (prices) and asset stocks of the monetary assets.[11] The theory provides strong reasons to use chained superlative statistical index numbers, which are formally defined in a section of the paper titled "Statistical Index Numbers".

The microeconomic foundations of monetary aggregation can be illustrated with models other than consumer utility maximization. Barnett (1987) discusses monetary aggregation theory in the context of a profit maximizing manufacturing firm that produces multiple products. In this model the firm maximizes profit subject to a production function that contains monetary assets as an input. If the structure of the production function is such that current period monetary assets are weakly separable from all other inputs, then there exists a theoretical aggregator function defined over current period monetary assets. Further, profit maximization implies that the allocation of resources over the weakly separable block will be efficient, and once again quantity rationing is ruled out. Under the additional assumption that the firm is a price taker in the factor market for monetary assets, observable user costs and asset stocks can be used to construct statistical index numbers that track the monetary aggregates, and these user costs will be identical to the user costs derived from the household model.[12] Additional generalizations of these models (which maintain a strong connection to microeconomic theory) are possible. In particular, utility (profit) maximization can be replaced by expenditure (cost) minimization versions of these models. Expenditure (cost) minimization will guarantee that allocation of resources over the weakly separable blocks is efficient. We do not pursue these models further in this paper.

The General Neoclassical Intertemporal Consumer's Choice Problem

We begin by describing a representative agent's intertemporal decision problem when monetary assets are included in the agent's utility function; the specific form of the problem is due to Barnett (1978). One argument for including money in the utility function is that money is used to facilitate exchange. A variety of general equilibrium models provide mechanisms that cause money to have positive market value in general equilibrium without recourse to devices such as cash-in-advance constraints (see Duffie, 1990). Arrow and Hahn (1971) showed that, if money has positive value in general equilibrium, then there exists a derived utility function containing money.[13] The intuition is straightforward. Monetary assets have positive value if economic agents willingly forego some amount of another good or service (either now or in the future) to hold monetary assets now. In this case, agents who hold stocks of monetary assets sacrifice future consumption of other goods and services, due to the foregone interest they could

---

[11] In some cases, relaxing the price taking assumption may require the use of marginal or shadow prices, as suggested in Diewert (1980). One additional problem, is that the existence of a representative firm in Debreu's (1959) proof depends on the assumption of perfectly competitive markets.

[12] In addition to these demand side aggregates, a supply side aggregate can be derived as in Barnett (1987).

[13] Although any motive for holding money is equivalent to the existence of a derived utility function containing money, the form of the utility function cannot be used to uniquely identify the reason money is valued. Feenstra (1986) derives the utility functions produced from several microeconomic models of money, including the cash in advance model (see Fischer, 1974, Philips and Spinnewyn, 1982, and Poterba and Rotemberg, 1987).

have received on an asset with a higher rate of return. Arrow and Hahn's results suggest that any model that does not include money in the utility function but produces a motive for holding money in equilibrium is functionally equivalent to a model that does include money in a derived utility function. Hence, no generality in modeling is lost (or gained) by including money in the utility function.

We begin by assuming that in each period the representative agent maximizes intertemporal utility over a finite planning horizon of T periods.[14] The agent's intertemporal utility function in any period t is

$$U(m_t, m_{t+1}, \ldots, m_{t+T}; q_t, q_{t+1}, \ldots, q_{t+T}; l_t, \ldots, l_{t+T}; A_{t+T})$$

where,

$m_s = (m_{1s}, \ldots, m_{ns})$ is a vector of real stocks of n monetary assets,

$q_s = (q_{1s}, \ldots, q_{ms})$ a vector of quantities of m non-monetary goods and services,

l_s is the desired number of hours of leisure, and

A_{t+T} is the real stock of a benchmark financial asset in the final period of the planning horizon at date $t+T$,

for all $s \in \{t, t+1, \ldots, t+T\}$

The representative agent is assumed to reoptimize in each period t, choosing values of $(m_t, \ldots, m_{t+T}; q_t, \ldots, q_{t+T}; l_t, \ldots, l_{t+T}; A_t, \ldots, A_{t+T})$ that maximize the intertemporal utility function subject to a set of $T+1$ multiperiod budget constraints. The set of multiperiod budget constraints, indexed by $s \in \{t, t+1, \ldots, t+T\}$, are

$$\sum_{i=1}^{m} p_{is} q_{is} = w_s L_s + \sum_{i=1}^{n} \left[(1 + r_{i,s-1}) p^*_{s-1} m_{i,s-1} - p^*_s m_{i,s} \right] + \left[(1 + R_{s-1}) p^*_{s-1} A_{s-1} - p^*_s A_s \right]$$

[14] See Barnett (1987), pp. 116-20.

where,

p_s^* is a true cost of living index

$p_s = (p_{1s},\ldots, p_{ms})$ is a vector of prices for the m non-monetary goods and services,

$r_s = (r_{1s},\ldots, r_{ns})$ is a vector of nominal holding period yields on the n monetary assets,

R_s is the nominal holding period yield on the benchmark asset,

w_s is the wage rate,

A_s is the real quantity of a benchmark asset which appears in the utility function only in the final period $t+T$,

and L_s is the number of hours of labor supplied,

for all $s \in \{t, t+1,\ldots, t+T\}$

Leisure time consumed by the household during each period is $l_s = H - L_s$, where H is the total number of hours in a period.

We assume the existence of the true cost of living index p_s^*, which can be shown to be equivalent to assuming that non-monetary goods and services are blockwise weakly separable in the current period from other decision variables in the model. In this model, all of the services provided to the agent by monetary assets, except for the intertemporal transfer of wealth, have been absorbed into the utility function. The benchmark asset, A_s, does not appear in the utility function, except in the final period. The agent therefore uses the benchmark asset only for the purpose of transferring wealth from one period to another, it does not furnish any other services (monetary or otherwise) to the agent except in the final period of the planning horizon.

To simplify notation let $x_t = (m_{t+1},\ldots,m_{t+T}; q_t ,\ldots,q_{t+T}; l_t ,\ldots,l_{t+T}; A_{t+T})$, and we remind the reader that $m_t = (m_{1t},\ldots,m_{nt})$. Let $m_t^* = (m_{1t}^*,\ldots,m_{nt}^*)$ and $x_t^* = (m_{t+1}^*,\ldots,m_{t+T}^*; q_t^*,\ldots,q_{t+T}^*; l_t^*,\ldots,l_{t+T}^*; A_{t+T}^*)$ denote the solution to the agent's maximization problem, or in other words, let m_t^* be the optimal holdings of current period monetary assets, and let x_t^* be the optimal holdings of all other decision variables in the model. Writing the utility function $U(m_t, \ldots, m_{t+T}; q_t, \ldots, q_{t+T}; l_t, \ldots, l_{t+T}; A_{t+T})$ as $U(m_t, x_t)$, the first-order conditions of this model imply that the marginal rate of substitution between current period monetary assets i and j evaluated at the optimum is

$$\frac{\partial U(m_t, x_t)/\partial m_{it}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}}{\partial U(m_t, x_t)/\partial m_{jt}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}} = \frac{p_t^* \dfrac{R_t - r_{it}}{1 + R_t}}{p_t^* \dfrac{R_t - r_{jt}}{1 + R_t}}.$$

The first order conditions also imply that the marginal rate of substitution between the current period monetary asset i and the current period non-monetary good k at the optimum is

$$\frac{\partial U(m_t, x_t)/\partial m_{it}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}}{\partial U(m_t, x_t)/\partial q_{kt}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}} = \frac{p_t^* \dfrac{R_t - r_{it}}{1 + R_t}}{p_{kt}}.$$

A general relationship in microeconomic optimization is that, at the optimal solution, marginal rates of substitution between goods will be equal to relative prices. In these expressions,

$p_t^* \dfrac{R_t - r_{it}}{1 + R_t}$ appears as the "price" (or opportunity cost) of the current period monetary asset i.

