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Learning Equilibria

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LEARNING EQUILIBRIA

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Abstract. This paper employs the Hopf bifurcation theorem to prove the existence of complicated equilibrium trajectories under least squares learning in a standard version of the overlapping generations model. The periodic and quasiperiodic learning equilibria exist when the locally unique perfect foresight equilibrium is the monetary steady state, and thus are induced by the introduction of learning alone. Learning equilibria can be stable or unstable depending on higher order derivatives of the underlying utility function not specified by economic theory; examples of both attracting and repelling invariant closed curves are provided. This research confirms the intuition of some previous authors, who have suggested that stationary equilibrium trajectories under learning may differ from those under rational expectations.

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1. Introduction

Rational expectations equilibrium is often viewed as a long run concept, where agents have already learned the law of motion governing the system in which they operate. A common research question, asked increasingly often in the recent literature, is how this learning takes place, and more importantly, if it makes any difference for inferences from dynamic general equilibrium models whether the learning is explicitly modeled. Some have argued that simple and plausible learning rules exist that do converge to rational expectations equilibria under general conditions, and that therefore the rational expectations concept is appropriate for naming the potential long run outcomes of a dynamic model. Lucas (1987, p. 231) suggests, for instance, that learning "lend[s] plausibility" to the theory of rational expectations, and Marcet and Sargent (1988, p. 171) comment that:

> It is remarkable that the "adaptive" least squares learning schemes are attracted to rational expectations equilibria, and that, naive and backward-looking as they are, they provide promising leads on superior ways for us economists to compute rational expectations equilibria. It is also comforting that these adaptive mechanisms seem not to be attracted to "bad" bubble equilibria as limit points.

In this view, the purpose of explicitly modeling learning is to select a single rational expectations equilibrium as the likely outcome of a model under actual expectations, thus helping to resolve the problem of multiplicity.[1]

On the other hand, some authors, such as Grandmont and Laroque (1990b, p. 2), have suggested that there might be more to the problem, in the sense that "... learning might generate endogenously complex nonlinear trajectories, along which forecasting errors would never vanish." In other words, instead of viewing the potential outcomes under learning as either convergence to rational expectations or explosive nonconvergence, one should recognize that these (generally speaking) nonlinear learning systems may possess attractors other than the steady state. This paper provides an example of such an

[1]See also the work on expectational stability by Evans (1985) and Evans and Honkapohja (1991).

outcome. In a standard version of the overlapping generations model with learning as represented by least squares autoregression, it is shown that the system need not be attracted to a rational expectations equilibrium as a limit point. The alternative attractors are dubbed *learning equilibria.*

The main result of the paper uses the Hopf bifurcation theorem to prove the existence of periodic and quasiperiodic trajectories that do not exist under perfect foresight. These equilibria occur near the monetary steady state and depend entirely on fundamental factors.[2] In a learning equilibrium, the system follows a complicated dynamic path, yet under the assumptions of the model, no perfect foresight *periodic* equilibria exist. Hence, explicitly specifying the learning process underlying rational expectations can imply new potential outcomes for the model that do not exist when it is assumed that "learning is complete," as is common practice.

The results of the paper can also be interpreted as demonstrating that learning alone can lead to complicated dynamics in a simple overlapping generations model. Several authors have shown that various propagation mechanisms can lead to endogenous competitive business cycles in this model under perfect foresight; examples in this class include Farmer (1987), Reichlin (1987), and Grandmont (1985). The analysis in this paper shows that a learning assumption can provide another route to endogenous cycles in a dynamic general equilibrium setting. The periodic equilibria are induced by learning alone in the sense that they are produced in a setting where it is known that no cycles exist under perfect foresight.

The bifurcation parameter is the gross rate of growth of the money stock, which is the only policy parameter in the model. For any utility function meeting general prescriptions, there is a rate of money growth "sufficiently high," made precise in the

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[2]That is, the equilibria under discussion are not induced by the introduction of frivolous variables into the agents' forecast functions, so these are not sunspot equilibria of the type discussed by Azariadis (1981) and others.

analysis, that causes a Hopf bifurcation to occur. The dynamic outcomes of the model may then lie entirely on an invariant closed curve. Hence, this is a model where some monetary policy rules are bad in the sense that they induce aggregate fluctuations, while other policy rules are good in the sense that they allow convergence to the monetary steady state. A Hopf bifurcation cannot occur under perfect foresight because in that case the dynamics of the model are described by a first order difference equation which cannot give rise to complex characteristic roots.

The next section outlines a version of the overlapping generations model under least squares learning. Section three discusses the Hopf bifurcation and presents the necessary and sufficient conditions for such an event to occur near the monetary steady state. Section four presents some results for simulated systems. Section five shows how forecast errors can be eliminated in a periodic learning equilibrium, and the final section provides a summary, draws conclusions, and suggests areas for further research.

2. An overlapping generations model

A standard version of the overlapping generations model is employed; for a detailed account see Sargent (1987). The presentation here is in terms of the gross inflation rate—the inverse of the gross rate of interest—in order to facilitate the introduction of least squares learning.

An infinite horizon economy is populated by agents who live for two periods. The agents are indexed $n = 1 \dots N(t)$, where $N(t)$ indicates the number of agents born at time t. There is no storage and there are no bequests. Agent n born at time t is endowed with $w_t^n(t)$ at time t and $w_t^n(t+1)$ at time $t+1$; taxed at $\tau_t^n(t)$, $\tau_t^n(t+1)$; and consumes $c_t^n(t)$, $c_t^n(t+1)$. The agents maximize utility $U_t^n[c_t^n(t),\ c_t^n(t+1)]$ where (i) indifference curves are convex; (ii) more is preferred to less; (iii) $U_{t1}^n / U_{t2}^n \longrightarrow \infty$ as $c_t^n(t)/c_t^n(t+1) \longrightarrow 0$ and $U_{t1}^n / U_{t2}^n \longrightarrow 0$ as $c_t^n(t)/c_t^n(t+1) \longrightarrow \infty$; and (iv) $c_t^n(t)$, $c_t^n(t+1)$ are normal goods. The government consumes $G(t) \geq 0$ units of the consumption good, levies lump–sum taxes $\tau^n_t(t)$, $\tau^n_{t-1}(t)$, and lends (borrows) $L^g(t) > 0$ (< 0) by negotiating one period loans at time t. The loans are repaid the next period in the amount of $R(t)L^g(t)$. The solution to the agent's problem requires that $U^n_{t2}/U^n_{t1} = 1/R(t)$. The individual agent's supply of savings $\ell^n_t = f^n_t[w^n_t(t) - \tau^n_t(t),\ w^n_t(t+1) - \tau^n_t(t+1),\ R(t)]$ is increasing in the first period after tax endowment, decreasing in the second period after tax endowment, and, generally speaking, ambiguous in the gross rate of interest $R(t)$.

In particular, the standard assumptions on utility given by (i)–(iv) are in general insufficient to require savings to depend positively on the interest rate in the overlapping generations model. An auxiliary assumption that savings does depend positively on the interest rate amounts to an assumption that $c^n_t(t)$ and $c^n_t(t+1)$ are gross substitutes.[3] The work of Sonnenschein (1973) and others has shown that any continuous function could be an excess demand function; full generality in this regard would allow complicated dynamics under rational expectations, and possibly even chaos, as has been shown by Grandmont (1985). Therefore, the simple overlapping generations model has, as a logical possibility, complicated and chaotic equilibrium paths based on perfect competition, perfect foresight and utility maximization.

However, in this paper, cyclic and chaotic perfect foresight equilibrium paths are ruled out by the following:

*A*1. *Savings depends positively on the rate of interest.*

Because this assumption is made and maintained throughout the paper, the complicated equilibrium paths derived will be induced by the introduction of learning alone.

The model can be completed by introducing a role for currency. Denote the nonnegative price level by $P(t)$, and suppose that at time $t = 1$ the government owns

[3]See, for instance, the discussion in Sargent (1987) and the references cited therein.

$H(1) > 0$ units of currency. The government budget constraint is

$$G(t) = R(t-1)L^g(t-1) - L^g(t) + [H(t) - H(t-1)]/P(t) + \sum_n \tau^n_{t-1}(t) + \sum_n \tau^n_t(t).$$

Arbitrage requires that the rate of return to loans equals the rate of return to holding currency, $R(t) = P(t)/P(t+1)$. The government budget constraint can therefore be rewritten

$$G(t) = R(t-1)\Big[L^g(t-1) - H(t-1)/P(t-1)\Big] - \Big[L^g(t) - H(t)/P(t)\Big] + \sum_n \tau^n_{t-1}(t) + \sum_n \tau^n_t(t)$$

where $H(0) = 0$ and $L^g(0) = 0$. Loan market equilibrium requires

$$\sum_n f^n_t\Big[R(t),\ w^n_t(t) - \tau^n_t(t),\ w^n_t(t+1) - \tau^n_t(t+1)\Big] = H(t)/P(t) - L^g(t).$$

While this more general framework could be retained in what follows, some simplifying assumptions will be applied in order to keep the thrust of the paper clear, and also to maintain comparability to other work—particularly Marcet and Sargent (1989b). First, suppose population is fixed and normalized to unity, and secondly set $L^g(t) = 0\ \forall\ t$. Use the arbitrage condition $R(t-1) = P(t-1)/P(t)$ to obtain

$$H_t/P_t = f[P_t/F_tP_{t+1}]\ . \tag{1}$$

where $f[\cdot]$ is the aggregate savings function, and where parenthetical notation is replaced with subscript notation for convenience. The following assumption is employed:

A2. Aggregate savings is positive.

The process for currency creation is given by

$$H_t = \theta H_{t-1} \tag{2}$$

where $\theta > 1$ is the gross rate of currency growth, a policy rule chosen by the authorities.[4]

[4]The model given by (1) and (2) corresponds to Sargent's (1987) example 7.6, p. 271, or equally well to Grandmont (1985) with the addition of the maintained gross substitutes assumption *A*1. Marcet and Sargent (1989b) use a slightly different version of the government budget constraint, $H_t = \theta H_{t-1} + \xi P_t$, where ξ is the fixed real government deficit; identical results can be obtained with this formulation (as will be shown in section four, example 2) viewing ξ as the bifurcation parameter. The system described by equations (1) and (2) can also be viewed as a version of the Cagan model of hyperinflation.

This paper is concerned with the analysis of the system defined by (1) and (2) under least squares learning.

The model can be closed under perfect foresight by introducing the following notation:

(3a) $$F_t P_{t+1} = \beta_t P_t$$

(3b) $$\beta_t = P_{t+1}/P_t.$$

Given a specification for agents' utility functions, the savings function can be derived and the system can then be reduced to a nonlinear difference equation in β_t.[5]

Learning can be introduced into the model by assuming that the forecast price is a function of past prices

$$F_t P_{t+1} = \varphi(P_t, \ldots, P_0).$$

The least squares autoregression used to describe learning in this paper is given by

(4a) $$F_t P_{t+1} = \beta_t P_t$$

(4b) $$\beta_t = \left[\sum_{s=1}^{t-1} P_{s-1}^2\right]^{-1}\left[\sum_{s=1}^{t-1} P_{s-1}P_s\right].$$

That is, agents form expectations by computing a first order autoregression using data available through time $t-1$. Versions on this theme have recently been studied extensively by Marcet and Sargent (1989abc) and Sargent (1991). In order to make the system under learning tractable, the formula in (4ab) can be written recursively as:[6]

(5a) $$\beta_t = \beta_{t-1} + C_{t-1}^{-1} P_{t-2}\left[P_{t-1} - P_{t-2}\beta_{t-1}\right]$$

(5b) $$C_{t-1} = \left[\sum_{s=1}^{t-1} P_{s-1}^2\right].$$

Combining (5ab) with (1) and (2) implies

[5]See section four for some examples of solutions under perfect foresight.

[6]See Harvey (1981) or Ljung and Soderstrom (1983) for details of deriving recursive least squares formulas.

(6) $$\beta_t = \beta_{t-1} + g_{t-1}\left[\theta \frac{f\,(\beta_{t-2}^{-1})}{f\,(\beta_{t-1}^{-1})} - \beta_{t-1}\right]$$

where

$$g_{t-1} = P_{t-2}^2 \left[\sum_{s=1}^{t-1} P_{s-1}^2\right]^{-1}.$$

The remainder of the paper is primarily concerned with the analysis of this system. Equation (6) has a fixed point at $\beta = \theta$; this is the monetary steady state of the model. Assumption 2 guarantees that money demand is positive at this stationary equilibrium. There is also an autarkic steady state in the model; this will be denoted by $\beta = \bar{\beta} > \theta$.

3. Learning equilibria

The heart of the analysis concerns a Hopf bifurcation near the monetary steady state. Let the characteristics of the economy, including the gross rate of money growth, the endowments, and the parameters of the utility function, be indexed by a single parameter μ in some open interval of μ_0, the critical value at which the Hopf bifurcation occurs. Denote the interval by (μ_a, μ_b), where $\mu_a < \mu_0 < \mu_b$.

The dynamics of the economy are described by

(7a) $$\beta_t = \beta_{t-1} + g_{t-1}\left[\theta \frac{f\,(\beta_{t-2}^{-1})}{f\,(\beta_{t-1}^{-1})} - \beta_{t-1}\right]$$

(7b) $$\beta_{t-1} = \beta_{t-1}$$

(7c) $$g_t = \left[g_{t-1}^{-1}\left[\theta \frac{f\,(\beta_{t-2}^{-1})}{f\,(\beta_{t-1}^{-1})}\right]^2 + 1\right]^{-1}$$

which is a rewrite of equation (6). Denote (7abc) by $\gamma_t = G_\mu(\gamma_{t-1})$, where $\gamma_t = [\beta_t, \beta_{t-1}, g_t]'$, $G\colon U \to \mathbb{R}^3$ is defined by the right hand side of (7), and U is a subset of $\mathbb{R}^3$ such that $\beta \in (0,\bar{\beta})$ and $g \in (0,1)$. A locally unique fixed point is given by $\gamma^\star = [\theta, \theta, \bar{g}]'$, where $\bar{g}$ is the limiting value of g_t as time tends to infinity. A lemma establishes the value of this limit when β_t converges to the steady state at $\beta = \theta$. The lemma also establishes that g_t remains in a neighborhood of $\bar{g}$ when β_t remains in a neighborhood of θ:

LEMMA 1. *(i) If* $\lim_{t\to\infty}\beta_t = \theta$, $\bar{g} \equiv \lim_{t\to\infty} g_t = 1 - \theta^{-2}$. *(ii) Let* ϵ, ϵ' *be positive real numbers, and let* μ *be the bifurcation parameter, where* μ_0 *denotes the bifurcation point. If, at some time* $\tau > 0$, $\{\beta_t\}_{t=\tau}^{\infty} \subset (\theta-\epsilon,\theta+\epsilon)$, *then* $\{g_t\}_{t=\tau}^{\infty} \subset (\bar{g}-\epsilon',\bar{g}+\epsilon')$. *Furthermore,* $|\mu - \mu_0| \longrightarrow 0 \implies \epsilon \longrightarrow 0 \implies \epsilon' \longrightarrow 0$.