Barnett (1978) proved this intuition formally, and we will discuss this result in more detail in the following section.

User Costs of Monetary Assets and the Durable Goods Aggregation Problem

In the model we have been discussing, monetary assets are treated as durable goods. Monetary assets appear in the utility function, and provide services to the agent. They are viewed as depreciating, but not fully over the period (this approach is similar to modeling durable physical assets). To aggregate over stocks of durable monetary assets, we need to derive equivalent rental prices, or user costs, for the services provided by a unit of each monetary asset.

Diewert (1974, 1980) discusses the general procedure for constructing the user cost of a durable good (or physical capital asset) from the purchase price of the good, the depreciation rate of the good, and a discount factor. If the agent bought one unit of a durable good, and then sold the non-depreciated part of that unit at the end of the period, the difference between the purchase price of the unit and the discounted resale value of the non-depreciated part of the unit would represent the price of renting the unit (and hence the services of that unit) for one period. This concept is sometimes called an "equivalent" rental price because an explicit rental market may not actually exist, and hence the agent may be viewed as renting the good to himself. Let p_t and p_{t+1} be the market prices of a durable good in periods t and $t+1$ respectively, δ be the depreciation rate, and D be the discount factor. The equivalent rental price of the durable good is

$$p_t - \left(\frac{1-\delta}{1+D}\right)p_{t+1}.$$

If the depreciation rate equals unity, as it does for nondurable goods that are fully consumed within a single period, then the rental price equals the purchase price.

Barnett (1978) derived the general form of the equivalent rental price, or user cost, of monetary assets. Combining the $T+1$ budget constraints in the general intertemporal model (by solving each equation for A_s and recursively back substituting this expression starting with A_{t+T}), the general form of the discounted nominal user cost of each monetary asset i in each period $s=\{t,t+1,\ldots t+T\}$ is

$$\pi_{is} = \left[\frac{p_s^*}{\rho_s} - \frac{p_s^*(1+r_{is})}{\rho_{s+1}}\right],$$

where r_{is} is the nominal holding period yield. The discount factor, ρ_s, is defined by

$$\rho_s = \begin{cases} 1, & s=t \\ \prod_{u=t}^{s-1}(1+R_u), & t+1 \le s \le t+T \end{cases}.$$

This general form may be specialized to the (current) period t nominal user cost of monetary asset i,

$$\pi_{it} = p_t^*\left(\frac{R_t - r_{it}}{1 + R_t}\right),$$

which may be interpreted as a "price" of current period monetary assets (see Barnett, 1978, and Donovan, 1978).[15] It can then be shown that monetary asset i is implicitly assumed to depreciate at the rate

$$\delta_{it} = \frac{p_{t+1}^* - p_t^*(1 + r_{it})}{p_{t+1}^*},$$

which for non-interest bearing monetary assets (such as cash) would equal the inflation rate (see Fisher, Hudson, and Pradhan, 1993). The user cost, π_{it}, represents the equivalent rental price of the services provided by a unit of monetary asset i, the amount $m_{it}^*\pi_{it}$ represents expenditure on monetary asset i in the current period t (at the optimum), and $\sum_{i=1}^{n} m_{it}^*\pi_{it}$ is total expenditure on the services provided by the monetary assets.

Monetary Aggregator Functions, Dual User Cost Aggregates, and Two Stage Budgeting

We can develop monetary aggregates that are consistent with the solution to the representative agent's decision problem by imposing additional assumptions on the structure of the model. Assume that the intertemporal utility function is weakly separable in the block of current period monetary assets, so the utility function has the following form:

$$U[u(m_t), m_{t+1}, \ldots, m_{t+T}; q_t, q_{t+1}, \ldots, q_T; l_t, \ldots, l_T; A_{t+T}],$$

[15] The current period user cost is in the form of discounted interest foregone by holding the particular monetary asset. Discounting reflects the payment of interest at the end of the period.

which can be written as $U(u(m_t),x_t)$, where x_t was defined previously. Note that only current period monetary assets $m_t = (m_{1t},\ldots,m_{nt})$ are included in the function u, which is called the "category subutility function" (defined over current period monetary assets). The separability assumption is not symmetric, so that weak separability of current period monetary assets does not imply that any other combination of decision variables is weakly separable.

In general, weak separability of a block of decision variables implies that marginal rates of substitution between variables in the weakly separable block are independent of the quantities of decision variables outside the block, (see Goldman and Uzawa, 1964 for a discussion of weak separability). Weak separability of current period monetary assets from the other decision variables in the utility function implies that the agent's marginal rates of substitution between current period monetary assets, are equal to

$$\frac{\partial u(m_t)/\partial m_{it}}{\partial u(m_t)/\partial m_{jt}},$$

which, evaluated at the optimum, reduce to the following form:

$$\frac{\partial u(m_t)/\partial m_{it}\big|_{m_t = m_t^*}}{\partial u(m_t)/\partial m_{jt}\big|_{m_t = m_t^*}} = \frac{\pi_{it}}{\pi_{jt}}.$$

Barnett (1980, 1981, 1987) used this result to show that the vector of current-period monetary assets that solves the general (weakly separable) intertemporal problem, $m_t^* = (m_{1t}^*,\ldots,m_{nt}^*)$, is exactly the same vector that would have been chosen if the agent had solved the simpler problem involving only current-period variables:

$$\underset{m}{Max}\, u(m) \text{ subject to } \sum_{i=1}^{n} m_{it}\pi_{it} = y_t,$$

where $y_t = \sum_{i=1}^{n} m_{it}^{*}\pi_{it}$ is the total expenditure on monetary services implied by the solution to the consumer's original intertemporal decision problem. Barnett's result establishes that the agent's intertemporal decision problem (under weak separability) is equivalent to a two stage budgeting problem. In the first stage, the agent chooses the total expenditure on the weakly separable block of current period monetary assets, y_t, and chooses the optimal quantities of other decision variables that are not in the weakly separable block of current period monetary assets. In the second stage, the chosen expenditure on monetary assets is optimally allocated among the individual current period monetary assets, (see Green, 1964 for a discussion of two stage budgeting). Interpreted as a two-stage budgeting problem, the second stage of the problem corresponds to maximizing the subutility function *u,* subject to the expenditure constraint implied by the first stage.

If *u* is first degree homogeneous, it can be interpreted as defining a monetary quantity aggregate.[16] The representative agent will view $u(m_t^*)$ as the optimal quantity of an elementary good, which we call "monetary services". This allows the first stage decision to be reinterpreted as the simultaneous choice of optimal quantities of monetary services and all other decision variables which are not in the weakly separable block of current period monetary assets, given prices and a budget constraint. This provides the justification for applying microeconomic demand theory to the monetary aggregate. In the remainder of this section, we formally discuss the first stage decision, and in the process define the dual opportunity cost of monetary services.

Let $M_t = u(m_t^*)$ denote the optimal quantity of monetary services chosen by the agent, and define the dual opportunity cost (or price) of monetary services, Π_t, as the minimum cost of one unit of M_t, or formally, as the unit expenditure function

$$\Pi_t = E(\pi_t, 1) = \min_m \{ \sum_{i=1}^{n} m_i \pi_{it} : u(m) = 1 \}$$

where $\pi_t = (\pi_{1t}, \ldots, \pi_{nt})$ is the vector of nominal user costs of current period monetary assets.