PROOF. *See Appendix* 1.

Focus will be placed on bifurcations near the monetary steady state $\beta = \theta$. A fact about the Hopf bifurcation is that the radius of the invariant closed curve is proportional to $|\mu - \mu_0|$, so that the cyclic outcomes can be induced to occur arbitrarily close to the steady state at $\beta = \theta$ by choosing μ arbitrarily close to μ_0 (Ruelle, 1989). Thus, part (ii) of lemma 1 applies to the case where the sequence $\{\beta_t\}_{t=\tau}^{\infty}$ is a limit cycle about θ when $\mu \in (\mu_a,\mu_b)$. Lemma 1 can be applied to approximate the system (7abc) by a linearization at the steady state, replacing g_t with its limiting value $\bar{g}$. The Jacobian is given by

$$(8) \qquad DG(\gamma^{\star},\mu) = \begin{bmatrix} 1-\bar{g}+J & -J & 0 \\ 1 & 0 & 0 \\ \partial g_{t-1}/\partial\beta_{t-1} & \partial g_{t-1}/\partial\beta_{t-2} & \theta^2\left[1+\bar{g}\theta^2\right]^{-2} \end{bmatrix}$$

where $J = -\bar{g}\theta f_{\beta}(\theta^{-1})/f(\theta^{-1})$. The notation is meant to emphasize that the derivative $f_{\beta}(\theta^{-1})$ is taken with respect to β_{t-1}, not β_{t-1}^{-1}, so that its sign is negative by assumption 1. Hence $J > 0$ by assumption.

The Hopf bifurcation theorem is stated in appendix 2. It applies to mappings indexed by a single parameter from an open subset U of $\mathbb{R}^2$ into $\mathbb{R}^2$. The bifurcation occurs when the Jacobian of the map, evaluated at the fixed point, has complex roots which pass through the unit circle at $\mu = \mu_0$. Heuristically, then, the first steps in applying the theorem are to find the conditions for complex roots and to reduce the $\mathbb{R}^3$ problem to a problem in $\mathbb{R}^2$.

Lemma 1 implies that when $lim_{t\to\infty}\beta_t = \theta$ one of the roots of (8) is given by $1/\theta^2$, which is always between zero and one. The second part of the lemma implies that this is also approximately true near the bifurcation point. This fact can be used to achieve a reduction in the dimension of the problem via an application of the center manifold theorem stated in appendix 3. Essentially, the theorem states that near a fixed point, recurrent behavior such as stationary states, cycles, and invariant closed curves must occur (locally) in a center manifold.[7] In the present case, since the Jacobian $DG(\gamma^\star,\mu)$ is available analytically (and since the third root of [8] is stable), the theorem implies that analysis can proceed as if the entire Jacobian consisted of the submatrix given by

(9) $$DF_\mu = \begin{bmatrix} 1-\bar{g}+J & -J \\ 1 & 0 \end{bmatrix}$$

where F_μ is defined by the right hand side of (7ab). This amounts to finding the zeros of the characteristic equation of (9), which, denoting the roots by ρ, is given by

(10) $$\rho^2 - [\theta^2 + J]\rho + J = 0\,.$$

In order to apply the Hopf bifurcation theorem, define $\phi_t = [\beta_t, \beta_{t-1}]'$, $\phi^\star = [\theta, \theta]'$ and denote $\phi_t = F_\mu(\phi_{t-1})$, $\phi^\star = F_\mu(\phi^\star)$. The theorem will be applied in three steps. First, conditions are derived under which the eigenvalues of DF_μ evaluated at the fixed point can be complex. Next, it will be shown that there is a critical value μ_0 for which these roots change stability (that is, a value μ_0 such that the roots have modulus one). Finally, conditions (ii) and (iii) of the theorem will be analyzed.

[7]See Grandmont (1988) for a discussion.

LEMMA 2. (*Complex roots*).

Let $\rho, \bar{\rho}$ be the roots of $DF_\mu(\phi^\star)$. Then a necessary condition for $\rho, \bar{\rho}$ to be complex conjugates is that $0 < J < 4$. A sufficient condition is that $J = 1$.

PROOF. *From equation* (9),

$$\rho, \bar{\rho} = \frac{(1 - \bar{g} + J) \pm \sqrt{(1 - \bar{g} + J)^2 - 4J}}{2}$$

which implies $\rho, \bar{\rho}$ complex iff

$$(1 - \bar{g} + J)^2 < 4J.$$

For this condition to hold, $J > 0$ is required. Because $0 < \bar{g} < 1$, $J < 4$ is also required, which establishes the necessary condition. Within this range, $J = 1$ is the only value of J for which the condition holds regardless of the value of $\bar{g}$, which establishes sufficiency. ∎

Lemma 2 implies that a Hopf bifurcation can occur about the steady state under least squares learning only when $f_\beta(\theta^{-1}) < 0$ (implying $J > 0$); hence only when cycles and chaos are ruled out under perfect foresight. That is, the existence of a Hopf bifurcation in this model *requires* the imposition of a gross substitutes assumption, which is known to be sufficient to eliminate cycles and chaos under perfect foresight.[8]

Next, the meaning of the critical value of the bifurcation parameter μ is investigated. The roots $\rho, \bar{\rho}$ are, at the point of the Hopf bifurcation, given by the complex conjugates $\rho, \bar{\rho} = u \pm vi$, where

$$u = (1 - \bar{g} + J)/2$$

$$v = \frac{\sqrt{4J - (1 - \bar{g} + J)^2}}{2}.$$

A bifurcation occurs if some value μ_0 exists such that when $\mu = \mu_0$ these roots pass

[8]See Sargent (1987) for a discussion of this fact.

through the unit circle. A simple calculation shows that modulus is one when $J = 1$, which is when

$$-\bar{g}\theta f_\beta(\theta^{-1})/f(\theta^{-1}) = 1.$$

This equality can hold so long as $f_\beta(\theta^{-1}) < 0$, which is again the gross substitutes condition in assumption 1. Substituting the limiting value of $\bar{g}$ shows that $J = 1$ when

$$f_\beta(\theta^{-1})/f(\theta^{-1}) = \theta/(1 - \theta^2).$$

The nature of the savings function $f(\cdot)$ is determined in part by the parameters of the utility function underlying the model; these parameters can be taken to be fixed for the present analysis. *Therefore the general bifurcation parameter* μ *can be thought of as the gross money growth rate* θ *in the present context.* The following theorem shows that there is, under general conditions, some value $\theta_0 \in (1,\infty)$ that makes $J = 1$. At this same value θ_0 the roots of equation (10) are complex and condition (iii) of the Hopf bifurcation theorem holds. Taken together, these facts imply the existence of a Hopf bifurcation in this system near the steady state with $\theta = \theta_0$.

THEOREM 1. (*Existence of a Hopf Bifurcation*).

Let $\beta = \bar{\beta}$ *be the autarkic equilibrium of the model. Under maintained assumptions, the monetary steady state occurs at* $\beta = \theta < \bar{\beta}$. Assume (*i*) *the limits* $lim_{\theta \to \bar{\beta}_-} f_\beta(\bar{\beta})$ *and* $lim_{\theta \to 1_+} f_\beta(1)$ *are finite, and* (*ii*)

$$f_{\beta\beta}(\theta_0) < f_\beta(\theta_0)\frac{(1 + \theta_0^{-2})}{(\theta_0^{-1} - \theta_0)} + \frac{f_\beta(\theta_0)^2}{f(\theta_0)},$$

where $f_{\beta\beta}$ *is the second derivative of the savings function with respect to* β. *Then a value* $\theta_0 \in (1,\bar{\theta})$ *exists such that the following conditions hold simultaneously:*

(*a*) $\rho, \bar{\rho}$ *are complex conjugates,*

(*b*) ρ *has modulus one, and*

(*c*) $\left.\frac{dR}{d\mu}\right|_{\mu=\mu_0} > 0, \qquad R \equiv \sqrt{u^2 + v^2}.$

PROOF. *By Lemma 2 the sufficient condition for complex roots is that* $J = 1$, *or*

$$f_\beta(\theta^{-1})/f(\theta^{-1}) = \theta/(1-\theta^2).$$

Denote $f^\star(\theta) = f_\beta(\theta^{-1})/f(\theta^{-1})$, *and* $z(\theta) = \theta/(1-\theta^2)$. *By the definition of modulus* R, $J = 1$ *is also the condition for the roots to cross the unit circle. Hence,* $(b) \Leftrightarrow (a)$.

The function $z(\theta)$ *is continuous and monotonically increasing in* θ $\forall$ $\theta \in (1,\infty)$. *Furthermore,* $lim_{\theta \to 1_+} z(\theta) = -\infty$ *and* $lim_{\theta \to \infty} z(\theta) = 0$.

Under the assumptions on utility and assumptions 1 *and* 2, $f_\beta(\theta^{-1}) < 0$ *and* $f(\theta^{-1}) > 0$, *so that* $f^\star(\cdot) < 0$. *Since* $f(\bar{\beta}) = 0$, $lim_{\theta \to \bar{\beta}_-} f^\star(\bar{\beta}) = -\infty$. *In addition, since* $f(\cdot)$ *is monotonically downward sloping, the point where* $f(\cdot) = 0$ *is unique; hence* $f^\star(1)$ *is finite. Therefore there exists a* $\theta_0 \in (1,\infty)$ *such that* $J = 1$. *This proves parts* (a) *and* (b) *of the theorem, and is illustrated in Figure* 1.

Part (c) *can be verified by direct calculation, noting that*

$$\left.\frac{dR}{d\mu}\right|_{\mu=\mu_0} = \left.\frac{dJ}{d\theta}\right|_{\theta=\theta_0}. \blacksquare$$

Neither of the assumptions (i) or (ii) are very restrictive, at least for the types of examples considered later in the paper. In particular, $f_{\beta\beta} < 0$ at the bifurcation point $\theta = \theta_0$ is sufficient for (ii).

<u>Strong resonance cases</u>

To understand condition (iii) of the Hopf bifurcation theorem, translate the roots ρ, $\bar{\rho}$ into polar coordinates as

$$\rho, \bar{\rho} = J(cos\ \alpha \pm i\ sin\ \alpha)$$

where

$$cos\ \alpha = (1 - \bar{g} + J)/2J$$

$$sin\ \alpha = \frac{\sqrt{4J - (1 - \bar{g} + J)^2}}{2J}$$

so that α is a function of θ, denoted $\alpha = \alpha(\theta)$. Condition (iii) requires $\alpha(\theta_0) \neq 2\pi/q$ for $q = 1, 2, 3, 4$. At $\theta = \theta_0$, $J = 1$ and the equations become

$$\cos \alpha = (1 + \theta_0^{-2})/2$$

$$\sin \alpha = \frac{\sqrt{3 - 2\theta_0^{-2} - \theta_0^{-4}}}{2}.$$

Then if, say, $\theta_0 = 1$ (which is not allowed anyway), $\cos \alpha = 1$ and $\sin \alpha = 0$ so that $\alpha = 2\pi$, which is a value of α cited in condition (iii). These situations are known as "strong resonance" cases, and are ruled out by assumption. That is, values of $\theta_0 \in (1,\infty)$ are allowed, excepting those that make condition (iii) of the Hopf bifurcation theorem true. Strong resonance cases can be analyzed using alternative methods (see Iooss, 1979), but are ignored here since they are exceptional in any event (Grandmont, 1988).

So far, this section has demonstrated that a Hopf bifurcation occurs in the model under general conditions at a critical value of the gross rate of monetary growth. Two topics remain concerning the nature of the dynamics generated as θ passes through θ_0. First, the closed invariant curve, the existence of which has now been established, may be either asymptotically stable or unstable; in the terminology of the mathematics literature, the Hopf bifurcation may be either supercritical or subcritical. Second, the nature of the periodicities on an invariant closed curve warrant discussion.

Stability of the invariant closed curve

From the last portion of the statement of the Hopf bifurcation theorem in appendix 2, the stability of the invariant closed curve is seen to depend on the value of a certain coefficient w_2 at $\theta = \theta_0$. The value of this coefficient can be calculated for the present model, although the calculations are quite arduous. Unfortunately, the evaluation of $w_2(\theta_0)$ involves second and third derivatives of excess demand $f(\cdot)$ and hence third and fourth derivatives of the utility function U. Since the signs of these derivatives are not given by theory, the conclusion must be that the invariant closed curve may be either attracting or repelling. Examples of both possibilities are provided in the next section.[9]

Periodic and quasi–periodic trajectories

When the utility function is such that an invariant closed curve is stable, it remains to understand the nature of the dynamics generated by the map $G_\mu(\cdot)$. There are two possibilities. One is that all points on the circle are periodic, with the period length denoted by k. Another is that none of the points on the circle are periodic. This latter case is not chaotic, however; the trajectories are said to be quasiperiodic, in that they nearly repeat every k periods. The motion can be visualized as a rotation of a certain angle $\tilde{\alpha}$ on the unit circle. If $\tilde{\alpha}/2\pi$ is rational, the periodic case is obtained, while if $\tilde{\alpha}/2\pi$ is irrational, the quasiperiodic case holds.[10]

Although the angle $\tilde{\alpha}$ can be calculated for the present model, it is not useful to do so. Any small change in the parameters of the model can change $\tilde{\alpha}/2\pi$ from rational to irrational and hence change the dynamics. The most that can be said, therefore, is that when a closed invariant curve exists and is asymptotically stable, the nonlinear map $G_\mu(\cdot)$ can have either of two types of dynamics, periodic or quasiperiodic.