The consumer is assumed to maximize $U(u(m_t), x_t)$ subject to the $T+1$ multi-period budget constraints which were discussed above. The first order conditions imply that the marginal rates of substitution between current period monetary assets and non-monetary goods and services can be written as

$$\frac{\partial U(u(m_t), x_t)/\partial m_{it}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}}{\partial U(u(m_t), x_t)/\partial q_{kt}\Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}} = \frac{\pi_{it}}{p_{kt}},$$

for all monetary assets i and non-monetary goods and services k. Blockwise weak separability of current period monetary assets from other decision variables and first degree homogeneity of the category subutility function u imply that these expressions can be combined into fewer expressions of the form

$$\frac{\partial U(u, x_t)/\partial u\Big|_{\substack{x_t = x_t^* \\ u = u(m_t^*) = M_t}}}{\partial U(u, x_t)/\partial q_{kt}\Big|_{\substack{x_t = x_t^* \\ u = u(m_t^*) = M_t}}} = \frac{\Pi_t}{p_{kt}}$$

for all non-monetary goods and services k. This expression has the interpretation that the marginal rate of substitution between a good (monetary services), whose price is Π_t, and a non-monetary good k, whose price is p_{kt}, is equal to the relative price ratio of the two goods at the optimum, and that M_t is the optimal quantity of monetary services. This result can be generalized, so that all of the first order conditions involving current period monetary assets can

[16] It is only necessary for the category subutility function u to be homothetic. We simply choose a first degree homogeneous (linearly homogeneous) cardinalization of the subutility function.

be rewritten as first order conditions involving only the aggregates (M_t and Π_t) and these new first order conditions will have standard microeconomic interpretations.[17]

The final step in the argument is to show that the budget constraint can be rewritten in terms of the aggregates. As already noted, Barnett (1978) showed that the *T+1* multi-period budget constraints could be combined into a single budget constraint. It can be shown that current period monetary assets enter this single budget constraint as total expenditure on current period monetary assets, $\sum_{i=1}^{n} m_{it}\pi_{it}$. First degree homogeneity of the function *u*, implies the following identity holds:

$$\Pi_t M_t = \sum_{i=1}^{n} \pi_{it} m_{it}^* = y_t ,$$

which is called factor reversal. The product of the optimal quantity of monetary services M_t and its dual opportunity cost (or price) Π_t, represents total expenditure on current period monetary services at the optimum. The preceding identity shows that this product equals the optimal total expenditure on the services provided by the individual current period monetary assets. The budget constraint at the optimum can therefore be rewritten in terms of the aggregates, M_t and Π_t. Because these aggregates satisfy factor reversal, the dual opportunity cost can be implicitly defined by:

$$\Pi_t = \frac{\sum_{i=1}^{n} m_{it}^* \pi_{it}}{M_t} .$$

The interpretation of the agent's optimization problem as a two stage decision problem can now be restated. The above discussion demonstrates formally that the first stage decision

[17] The exceptions are the first order conditions which involve only current period monetary assets. These are the first order conditions for the second stage allocation decision.

can be interpreted as the simultaneous choice of optimal quantities of monetary services, M_t, and all other decision variables outside the weakly separable block of current period monetary assets, subject to prices and a budget constraint, where the price of monetary services is given by the dual opportunity cost, Π_t. The first stage decision produces $y_t = M_t \Pi_t$, the optimal total expenditure on current period monetary services, and this optimal expenditure is allocated among the current period monetary assets in a second stage decision. Any current period monetary portfolio substitution which does not change the level of the monetary aggregate is irrelevant to other decision variables in the model. The monetary aggregates (M_t and Π_t) contain all information about the portfolio of current period monetary assets held by the agent that are relevant to other aspects of the representative agent's decision.

STATISTICAL INDEX NUMBERS

In the previous section, microeconomic theory was used to identify monetary services (quantity) and dual user cost aggregates for current period monetary assets. In empirical research, however, usually neither the functional forms of the aggregator functions nor the values of their parameters are known. In order to estimate the aggregator functions, specific assumptions must be made about the functional forms of the expenditure and utility functions.

Statistical index numbers are approximations to aggregator functions that contain no unknown parameters are specification and estimation free,and utilize observed data on both prices and quantities. A statistical index number is said to be *exact* for an aggregator function if it tracks the aggregator function without error. We begin our discussion of index number theory with the result that the Divisia quantity index (first suggested as an index number by Divisia, 1925) is exact for the monetary services (quantity) aggregate in continuous time. The continuous time Divisia quantity index, M_t^D, is defined for monetary assets by the differential equation:

$$\frac{d\,log(\,M_t^D\,)}{dt} = \sum_{i=1}^{n} s_{it} \frac{d(log(\,m_{it}^*\,))}{dt},$$

where the expenditure shares are defined as $s_{it} = \frac{m_{it}^*\pi_{it}}{\sum_{i=1}^{n} m_{it}^*\pi_{it}}$.[18] In addition, the continuous time Divisia user cost index, Π_t^D, is defined by

$$\frac{d\,log(\,\Pi_t^D\,)}{dt} = \sum_{i=1}^{n} s_{it} \frac{d(log(\pi_{it}\,))}{dt}.$$

Note that the continuous time Divisia quantity and user cost indices satisfy factor reversal, i.e., $M_t^D\Pi_t^D = \sum_{i=1}^{n} m_{it}^*\pi_{it}$ (see Leontief, 1936).

Although the true functional form of the monetary services (quantity) aggregator function $M_t = u(m_t^*)$ is unknown, it is possible to describe its path in continuous time using only the property of homotheticity and the first order conditions for utility maximization. Let $M_t = u(m_t^*)$ to be the solution of the optimization problem, which includes only current-period values:

$$\underset{m}{Max}\, u(\,m\,) \text{ subject to } \sum_{i=1}^{n} m_i\pi_{it} = y_t\,.$$

The first order conditions imply that $\frac{\partial u(\,m_t^*\,)}{\partial m_{it}} = \lambda\pi_{it}$ for each i, where λ is the Lagrange multiplier for the budget constraint. The quantity aggregate is first degree homogeneous by assumption and has the following property known as Euler's law:

$$M_t = u(\,m_t^*\,) = \sum_{j=1}^{n} \frac{\partial u(\,m_t^*\,)}{\partial m_{jt}} m_{jt}^* = \sum_{j=1}^{n} \lambda\pi_{jt} m_{jt}^* = \lambda \sum_{j=1}^{n} \pi_{jt} m_{jt}^* \quad .$$

[18] In this paper log always denotes the base e (natural) logarithm.

To study the time path of the quantity aggregate, we take its derivative with respect to time:

$$\frac{dM_t}{dt} = \frac{du(m_t^*)}{dt} = \sum_{i=1}^{n} \frac{\partial u(m_t^*)}{\partial m_{it}} \frac{dm_{it}^*}{dt} = \sum_{i=1}^{n} \lambda \pi_{it} \frac{dm_{it}^*}{dt} = \lambda \sum_{i=1}^{n} \pi_{it} m_{it}^* \frac{d\,log(m_{it}^*)}{dt} .$$

Dividing this expression by the previous one we obtain,

$$\frac{d\,log(M_t)}{dt} = \sum_{i=1}^{n} s_{it} \frac{d(log(m_{it}^*))}{dt},$$

where $s_{it} = \frac{\pi_{it} m_{it}^*}{\sum_{i=1}^{n} \pi_{it} m_{it}^*}$ is the expenditure share of good i in period t. This expression is the growth rate of the Divisia quantity index. Thus, the continuous time Divisia quantity index is exact for the quantity aggregate, and is a direct implication of economic theory, rather than an approximation. Hulten (1973) proved that the resulting line integral that solves the differential equation for the index is path independent under the previously maintained assumption of weak separability.