This section has shown that a Hopf bifurcation occurs in a simple overlapping generations model under least squares learning. It is assumed that the money supply is growing and that no equilibrium cycles exist under perfect foresight. The bifurcation occurs at a value of the gross rate of money growth that is "sufficiently high." Such a value of the bifurcation parameter exists under general conditions. The analysis is local and concerns a neighborhood of this critical point, $\theta_0 \in (\theta_a,\theta_b)$. For values of θ such that $\theta_a < \theta < \theta_0$, the monetary steady state is locally stable under least squares learning. As θ increases through $\theta = \theta_0$, either an unstable invariant closed curve vanishes (subcritical

[9]See also Farmer (1987) and Reichlin (1987).

[10]Grandmont (1988) provides details.

Hopf bifurcation) or a stable invariant closed curve emerges (supercritical Hopf bifurcation). In either case, the steady state becomes unstable. In the subcritical case, the system simply diverges. However, in the supercritical case the equilibrium trajectories now become periodic or quasiperiodic. Whether the bifurcation is supercritical or subcritical depends on characteristics of the underlying utility function which are theoretically unspecified. Hence, the main result is that in this model there is always some critical value of the gross rate of growth of the money stock such that beyond this rate, the model under least squares learning can attain an equilibrium that does not exist under perfect foresight.

4. Parameterization and simulation

The results of the previous section can be made more concrete by parameterizing the model and simulating the implied system. Several versions of the model studied by Marcet and Sargent (1989b) are employed.

Example 1

In this example, the aggregate savings function is linearized as

$$H_t/P_t = 1 - \lambda F_t P_{t+1}/P_t \tag{11}$$

for $\lambda \in (0,1)$, while currency growth is given by equation (2), reproduced for convenience as

$$H_t = \theta H_{t-1}, \tag{12}$$

where $\theta \in (1,\lambda^{-1})$.[11] The savings function in equation (11) can be derived from the case where agents have logarithmic utility $U = ln\ c_1 + ln\ c_2$ with the first period endowment set to 2 and the second period endowment set to 2λ. The parameter λ therefore represents the size of the second period endowment relative to the first period endowment. The linear specification is also more than sufficient to rule out cycles and chaos under

[11]This amounts to Marcet and Sargent's (1989b) model with their $\gamma = 1$ and $\xi = 0$.

perfect foresight (that is, to impose assumption 1).

If the model is closed using perfect foresight, the stationary equilibria of the system are solutions to

$$\beta_{t+1} = \lambda^{-1} + \theta - \lambda^{-1}\theta\beta_t^{-1} . \tag{13}$$

There are two stationary equilibria, $\beta_1^\star = \theta$ and $\beta_2^\star = \lambda^{-1}$, and $\theta < \lambda^{-1}$. The stationary equilibrium at θ is the monetary steady state, while the equilibrium at λ^{-1} is the autarkic outcome where zero real balances are demanded. The qualitative graph of (13) is given in figure 2. Equilibrium sequences $\{\beta_t\}_{t=0}^{\infty}$ are indexed by β_0. All equilibrium sequences save one ($\beta_0 = \beta_1^\star$) converge to the autarkic equilibrium.

Under least squares learning the system is described by

$$\beta_t = \beta_{t-1} + g_{t-1}\left[\theta\frac{(1 - \lambda\beta_{t-2})}{(1 - \lambda\beta_{t-1})} - \beta_{t-1}\right] \tag{14}$$

where

$$g_{t-1} = P_{t-2}^2\left[\sum_{s=1}^{t-1} P_{s-1}^2\right]^{-1} .$$

The asymptotic behavior of the system can be analyzed by simulating (14) with initial conditions for β_{-1} and β_0 near the monetary steady state $\beta = \theta$.[12] According to Theorem 1, a value of θ exists such that a Hopf bifurcation occurs.

For this simulation, $\lambda = .9$ and $\theta \approx 1.0374$, which is near the bifurcation point. The autarkic equilibrium occurs at $\lambda^{-1} = 1.11\bar{1}$. Two cases are illustrated, one where θ is less than the bifurcation point θ_0, and one where $\theta \to \theta_0$ from the left.

In Figure 3, $\theta = 1.037$. Both the perfect foresight and the least squares learning systems are provided with the same initial condition at $\beta_0 = 1.047$. The perfect foresight system converges to the high inflation stationary state, while the system under learning converges to the steady state at $\beta = \theta$. This illustrates a theme in the recent literature on learning, that the stability properties of stationary equilibria can be reversed under

[12] In all of the simulations that follow, $\beta_{-1} = \beta_0$. The recursive formulae (7abc) were employed.

learning relative to perfect foresight.[13] When the value of θ is closer to 1 or the initial condition is closer to the monetary steady state, or both, the system under learning converges more rapidly to the monetary steady state at $\beta_1^\star = \theta$.

Figure 4 illustrates the case where the money growth rate is increased to $\theta = 1.0374885$ and the initial condition is $\beta_0 = 1.0474885$. The perfect foresight dynamics are qualitatively unchanged, but the system under learning has undergone a bifurcation and now orbits about the steady state at $\beta_1^\star = \theta$. Figure 5 plots the invariant closed curve in (β_t, β_{t+1}) space for $\theta = 1.0374885$. In this diagram the steady state is the point in the ellipse at (1.0374885,1.0374885). Although it is not apparent from the diagram, this closed curve is repelling.

Example 2

This example is exactly the same as example 1 except that equation (12) is replaced with the Marcet and Sargent (1989b) specification

$$H_t = \theta H_{t-1} + \xi P_t. \tag{12'}$$

The purpose is to show that the same bifurcation can be obtained by setting $\theta = 1$ and examining the case of positive fiscal deficits ξ, now taking ξ to be the bifurcation parameter. With the given parameter values, the existence of a stationary equilibrium requires that $\xi < \xi_{max} = .002633403$. The bifurcation occurs near $\xi \approx .0024$. The periodic equilibrium is given by the graph in Figure 6. As in example 1, this closed curve is repelling.

Example 3

This example produces an attracting closed curve. The logarithmic utility function of examples 1 and 2 is replaced with a CES utility function $U = (c_1^\rho + c_2^\rho)^{1/\rho}$.

[13] See especially Grandmont and Laroque (1990) and Marcet and Sargent (1989b).

Endowments are set as in example 1. The system is given by

$$H_t/P_t = 2 - \left[\frac{2 + 2\lambda\beta_t}{1 + \beta_t^{\rho/(\rho-1)}}\right] \tag{15a}$$

$$H_t = \theta H_{t-1}\,. \tag{15b}$$

Money demand is zero when $\beta = \lambda^{\rho-1}$. The parameter ρ is set to .8 for this example. If the system is closed using perfect foresight, and if the second period endowment is set to zero ($\lambda = 0$), the system has a closed form representation as the difference equation

$$\beta_t = [\theta^{-1}\beta_{t-1}(1 + \beta_{t-1}^4) - 1]^{1/4}. \tag{16}$$

This equation has a fixed point at $\beta = \theta$. Because $\lambda = 0$, the autarkic equilibrium is represented by a sequence $\{\beta_t\}_{t=0}^{\infty}$ with $lim_{t\to\infty}\beta_t = \infty$. The system under learning can be simulated by substituting (15a) into equation (6).

In this example, the bifurcation point occurs at $\theta_0 \approx 1.24$. When $\theta < \theta_0$, the steady state at $\beta = \theta$ is the attractor under least squares learning. When $\theta > \theta_0$, the attractor for the system is a closed curve. Figure 7 illustrates the learning dynamics after the bifurcation point. In the diagram, $\theta = 1.243$ is the monetary steady state. The initial condition is very close to the steady state, $\beta_0 = 1.246$. At this value of θ, the steady state is unstable and the system is attracted to the closed curve. The periodicity in this case is 11, and can be computed directly from the diagram.