Although the Divisia quantity index is exact for arbitrary quantity aggregators in continuous time, in discrete time there is no index number which is exact for arbitrary aggregator functions. Consequently, we must rely on an approximation. This leads to the definition of a *locally flexible functional form* as a function that can provide a local second order approximation to an arbitrary discrete time aggregator function.[19] Diewert (1976) showed that there exists a class of statistical index numbers, which he called superlative, that are exact for such flexible functional forms. Thus, superlative statistical index numbers can provide second-order approximations to arbitrary, unknown aggregator functions in discrete time. Provided that price and quantity changes are small, different superlative indices will remain close to each other, because of their second order approximation properties.

An index is said to be *chained* if the prices and quantities used in the index number formula are the prices and quantities of adjacent periods, and is said to be *fixed base* if the prices and quantities used in the index number formula are those of current and base periods. When an index number is chained, the center of the second order approximation moves such that the remainder term is relative to the changes between successive periods, rather than from the current period back to the fixed base period. Chained indices will provide better approximations provided that changes in prices and quantities in adjacent periods are smaller than changes in prices and quantities relative to a fixed base period; see Diewert (1978). All of the statistical index numbers presented in this paper are in their chained forms, as are the indices in the MSI database which are described in Anderson, Jones, and Nesmith (1997).

Many familiar index numbers are contained in the class of Diewert superlative indices. Any member of the Diewert superlative class provides a second order approximation to the true economic aggregate, and all Diewert superlative index numbers are equivalent up to their second order expansion terms.[20] In particular, Diewert (1976) showed that the Fisher ideal index is exact for a homogeneous quadratic functional form, see also Konüs and Byushgens (1926). The Fisher ideal quantity index (for the monetary aggregation case), M_t^F, is defined by

$$M_t^F = M_{t-1}^F \sqrt{\frac{\sum_{i=1}^{n} m_{it}^* \pi_{it}}{\sum_{i=1}^{n} m_{i,t-1}^* \pi_{it}} \cdot \frac{\sum_{i=1}^{n} m_{it}^* \pi_{i,t-1}}{\sum_{i=1}^{n} m_{i,t-1}^* \pi_{i,t-1}}}\ .$$

[19] Barnett (1983) showed that the mathematical definition of a second order approximation is equivalent to Diewert's (1971) definition. Flexible functional forms have the property that they can attain arbitrary elasticities of substitution at a single point.

[20] Diewert (1978,1980) provides theorems which suggest that using the chain principle will minimize the differences between various index number formulas, because the changes in prices and quantities will generally be small between adjacent periods. These theorems are based on numerical analysis and do not require optimization.

The Fisher ideal index is the geometric mean of the well known Paasche and Laspeyres quantity index numbers that have been the basis for many government produced aggregates. Paasche and Laspeyres index numbers can be shown to produce only first order approximations to the underlying quantity aggregate.

One widely-used superlative index number is the Törnqvist-Theil discrete time approximation to the continuous time Divisia quantity index. For monetary aggregation, the index is defined as M_t^{TT} :

$$M_t^{TT} = M_{t-1}^{TT} \prod_{i=1}^{n} \left(\frac{m_{it}^*}{m_{i,t-1}^*}\right)^{\frac{1}{2}(s_{it}+s_{i,t-1})}$$

and in log changes

$$\Delta \log(M_t^{TT}) = \sum_{i=1}^{n} \bar{s}_{it} \Delta \log(m_{it}^*),$$

where the average expenditure shares are $\bar{s}_{it} = \frac{1}{2}(s_{it} + s_{i,t-1})$ for all i.[21] Diewert (1976) demonstrated that the Törnqvist-Theil index is exact for the translog flexible functional form, and thus it can be used to track the unobservable true monetary aggregate without error up to the second order. The user cost index which is dual to the Törnqvist-Theil monetary services (quantity) index, Π_t^{Dual} is defined (implicitly) by

$$\Pi_t^{Dual} = \Pi_{t-1}^{Dual} \cdot \left(\frac{\sum_{i=1}^{n} \pi_{it} m_{it}^* / \sum_{i=1}^{n} \pi_{i,t-1} m_{i,t-1}^*}{M_t^{TT} / M_{t-1}^{TT}} \right).$$

The Törnqvist-Theil monetary services (quantity) index and its dual user cost (price) index can therefore be used as high quality statistical approximations of the true, but unknown, aggregates which were discussed in the preceding section.

[21] Mathematically, the Törnqvist-Theil discrete time index is the Simpson's rule approximation of the continuous time Divisia index.

THE FLOW OF MONETARY SERVICES AND THE STOCK OF MONETARY WEALTH

In the preceding section, we developed the Törnqvist-Theil index as a measure of the monetary service flow in the economy. There is a demand for and supply of monetary service flows, and changes in the price of monetary services will affect the demand and supply for all other goods purchased by the representative agent. In addition to these substitution and income effects, there may be wealth effects associated with monetary assets. In this section, we explicitly derive an expression for the stock of monetary wealth as the discounted present value of expenditure on monetary service flow, and discuss a possible quantitative measure of the concept.

We discussed above Barnett's (1978, 1987) result that the multi-period budget constraints for the intertemporal decision, indexed by $s \in \{t, t+1, \ldots t+T\}$,

$$\sum_{i=1}^{m} p_{is} q_{is} = w_s L_s + \sum_{i=1}^{n} \left[(1 + r_{i,\tau-1}) p^*_{s-1} m_{i,s-1} - p^*_s m_{i,s} \right] + \left[(1 + R_{s-1}) p^*_{s-1} A_{i,s-1} - p^*_s A_s \right]$$

could be combined into a single budget constraint. In this single budget constraint, monetary assets enter the single budget constraint through the term:

$$V_t = \sum_{s=t}^{\infty} \sum_{i=1}^{n} \left[\frac{p^*_s}{\rho_s} - \frac{p^*_s (1 + r_{is})}{\rho_{s+1}} \right] m_{is} = \sum_{s=t}^{\infty} \sum_{i=1}^{n} \pi_{is} m_{is}$$

where the discount factor, ρ_s

$$\rho_s = \begin{cases} 1 & s = t \\ \Pi_{u=t}^{s-1} (1 + R_u) & t+1 \le s \le t+T \end{cases}$$

and the discounted nominal user cost, π_{is}, were discussed previously.

The economic interpretation of V_t is relatively straightforward. Letting T go to infinity and evaluating V_t at the optimum yields,

$$V_t = \sum_{s=t}^{\infty}\sum_{i=1}^{n}\pi_{is}m_{is}^{*} = \sum_{s=t}^{\infty}y_s ,$$

where, y_s is the discounted expected total expenditures on monetary assets in period s. V_t can thus be interpreted as the discounted present value of all current and future expenditure on monetary services, and is the stock of monetary wealth.

Unfortunately, V_t is an infinite forward sum of discounted expenditures and hence cannot be directly computed. In order to use this definition of the stock of monetary wealth, we assume that economic agents form static expectations of the future price and own rate variables. Specifically, the assumption of static expectations means assuming that the agent expects all future interest rates including the benchmark rate to equal current interest rates (i.e. $r_{it} = r_{is}$, and $R_t = R_s$ for all $s \in \{t, t+1, t+2, \ldots\}$), and that the expected optimal holdings of all monetary assets in all future periods equal current holdings, $m_{it}^{*} = m_{is}^{*}$, for all $s \in \{t, t+1, t+2, \ldots\}$. Recall that the benchmark rate is the yield on an asset that furnishes no monetary services, and is default risk-free. Under this assumption, Barnett (1991) has shown that the stock of monetary wealth is equal to the Rotemberg currency equivalent (CE) index:

$$CE_t = p_t^{*}\sum_{i=1}^{n}\frac{R_t - r_{it}}{R_t}m_{it}^{*}$$

(see Rotemberg, 1991, and Poterba, Rotemberg and Driscoll, 1995). In this case, the CE index is a measure of the stock of monetary wealth and can be used to study the wealth effects of money.