<u>Example 4</u>

This example makes use of a projection facility. Some authors have suggested that the use of such facilities can improve the convergence properties of least squares learning mechanisms by keeping the system in a region that has an economic interpretation. In the model analyzed here, the only interesting cases are ones where the sequences for currency and prices are strictly positive. To guarantee this outcome, one might assume that agents never forecast negative prices. This can be accomplished as follows. Let the system under learning in example 1 be given by equation (14) when $\beta_t \in (0,\lambda^{-1})$, and otherwise be given by $\beta_t = \beta_{t-1}$. The projection facility has the effect of containing the system when it threatens to enter a region that does not make economic sense.[14]

When the model of example 1 is augmented with a projection facility, very erratic dynamics are possible under learning. In Figure 8, the gross rate of money growth is set to 1.1, which is past the bifurcation point. The time series presented consists of the observations from 50,000 to 50,500. The system shows no tendency to converge either to the monetary steady state at $\beta = 1.1$ or to any other attractor. Heuristically, the attractor for the system lies in a region of the space where the model is not defined, and is prevented from moving toward the attractor by the projection facility. Such systems are unlikely to converge.

5. Eliminating Forecast Errors

The idea that agents learn over time to eliminate systematic forecast errors lies at the heart of the concept of rational expectations. A sensible question therefore is whether, in the periodic learning equilibrium, systematic forecast errors exist which could be eliminated. In fact such errors do exist, and they can be easily eliminated in a way that makes the learning equilibrium also a perfect foresight equilibrium.

To see this, consider the actual versus the perceived law of motion for prices in this model:

Actual: $$P_{t+1} = \theta \frac{f(\beta_{t-1})}{f(\beta_t)} P_t$$

Perceived: $$P_{t+1} = \beta_t P_t .$$

The actual law of motion is derived from equations (1) and (2), while the perceived law of motion is given by equation (3). When $\beta = \theta$, the actual law of motion equals the perceived law of motion, and rational expectations equilibrium is attained.

Denote a strictly periodic learning equilibrium of order k, attained at some time τ,

[14]See Marcet and Sargent (1989abc) and Grandmont and Laroque (1990b).

by $\{\beta\}_{t=\tau}^{\infty} \in \{\beta_1, \beta_2, ..., \beta_k\}$ where $\beta_0 = \beta_k$, $\beta_1 = \beta_{k+1}$, and so on. The case of a quasiperiodic learning equilibrium will be set aside for the moment. Then considering the actual law of motion, the relationship between adjacent prices cannot be fixed in general, since the ratio in the savings function cannot be constant. But the perceived law of motion, representing a regression of prices in period t on those in period $t-1$, is specified to find a fixed relationship between adjacent prices—a relationship of exactly the sort that does not exist when the system converges to a cycle. Hence, forecast errors, calculated as perceived P_{t+1} less actual P_{t+1}, do not tend to zero under this regression specification.

However, the following lemma demonstrates that in a periodic equilibrium there is a fixed relationship between certain prices, and hence also an alternative regression specification, that can be exploited by the agents in this model in order to completely eliminate all forecast errors and thus make the learning equilibrium into a rational expectations equilibrium. In particular, along a periodic equilibrium of order k, there is a fixed relationship between prices of period t and those of period $t-k$.

LEMMA 3. *Suppose* $\{\beta_t\}_{t=\tau}^{\infty}$ *follows a periodic path denoted* $\{\beta_1, \beta_2, ..., \beta_k\}$, *where* $\beta_0 = \beta_k$, $\beta_1 = \beta_{k+1}$, *and so on. Then* $P_{t+k} = \theta^k P_t$.

PROOF. *From equation* (1),

(i) $H_t = P_t f(\beta_t)$

(ii) $H_{t+k} = P_{t+k} f(\beta_{t+k})$.

Since $\beta_t = \beta_{t+k}$, $f(\beta_t) = f(\beta_{t+k})$. *From equation* (2), $H_{t+k} = \theta^k H_t$. *Hence* $P_{t+k} = \theta^k P_t$.∎[15]

[15]For the quasiperiodic case, the relationship between β_t and β_{t+k} is not exact. Then P_{t+k} is only approximated by $\theta^k P_t$, since $P_{t+k} = \theta^k P_t[f(\beta_t)/f(\beta_{t+k})] \approx \theta^k P_t$. Approximating the ratio of the savings functions in this expression by $1+\epsilon$ implies that the periodic case occurs as $\epsilon \to 0$. As ϵ deviates from zero, there is less and less sense in which there is a systematic relationship between the k^{th} prices that can be exploited by agents, and also less and less sense in which forecast errors are systematic.

In other words, in the periodic learning equilibrium the agents are searching for a fixed relationship between adjacent prices, when the actual fixed relationship is between the k^{th} prices. A regression specification that will detect this fixed relationship can be found as follows. Let

$$F_t P_{t+1} = b_t P_{t-k+1}$$

where b_t is calculated at time t via

$$b_t = \left[\sum_{s=k}^{t-1} P_{s-k}^2\right]^{-1}\left[\sum_{s=k}^{t-1} P_{s-k} P_s\right],$$

but represents the perceived gross inflation rate over the entire k periods that constitute the cycle. Since agents only live for two periods, they care only about β_t, the perceived gross inflation rate from period t to period $t+1$. This is related to the coefficient b_t by

$$\beta_t = \frac{b_t P_{t-k+1}}{P_t}.$$

The recursive formula for b_t is given by

$$b_t = b_{t-1} - C_{t-1}^{-1} P_{t-k-1}\left[P_{t-1} - P_{t-k-1} b_{t-1}\right] \tag{17}$$

where

$$C_{t-1} = \sum_{s=k}^{t-1} P_{s-k}^2 .$$

From equations (1) and (2), and by repeated substitution,

$$P_{t-1} = \theta^k \frac{f(\beta_{t-k-1})}{f(\beta_{t-1})} P_{t-k-1}$$

so that substitution into (17) yields

$$b_t = b_{t-1} - g_{t-1}\left[\theta^k \frac{f(\beta_{t-k-1})}{f(\beta_{t-1})} - b_{t-1}\right] \tag{18}$$

where

$$g_{t-1} = P_{t-k-1}^2\left[\sum_{s=k}^{t-1} P_{s-k}^2\right]^{-1} \in (0,1) .$$

Equation (18) describes the dynamics of the system when agents form expectations by regressing period t prices on period $t-k$ prices.

LEMMA 4. *Suppose* $\{\beta_t\}_{t=0}^{\infty}$ *follows a periodic path denoted* $\{\beta_1, \beta_2, ..., \beta_k\}$, *where* $\beta_0 = \beta_k$, $\beta_1 = \beta_{k+1}$, *and so on. Then the system described by* (18) *has a fixed point at* $b = \theta^k$.

PROOF. *On the attractor,* $\beta_{t-1} = \beta_{t-k-1}$.