Note that it is possible for both the Törnqvist-Theil index and the CE index to be contained in the same model because they are measures of different concepts. The Törnqvist-Theil index measures the flow of monetary services; the discounted present value of current and future expenditures on monetary services equals the stock of monetary wealth, which can be measured (under certain assumptions) by the CE index. Equivalently, the Törnqvist-Theil index is a measure of the demand for monetary service flow and the CE index is a measure of a term in the household's budget constraint (see Barnett, 1991). The two indexes coincide if the category subutility function is quasi-linear in a monetary asset whose own rate is always zero.

The simple sum index, SS_t is defined by:

$$SS_t = \sum_{i=1}^{n} p_t^* m_{it}^* .$$

The simple sum index does *not*, however, provide a measure of the flow of monetary services. The reason is that its linear form would imply that the indifference curves for monetary assets were lines, and hence all monetary assets would be perfect substitutes. If the assets have different prices, perfect substitutability implies that the agent would choose a corner solution and hold only one monetary asset in equilibrium, which is clearly counterfactual.[22]

Simple sum monetary aggregates may also be interpreted as stock variables in the context of this model, but not as a stock of monetary wealth. The following relationship is useful in describing the stock concept which the simple sum index measures.

$$SS_t = \sum_{i=1}^{n} p_t^* m_{it}^* = p_t^* \sum_{i=1}^{n} \frac{R_t - r_{it}}{R_t} m_{it}^* + p_t^* \sum_{i=1}^{n} \left\{ \frac{r_{it} m_{it}^*}{1+R_t} [1 + (\frac{1}{1+R_t}) + (\frac{1}{1+R_t})^2 + \ldots] \right\} .$$

The simple sum index can, with this expression, be decomposed into two terms. The first is the CE index, which we have already argued may be interpreted as the discounted present value of current and future expenditures on monetary services under the assumption of static expectations. The second term is the discounted present value of all current and future interest received on monetary assets under the assumption of static expectations. Thus, the simple sum index is the discounted present value of expenditures on monetary services plus the discounted present value of interest income from monetary assets, under the static expectations assumption (see Barnett,

[22] The existence of a corner solution identifies the price index dual to the simple index as Leontief, i.e. the smallest user cost over the weakly separable block of monetary assets. For arguments against the use of simple sum indexes, see Fisher (1922). The reader is cautioned that the conclusion in the text assumes a

1991). The latter term is a discounted investment return flow rather than a discounted monetary service flow and thus cannot be part of the monetary capital stock.

LIMITATIONS AND EXTENSIONS OF AGGREGATION AND STATISTICAL INDEX NUMBER THEORY

The discussion in the previous sections of this paper has been based on very strong microeconomic assumptions. In particular, we have assumed (1) the existence of a representative agent, (2) blockwise weak separability of current period monetary assets, (3) homotheticity of the category subutility function, and (4) perfect certainty.[23] In this section, we will discuss violations of the assumptions and recent advances in the theory which attempt to address these problems.

Representative Agent, Weak Separability, and the "Divisia" Second Moments

The microeconomic theory of monetary aggregation, which has been discussed in this paper, is built on the maintained hypothesis of a representative agent with an intertemporal utility functionthat is weakly separable in current period monetary assets. In this case, the monetary services (quantity) aggregate is the agent's category subutility function defined over monetary assets. The demand functions and other decision rules derived for a representative agent will be exactly correct for the behavior of aggregate data if and only if all economic agents have identical preferences up to an affine transformation (see Gorman, 1953).[24]

Under the aggregation assumptions, we have demonstrated that the Törnqvist-Theil discrete time approximation to the continuous time Divisia index provides a second order approximation to the unknown economic aggregate, and thus summarizes all of the relevant

---

single representative agent and that the portfolios of all agent(s) are in equilibrium. On the latter, see Spencer (1994).

[23] We also have assumed that each period's data reflects complete portfolio adjustment by the household. For discussion, see Spencer (1994).

[24] Once again, Muellbauer (1976) has provided a generalization of Gorman's (1954) result. An additional difficulty is that we do not know which block of monetary assets is weakly separable. In particular, the construction of nested monetary aggregates such as M1, M2, M3, and L implies very strong separability assumptions. Typically, in the empirical literature, separability assumptions are simply maintained. An exception is Serletis (1987).

information for the first stage decision up to the second order expansion terms. If economic agents do not have (nearly) identical preferences, as seems likely in a large economy, then the representative agent's decision rules may not well approximate the economic processes governing the evolution of aggregate economic data. When the conditions for the existence of a representative agent and/or for the existence of weak separability are violated for some periods in the sample, then the component monetary assets $m_t^* = (m_{1t}^*, \ldots, m_{nt}^*)$ may contain economic information in addition to the information contained in the aggregate

$$M_t^{TT} = M_{t-1}^{TT} \prod_{i=1}^{n} \left(\frac{m_{it}^*}{m_{i,t-1}^*}\right)^{\frac{1}{2}(s_{it}+s_{i,t-1})}.$$

Based on the above observation, Barnett and Serletis (1991) proposed the *dispersion dependency test* as a test of the aggregation assumptions, or equivalently, as a test of aggregation error. The test is based on Theil's statistical interpretation of the Divisia index, and is based on the dispersion of the growth rates of stocks of individual assets, user costs, and expenditure shares.

The log change (growth rate) of the Törnqvist-Theil quantity index is:

$$\Delta\, log(M_t^{TT}) = \sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(m_{it}),$$

which is in the form of an average share weighted mean of the log change of component quantities (component growth rates). Theil (1967) pointed out that the growth rate of the Törnqvist-Theil quantity index has a natural interpretation as the mean of the component quantity growth rates, where the average shares induce a valid measure of probability. Thus, the growth rate of the Törnqvist-Theil quantity index is the first moment of a distribution.

The growth rate of the Törnqvist-Theil user cost index, P_t^{TT}, is in the form of an average share weighted mean of component user cost growth rates:

$$\Delta\, log(P_t^{TT}) = \sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(\pi_{it})\,.$$ [25]

Similarly, the growth rate of the Törnqvist-Theil expenditure share index, S_t^{TT}, is in the following form:

$$\Delta\, log(S_t^{TT}) = \sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(s_{it})\,.$$

Thus, the growth rates of the Törnqvist-Theil user cost and expenditure share indices can also be interpreted as the first moments of an underlying probability distribution. [26]

Theil (1967) showed that the growth rates of the three indices are related by the following identity:

$$\sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(s_{it}) + \Delta\, log(y_t) = \sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(m_{it}) + \sum_{i=1}^{n} \bar{s}_{it} \Delta\, log(\pi_{it})\,.$$ [27]

The stochastic interpretation of the Törnqvist-Theil indices as first moments can be generalized to higher moments of the underlying distributions, which are usually called Divisia higher moments. The Divisia quantity growth rate variance, K_t, is defined by:

$$K_t = \sum_{i=1}^{n} \bar{s}_{it} (\Delta\, log(m_{it}) - \Delta\, log(M_t))^2\,,$$

which is the variance of the growth rates of the individual quantities. Analogously, the Divisia user cost variance, J_t, is defined by:

---

[25] Note that this index is not the user cost index which is dual to the Törnqvist-Theil quantity index. As Theil (1967) noted the discrete time Törnqvist-Theil index is not self dual. It fails factor reversal due to the existence of an average share weighted mean of component expenditure share growth rates. As noted previously the continuous time Divisia index is self dual.

[26] Clements and Izan (1987) develop an alternative interpretation of the Divisia index. In a model of statistical Hicksian aggregation, the Divisia price index has a direct interpretation. It can be interpreted as the GLS estimate of the common trend in a price formation function provided the variances of the individual component price estimates are inversely related to their expenditure share.

[27] On average, the Törnqvist-Theil share mean will be zero, thus the Törnqvist-Theil index is almost self dual. In applications, the Törnqvist-Theil quantity index should be used with its dual user cost index, not the Törnqvist-Theil user cost index.

$$J_t = \sum_{i=1}^{n} \bar{s}_{it}(\Delta\, log(\pi_{it}) - \Delta\, log(P_t))^2 ,$$

which is the variance of the growth rates of the component user costs. The growth rates of the component user costs and quantities have a covariance, Γ_t, defined by:

$$\Gamma_t = \sum_{i=1}^{n} \bar{s}_{it}(\Delta\, log(\pi_{it}) - \Delta\, log(P_t))(\Delta\, log(m_{it}) - \Delta\, log(M_t)) .$$

The Divisia expenditure share growth rate variance, Ψ_t, is defined by:

$$\Psi_t = \sum_{i=1}^{n} \bar{s}_{it}(\Delta\, log(s_{it}) - \Delta\, log(S_t))^2 .$$

Theil (1967) showed that these four second moments are related by an identity:

$$\Psi_t = K_t + J_t + 2\Gamma_t .$$

As has been argued above, if the aggregation assumptions are false, the dispersion of the component growth rates may contain information not contained in the growth rates of the aggregates. The Divisia higher moments, which are measures of component dispersion, can then be used to detect the remaining information. Barnett and Serletis (1991) advocate the use of Divisia second moments for dispersion dependency testing.

Dispersion dependency tests based on Divisia second moments are presented in Barnett and Serletis (1991) and Barnett, Jones, and Nesmith (1995,1996). The evidence in these studies for United States monetary data suggests that Divisia second moments contain additional economic information not contained in the corresponding monetary aggregates. In other words, for at least some time periods, movements in the various data are not consistent with the movements that would be implied by a representative agent with a weakly separable utility function. In this case, Barnett and Serletis (1991) suggest that including Divisia second moments in macroeconomic models might provide a correction for the aggregation error.[28]

Homothetic Preferences

The fundamental theoretical results presented in this paper have all been derived under the assumption that the category subutility function is homothetic. If homotheticity is violated, then the aggregator functions are not the category subutility function and the unit expenditure function, and the Divisia index will not track the utility function in continuous time. In this section, we will discuss economic aggregates which are correct in general (even if homotheticity is violated), and the implications of homotheticity for index number theory.

Assuming that current period monetary assets are weakly separable from other decision variables, we can define quantity and user cost aggregator functions – the Konüs and Malmquist indices -- which are correct aggregators even in the absence of homotheticity.

Let u be the category subutility function, which is not necessarily homothetic. The monetary services (quantity) aggregate may be defined as the distance function $d(m_t^*, \tilde{u})$, which is defined implicitly by

$$u\left(\frac{m}{d(m,\tilde{u})}\right) = \tilde{u} .$$

The user cost aggregate dual to the distance function is $e(\pi_t, \tilde{u})$, which is defined by

$$e(\pi_t, \tilde{u}) = \min_m \{ \sum_{i=1}^{n} \pi_{it} m_i : u(m) = \tilde{u} \}$$ (see Barnett, 1987, and Deaton and Muelbauer, 1980).

Normalizing these quantity and price aggregates to equal one in a base period produces what are called *exact economic indices* and allows us to define the Malmquist (1953) quantity index

---

[28] If aggregation conditions do not hold, in theory any higher moment of the aggregates distribution could contain additional information. Dispersion dependency testing could therefore be extended to testing other moments.

$$M(m_t^*, m_0^*, \tilde{u}) = \frac{d(m_t^*, \tilde{u})}{d(m_0^*, \tilde{u})}$$

and the Konüs (1939) user cost index,

$$K(\pi_t, \pi_0, \tilde{u}) = \frac{e(\pi_t, \tilde{u})}{e(\pi_0, \tilde{u})}$$

where both indices are normalized to unity in period *0*, and both indices are defined specifically for the monetary case.

Although these indices are correct regardless of the homotheticity of the utility function, a shortcoming is that both depend on the reference utility level, $\tilde{u}$. Konüs (1939) showed that the user cost index can be bounded above and below at different reference utility levels. The upper bound in the case of monetary assets is a Laspeyres index given by

$$\frac{\sum_{i=1}^{n} \pi_{it} m_{i0}^*}{\sum_{i=1}^{n} \pi_{i0} m_{i0}^*} \geq K(\pi_t, \pi_0, u(m_0^*)),$$

where m_{it}^* is the optimal quantity of monetary asset *i* in period *t*, and m_{i0}^* is the optimal quantity of monetary asset *i* in the base period *0*. The lower bound is a Paasche index given by

$$\frac{\sum_{i=1}^{n} \pi_{it} m_{it}^*}{\sum_{i=1}^{n} \pi_{i0} m_{it}^*} \leq K(\pi_t, \pi_0, u(m_t^*))$$

(see Frisch, 1936, and Leontief, 1936 for related early discussions of index number theory).

It can now be seen why homotheticity is such a valuable property. The general quantity and price aggregator functions described in this section are dependent on a reference utility level, as are the related Konüs and Malmquist indices. When the category subutility function is first degree homogeneous (or homothetic), the Konüs and Malmquist indices will be independent of the reference utility level.

Under first degree homogeneity, the distance function is proportional to the utility function with the proportionality factor equal to the reference utility level. Thus, we can rewrite the Malmquist index as $M(m_t^*, m_0^*, \tilde{u}) = \frac{d(m_t^*, \tilde{u})}{d(m_0^*, \tilde{u})} = \frac{u(m_t^*) \cdot \tilde{u}}{u(m_0^*) \cdot \tilde{u}} = \frac{u(m_t^*)}{u(m_0^*)}$. First degree homogeneity also implies that $e(\pi, \tilde{u}) = e(\pi, 1)\tilde{u}$, and therefore that the Konüs index equals $K(\pi_t, \pi_0, \tilde{u}) = \frac{e(\pi_t, 1)}{e(\pi_0, 1)}$ and is independent of the reference utility level.

First degree homogeneity also implies the important result that the Konüs index is bounded by the Paasche and Laspeyres indices

$$\frac{\sum_{i=1}^{n} \pi_{it} m_{i0}^*}{\sum_{i=1}^{n} \pi_{i0} m_{i0}^*} \geq K(\pi_t, \pi_0, \tilde{u}) \geq \frac{\sum_{i=1}^{n} \pi_{it} m_{it}^*}{\sum_{i=1}^{n} \pi_{i0} m_{it}^*}$$

for any $\tilde{u}$. This is a formal statement (in the monetary case) of the often quoted result that base period weighted (Laspeyres) price indices overstate the true increase in prices, and current period weighted (Paasche) price indices understate the true change in prices, and this result depends *critically* on the assumption of homotheticity. The upward (downward) bias resulting from the use of Laspeyres (Paasche) price indices is discussed by Triplett (1992). In general, if homotheticity is violated the Paasche price index may actually exceed the Laspeyres price index, see Deaton and Muellbauer (1980).