This result can be interpreted as follows. Agents regress prices from period t on prices from period $t-1$ in order to try to learn to have perfect foresight. Provided θ is sufficiently small, the agents will succeed and the system will converge to the monetary steady state. However, if the policy rule is for too rapid a rate of money growth, the system under this learning rule will not converge to the monetary steady state and instead may follow a periodic path after some time τ. This is a periodic "learning equilibrium." In this case, the agents will observe systematic forecast errors and will want to eliminate them by switching to a new forecast function. They can do this by switching, at some time $\tau' > \tau + k$, to a regression of period t prices on period $t-k$ prices. Such a switch implies the dynamic system described by equation (18). So long as agents use information from period τ to period τ', this system will immediately yield the fixed point $b = \theta^k$, and agents will completely eliminate the forecast errors.

6. Discussion

Imposing rational expectations on economic models often implies multiple equilibria, and which of these is the outcome under actual expectations is in doubt. Authors have sometimes invoked the casual argument that agents will learn over time to form rational expectations, and that the outcome of the learning process will be a rational expectations equilibrium. Lucas (1987), for instance, has suggested that the outcome of a learning process in the overlapping generations model will be the monetary steady state, and that learning lends plausibility to the study of rational expectations equilibria. Marcet and Sargent (1989abc) have studied systems governed by least squares learning and conclude that such systems, when they converge, converge to rational expectations equilibria. Grandmont and Laroque (1990b) have suggested that systems with learning may possess stationary equilibrium dynamics unrelated to stationary perfect foresight dynamics. The results of this paper shed some light on these claims.

Specifically, a general equilibrium system with least squares learning can converge to a periodic trajectory that does not exist under perfect foresight. Even in the neighborhood of the monetary steady state, there can be no presumption that plausible learning rules "select" the stationary equilibrium where money has value. Once the bifurcation parameter is past the bifurcation point, there is *no neighborhood* that is attracted to this steady state. Nor is it true that systems with least squares learners, when they converge, necessarily converge to stationary rational expectations equilibria. Instead, support is found for the idea that systems with learning can possess attractors independent of those that exist under perfect foresight.

A number of researchers have recently followed the lead of Azariadis (1981) and others who have shown that widely used macroeconomic models, such as the overlapping generations model analyzed here, can possess sunspot equilibria when frivolous variables enter the forecast functions of agents. Woodford (1990) has shown that agents might "learn to believe" in stationary sunspot equilibria. These models are sometimes advocated as systems where macroeconomic complexity is explained. The same sort of claim can be made for the model analyzed in this paper: for a certain values of the policy parameter, in particular those that are "too high," the dynamics of the model become complicated under least squares learning. However, learning equilibria are based entirely on fundamentals and do not require agents to "believe" in frivolous variables.

Although a standard version of the overlapping generations model is employed, the results of this paper are apparently unrelated to the existence of complicated perfect foresight dynamics. The assumptions needed to admit a Hopf bifurcation near the monetary steady state under least squares learning are such that the perfect foresight cycles do not exist. In particular, a maintained assumption has been that first and second period consumption of agents are gross substitutes.

Appendix 1

This appendix proves lemma 1.

LEMMA 1. *(i) If* $lim_{t\to\infty}\beta_t = \theta$, $\bar{g} = lim_{t\to\infty} g_t = 1 - \theta^{-2}$. *(ii) Let* ϵ, ϵ' *be positive real numbers, and let* μ *be the bifurcation parameter, where* μ_0 *denotes the bifurcation point. If, at some time* $\tau > 0$, $\{\beta_t\}_{t=\tau}^{\infty} \subset (\theta-\epsilon, \theta+\epsilon)$, *then* $\{g_t\}_{t=\tau}^{\infty} \subset (\bar{g}-\epsilon', \bar{g}+\epsilon')$. *Furthermore,* $|\mu - \mu_0| \longrightarrow 0 \implies \epsilon \longrightarrow 0 \implies \epsilon' \longrightarrow 0$.

PROOF. *This proof follows the approach of Marcet and Sargent* (1989*b*). *Using equations* (1) *and* (2), *repeated substitution shows that*

$$P_{t-1} = \theta^{t-1} \frac{f\,(\beta_{t-2}^{-1})}{f\,(\beta_{t-1}^{-1})} P_0 \,.$$

Since

$$g_t = P_{t-1}^2 \left[\sum_{s=1}^{t-1} P_{s-1}^2\right]^{-1},$$

we have

$$g_t = \left[\sum_{s=1}^{t-1} \left[\theta^{s-t+1} \frac{f\,(\beta_{t-2}^{-1})}{f\,(\beta_{t-1}^{-1})}\right]^2\right]^{-1}.$$

Part (*i*). *Let* $lim_{t\to\infty}\beta_t = \theta$. *Choose some small positive real* δ *exists such that*

$$lim\ sup_{t\to\infty} \frac{f\,(\beta_{t-2}^{-1})^2}{f\,(\beta_{t-1}^{-1})^2} \le (1+\delta)$$

and

$$lim\ inf_{t\to\infty} \frac{f(\beta_{t-2}^{-1})^2}{f(\beta_{t-1}^{-1})^2} \ge (1-\delta).$$

Then $\delta \longrightarrow 0$ *when* $\beta_t \longrightarrow \theta$. *Therefore,*

$$\lim_{\substack{t\to\infty \\ \beta\to\theta}} g_t = \left[\sum_{s=1}^{t-1} \left[\theta^{s-t+1}\right]^2 \right]^{-1},$$

or

$$\lim_{\substack{t\to\infty \\ \beta\to\theta}} g_t = \left[1 + \theta^{-2} + \theta^{-4} + \dots + \theta^{-2(t-1)}\right]^{-1}.$$

Since $\theta > 1$,

$$\lim_{\substack{t\to\infty \\ \beta\to\theta}} g_t = 1 - \theta^{-2}.$$

Part (*ii*). *Suppose that at some time* τ, *the sequence* $\{\beta_t\}_{t=\tau}^{\infty}$ *is described by motion on an invariant closed curve, induced by a Hopf bifurcation at* $\mu = \mu_0$. *Denote the period* k *cycle by* $B = \{\beta_1, \dots, \beta_k\}$, *where* $\beta_0 = \beta_k$, $\beta_1 = \beta_{k+1}$, *and so on, in the periodic case, and* $\beta_0 \approx \beta_k$, $\beta_1 \approx \beta_{k+1}$, *and so on, in the quasiperiodic case. The following fact about Hopf bifurcations is employed: as* $|\mu - \mu_0| \longrightarrow 0$, *the radius of the invariant closed curve approaches zero* (*Ruelle,* 1989). *Therefore, at time* $\tau > 0$, *the sequence* $\{\beta_t\}_{t=\tau}^{\infty} \subset (\theta-\epsilon, \theta+\epsilon)$, *for some positive real* ϵ, *and the points on the k–cycle can be made arbitrarily close to* θ *by choice of* μ.