Although homotheticity produces attractive simplifications of aggregation theory, it is implausible that any population is well characterized by an assumption of identical homothetic utility functions, Samuelson and Swamy (1974) label this a "Santa Claus" assumption. If homotheticity is implausible, how serious is the damage to aggregation theory caused by its failure? Recent research suggests that the damage is small when the Törnqvist-Theil index is used. We discussed above the ability of Diewert superlative statistical index numbers to track unknown aggregator functions when the category subutility functions are homothetic. When homotheticity is absent, the Törnqvist-Theil discrete time approximation to the continuous time Divisia index has similar tracking capabilities for the distance function. Specifically, Caves, Christensen, and Diewert (1982) proved that the Törnqvist-Theil index is superlative in the sense that it can provide a second order approximation to the Malmquist quantity index, even when homotheticity is violated. No other statistical index number is known to have this important property, and thus use of the Törnqvist-Theil index likely reduces the sensitivity of the monetary services indices to violations of the assumption of homotheticity.

Perfect Certainty

Recently, economists have become interested in constructing monetary aggregates that include assets with risky returns (see Collins and Edwards, 1994, Orphanides, Reid, and Small, 1994, and Barnett, 1994). To this point, our theoretical discussion has included only perfect certainty models even though some of the proposed monetary aggregates (MSI-L, for example) include monetary assets (money market instruments) such as Treasury bills that are capital uncertain. An extension of the model to include household's with preferences over risk is necessary.

The extension of the model to include risk neutral households is straightforward. Barnett (1994) has shown that under risk neutrality the Divisia index continues to provide a second order approximation to the unknown aggregator function in discrete time, where the user costs are defined as the expected value of the nominal current period perfect certainty user costs:

$$\pi_{it} = E_t\{ p_t^* \frac{R_t - r_{it}}{1 + R_t} \} .$$

The case of risk averse households is more difficult; Rotemberg (1991) noted that under risk aversion the Divisia index does not furnish the approximation derived by Barnett under risk neutrality because the user costs are correlated with marginal utility. As a result more general models have been considered to deal with risk aversion. Barnett and Liu (1995) produce a generalized Divisia quantity index for the monetary services flow, where the user cost is adjusted to account for risk aversion.[29] The adjustment depends on the degree of risk aversion and the covariance between the asset's rate of return and the agent's consumption stream. Empirically, Barnett and Liu (1995) find that there is negligible difference between their generalized Divisia index and the standard index for aggregates constructed over the set of monetary assets included in the official monetary aggregates. Hence, risk aversion is unlikely to be empirically important for the indices constructed in this research.

Conclusion

This paper has surveyed the microeconomic theory of monetary aggregation. In general, this theory is built from the aggregator function of a representative agent, which in the case of consumer demand requires (homothetic) weak separability of current period monetary assets from other goods in the utility function, and in the case of firm factor demand requires weak separability of current period monetary assets from other inputs in the production function. As such, its usefulness is potentially vulnerable to the (obvious) failure of these assumptions in an aggregate economy. Recent research suggests, however, that the aggregation results may be fairly robust to violations of these assumptions. Regardless, the aggregation methods that underlie the construction of the monetary services indices (MSI) are the same as those that form the basis for the Department of Commerce's methods for measuring real economic activity and for the construction of general equilibrium business cycle models. Thus, the inclusion of the MSI and their dual user cost indices in any model with other superlative chained indices (such as

[29] In addition, Barnett and Zhou (1994) derive a supply version of the model under risk aversion.

those produced by the Department of Commerce) does not require any stronger assumptions than those already implicitly accepted.

REFERENCES

Ando, Albert. "On a Theoretical and Empirical Basis of Macroeconometric Models," in J. Kmenta and J. B. Ramsey, eds., *Large-Scale Macro-Econometric Models: Theory and Practice*. Amsterdam: North-Holland, 1981.

Anderson, Jones, and Nesmith. (1997) "Building New Monetary Services Indices: Concepts, Methodology and Data," Federal Reserve Bank of St Louis, *Review*.

Arrow, K. J. and F.H. Hahn. *General Competitive Analysis*, San Francisco: Holden Day, 1971.

Barnett, William A. "The User Cost of Money," *Economic Letters*. vol 1 (1978), pp. 145-149.

______________. "Economic Monetary Aggregates: An Application of Index Number and Aggregation Theory," *Journal of Econometrics*. vol 14 n 1 (Summer 1980), pp. 11-48.

______________. *Consumer Demand and Labor Supply: Goods, Monetary Assets, and Time*. Amsterdam: North Holland Publishing Company, 1981.

______________. "Definitions of 'Second Order Approximation' and 'Flexible Functional Form'," *Economic Letters*. vol 12 (1983), pp. 31-35.

______________. "The Microeconomic Theory of Monetary Aggregation," in William A. Barnett and Kenneth J. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge, MA: Cambridge University Press, 1987.

______________. "Developments in Monetary Aggregation Theory," *Journal of Policy Modeling*. vol 12 n 2 (Summer 1990), pp. 205-257.

______________. "Reply (to Julio Rotemberg)," in Michael T. Belongia, ed., *Monetary Policy on the 75th Anniversary of the Federal Reserve System*. Boston, MA: Kluwer Academic Publishers, 1991.

______________. "Exact Aggregation Under Risk," Washington University *Working Paper Series*, Dept of Economics. Working Paper #183, (February 1994).

______________, Barry E. Jones and Travis Nesmith. "Divisia Second Moments: An Application of Stochastic Index number Theory." *International Review of Comparative Public Policy* (1996), forthcoming.

______________, Barry E. Jones and Travis Nesmith. "Time Series Cointegration Tests and Nonlinear Stationary Residuals," working paper, 1996.

______________, and Yi Liu. "Beyond the Risk Neutral Utility Function." forthcoming in Michael Belongia ed. *Proceedings of the University of Mississippi Conference on Divisia Monetary Aggregation* (title tentative), 1996.

_____________ and Apostolos Serletis. "A Dispersion Dependency Diagnostic Test for Aggregation Error: With Applications to Monetary Economics and Income Distribution," *Journal of Econometrics*. vol 43 (Jan/Feb 1990), pp. 5-34.

_____________, and Ge Zhou. "Financial Firms' Production and Supply Side Monetary Aggregation Under Dynamic Uncertainty," Federal Reserve Bank of St. Louis, *Review*. vol 76 n 2 (March/April 1994), pp. 133-165.

_____________, Melvin J. Hinich, and Warren E. Weber. "The Regulatory Wedge Between the Demand-Side and Supply-Side Aggregation-Theoretic Monetary Aggregates." *Journal of Econometrics*. vol 33 n 1/2 (Oct/Nov 1986) pp. 165-85.

Belongia, Michael T. "Weighted Monetary Aggregates: A Historical Survey," *Journal of Interantional and Comparative Economics*. vol 4 (1995), pp. 87-114.

_____________. "Measurement Matters: Recent Results from Monetary Economics Reexamined," *Journal of Political Economy*. vol 104 n5 (October 1996), pp. 1065-1083.

Caves, Douglas W., Laurits R. Christensen, and W. Erwin Diewert. "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity," *Econometrica*. vol 50 n6 (November 1982), pp. 1393-1414.

Chetty, V. Karuppan. "On Measuring the Nearness of the Near-Moneys," *American Economic Review*. vol 59 n3 (June 1969), pp. 270-281.

Clements, Kenneth W. and H. Y. Izan. "The Measurement of Inflation: A Stochastic Approach," *Journal of Business and Economic Statistics*. vol 5 n3 (July 1987), pp. 339-50.

Collins Sean, and Cheryl L. Edwards. "An Alternative Monetary Aggregate: M2 Plus Household Holdings of Bond and Equity Mutual Funds," Federal Reserve Bank of St. Louis, *Review*. vol 76 (November/December 1994), pp. 7-29.

Deaton, Angus, and John Muellbauer. *Economics and Consumer Behavior*. Cambridge, MA: Cambridge University Press, 1980.

Debreu, Gerard. Theory of Value: An Axiomatic Analysis of Economic Equilibrium. New York: Wiley, 1959.

Diewert, W. Erwin. "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function," *Journal of Political Economy*. vol 79 n3 (May/June 1971), pp. 481-507.

_____________. "Intertemporal Consumer Theory and the Demand for Durables," *Econometrica*. vol 42 n3 (May 1974), pp. 497-516.

_____________. "Exact and Superlative Index Numbers," *Journal of Econometrics*. vol 4 n 2 (May 1976), pp. 115-145.

_____________. "Superlative Index Numbers and Consistency in Aggregation," *Econometrica*. vol 46 n4 (July 1978), pp. 883-900.

_____________. "Aggregation Problems in the Measurement of Capital," in Dan Usher, ed., *The Measurement of Capital*. 1980.

Divisia, Francois. "L'Indice Monêtaire et la Thêorie de la Monnaie," *Revue d'Economie Politique*. vol 39 (1925), pp. 980-1008.

Donovan, Donald L. "Modeling the Demand for Liquid Assets: An Application to Canada," *International Monetary Fund Staff Paper*. vol 25 n 4 (December 1978), pp. 676-704.

Duffie, Darrell. "Money in General Equilibrium Theory," in Benjamin M. Friedman and Frank H. Hahn, eds., *Handbook of Monetary Economics,* Vol. 1. Amsterdam: North-Holland, 1990.

Fama, Eugene F. "Banking in the Theory of Finance," *Journal of Monetary Economics*. vol 6 n 1 (January 1980) pp. 39-57.

Feenstra, Robert C. "Functional Equivalence Between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics*. vol 17 n2 (March 1986), pp. 271-291.

Fischer, Stanley. "Money and the Production Function," *Economic Inquiry*. vol 12 n4 (December 1974), pp. 517-533.

Fisher, Irving. *The Making of Index Numbers: A Study of Their Varieties, Tests, and Reliability*. Cambridge, MA: Houghton Mifflin Company, 1922.

Fisher, Paul, Suzanne Hudson, and Mahmood Pradhan. "Divisia Indices for Money: An Appraisal of Theory and Practice," Bank of England *Working Paper Series* no.9. (April 1993).

Friedman, Benjamin. "Discussion [of Lucas and Sargent (1978)]." *After the Phillips Curve: Persistence of High Inflation and High Unemployment*. in Federal Reserve Bank of Boston Conference Series Number 19, 1978.

Friedman, M. and A. Schwartz. *Monetary Statistics of the United States: Estimates, Sources, Methods*. NewYork: Columbia University Press, 1970.

Frisch, Ragnar. "Annual Survey of General Economic Theory: The Problem of Index Numbers," *Econometrica*. vol 4 (January 1936), pp. 1-38.

Goldman, S. M. and H. Uzawa. "A Note on Separability in Demand Analysis," *Econometrica*. vol 32 (July 1964), pp. 387-398.

Gorman, W. M. "Community Preference Fields," *Econometrica*. vol 21 (January 1953), pp. 63-80.

Green, H. A. John. *Aggregation in Economic Analysis: an Introductory Survey*. Princeton, NJ: Princeton University Press, 1964.

Hancock, Diana. " The Financial Firm: Production With Monetary and Non Monetary Goods," *Journal of Political Economy.* vol 93 n 5 (October, 1985), pp. 859-80.

______________ ."A Model of the Financial Firm with Imperfect Asset and Deposit Elasticities," *Journal of Banking and Finance.* vol 10 n 1 (March 1986), pp. 37-54.

Hulten, Charles R. "Divisia Index Numbers," *Econometrica.* vol 63 (November 1973), pp. 1017-1025.

Hutt, W. H. *Keynesianism - Retrospect and Prospect: A Critical Restatement of Basic Economic Principles.* Chicago, IL: Henry Regnery Company, 1963.

Konüs, A.A. "The Problem of the True Index of the Cost of Living," *Econometrica.* vol 7 (January 1939), pp. 10-29.

Leontief, Wassily W. "Composite Commodities and the Problem of Index Numbers," *Econometrica.* vol 4 (January 1936), pp. 39-59.

Lucas, Robert E. and Thomas J. Sargent. "After Keynesian Macroeconomics" and "Response to Friedman" in *After the Phillips Curve: Persistence of Inflation and High Unemployment*. Federal Reserve Bank of Boston Conference, Series Number 19, 1978a.

Lucas, Robert E. and Thomas J. Sargent. "After Keynesian Macroeconomics" and "Response to Friedman" in *After the Phillips Curve: Persistence of Inflation and High Unemployment*. Federal Reserve Bank of Boston Conference, Series Number 19, 1978b.

Malmquist, S. "Index Numbers and Indifference Surfaces." *Tradajos de Estatistica*, vol 4 (1953), pp. 209-242.

Muellbauer, John. "Community Preferences and the Representative Consumer," *Econometrica.* vol 44 5 (September, 1976), pp. 979-999.

Niehans, Jurg. The Theory of Money Baltimore: Johns Hopkins UniversityPress, 1978.

Orphanides, Athanasios, Brian Reid, and David H. Small. "The Empirical Properties of a Monetary Aggregate that Adds Bond and Stock Funds to M2," Federal Reserve Bank of St. Louis, *Review.* (November/December 1994), pp. 31-51.

Pesek, Boris P. and Thomas R. Saving. *Money, Wealth, and Economic Theory*. New York: The Macmillan Company, 1967.

Philips, Louis and Frank Spinnewyn. "Rationality Versus Myopia in Dynamic Demand Systems," in R.L. Basmann and George F. Rhodes, eds., *Advances in Econometrics: A Research Annual*, vol 1. Greenwood, CT: JAI Press Inc., 1982.

Poterba, James M., and Julio J. Rotemberg. "Money in the Utility Function: An Empirical Implementation," in William A. Barnett and Kenneth J. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge, MA: Cambridge University Press, 1987.

Rotemberg, Julio. "Commentary: Monetary Aggregates and Their Uses,"in Michael T. Belongia, ed., *Monetary Policy on the 75th Anniversary of the Federal Reserve System*, Boston, MA: Kluwer Academic Publishers, (1991).

Rotemberg, Julio J., John C. Driscoll, and James M. Poterba. "Money, Output, and Prices: Evidence From a New Monetary Aggregate," *Journal of Business and Economic Statistics*. vol 13 n1 (January 1995), pp. 67-83.

Samuelson, Paul A. "What Classical and Neoclassical Monetary Theory Really Was," *Canadian Journal of Economics*, (February 1968). Reprinted in Robert C. Merton, ed., The Collected Scientific Papers of Paul A. Samuelson, Vol.3 Cambridge: MIT Press, 1972.

Samuelson, Paul A. and S. Swamy. "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis," *American Economic Review*. vol 64 n4 (September 1974), pp. 566-593.

Serletis, Apostolos. "Monetary Asset Separability Tests," in William A. Barnett and Kenneth J. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge, MA: Cambridge University Press, 1987.

Spencer, Peter. "Portfolio Disequilibrium: Implications for the Divisia Approach to Monetary Aggregation," *The Manchester School of Economic and Social Studies*. vol 62 n2 (June 1994), pp. 125-150.

Theil, Henri. *Economics and Information Theory*. Amsterdam: North Holland, 1967.

Triplett, Jack E. "Economic Theory and BEA's Alternative Quantity and Price Indexes," *Survey of Current* Business. vol 72 no 4 (April 1992), pp. 49-52.

Walras, L. *Elements of Pure Economics*. William Jaffé, trans. London: Allen and Unwin, 1954.