Denote

$$F_{min} \equiv \min_{\beta_t \epsilon B} \frac{f\,(\beta_{t-2}^{-1})^2}{f\,(\beta_{t-1}^{-1})^2} < 1\,.$$

By the continuity of the savings function and choice of μ, F_{min} *can be made arbitrarily close to* 1. *Therefore, using the argument in part* (*i*),

$$\lim\sup_{t\to\infty} g_t \geq \frac{1 - \theta^{-2}}{F_{min}} \longrightarrow 1 - \theta^{-2} \;\; as \;\; \epsilon \longrightarrow 0.$$

Similarly, let

$$F_{max} \equiv \max_{\beta_t \epsilon B} \frac{f\,(\beta_{t-2}^{-1})^2}{f\,(\beta_{t-1}^{-1})^2} > 1$$

so that

$$\lim\inf_{t\to\infty} g_t \leq \frac{1 - \theta^2}{F_{max}} \longrightarrow 1 - \theta^2 \quad as \quad \epsilon \longrightarrow 0.$$

Hence, a positive real ϵ' *can be chosen such that* $\{g_t\}_{t=T}^{\infty} \subset (\bar{g}-\epsilon', \bar{g}+\epsilon')$, *and* $\epsilon \longrightarrow 0$ *implies* $\epsilon' \longrightarrow 0$. ∎

Appendix 2

The Hopf Bifurcation Theorem is applied in the text.

THEOREM. (*Hopf Bifurcation*). *Let a* C^s, $s \geq 6$, *one parameter family of mappings be given by* $F_\mu: I \times U \longmapsto \mathbb{R}^2$, *where* U *is an open subset of* $\mathbb{R}^2$, $\mu_0 \in I$, *and* I *is an open subset of* $\mathbb{R}$. *Let* $\phi^\star(\mu_0) \in U$ *be a fixed point of* F_μ *at which the eigenvalues of* DF_μ *are the complex conjugates* $\rho(\mu_0)$, $\bar{\rho}(\mu_0)$. *Let* F_μ *be* C^2 *in* μ. *Assume*

(*i*) $|\rho(\mu_0)| = 1$

(*ii*) $\frac{d}{d\mu}(|\rho(\mu_0)|) > 0.$

Let (r,α) *represent the radius and angle of a polar coordinate, where* $\alpha = \alpha(\mu_0)$ *indicates a function of* μ_0. *Assume*

(*iii*) $\alpha(\mu_0) \neq 2\pi/q, \qquad q = 1, 2, 3, 4.$

Let w_i, $i = 1,2,3,4$ *represent constants. Then there exists a* C^{s-4} *change of coordinates* h *such that* $hF_\mu h^{-1}$ *has the polar coordinates form*

$$hF_\mu h^{-1}(r,\alpha) = \left[r(1 + w_1(\mu - \mu_0) + w_2 r^2),\ \alpha + w_3 + w_4 r^2\right] + \textit{higher order terms.}$$

If $w_2 < 0$ $(w_2 > 0)$ *there exists a right (left) neighborhood of* μ_0 *in which there is an invariant attracting (repelling) closed curve for* F_μ *in* V.

Statements of the theorem can be found in Grandmont (1988), Guckenheimer and Holmes (1983), or Ruelle (1989).

Appendix 3

The Center Manifold Theorem is applied in the text.

THEOREM. (*Center manifold*). *Consider the nonlinear map* $G\colon U \to \mathbb{R}^m$, *where* U *is a subset of* $\mathbb{R}^m$. *Denote the fixed point by* $\gamma^\star$, *such that* $G(\gamma^\star) = \gamma^\star$. *Assume the map is twice continuously differentiable. Then a neighborhood* V *of* $\gamma^\star$ *exists such that there is a local center manifold* W^c *in* V *with first order continuous derivatives. The manifold is locally attracting, in that for* γ, $G^n(\gamma)$ (n *indicating the* n^{th} *iterate of* G) *in* V $\forall$ n, *the distance between* $G^n(\gamma)$ *and* W^c *tends to zero as* $n \to \infty$.

For detailed discussions see Grandmont (1988), Iooss (1979), Guckenheimer and Holmes (1983), or Carr (1981).

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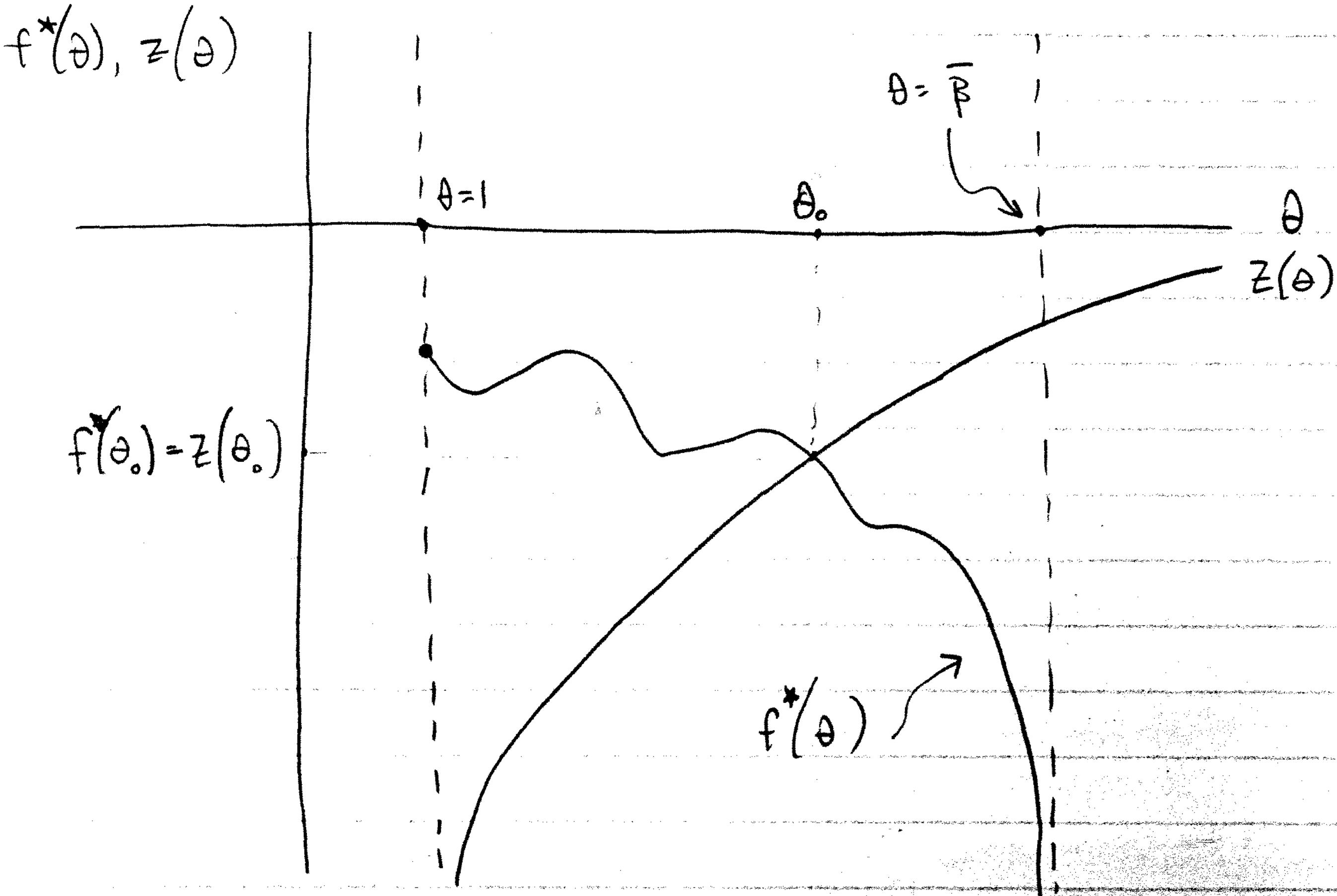
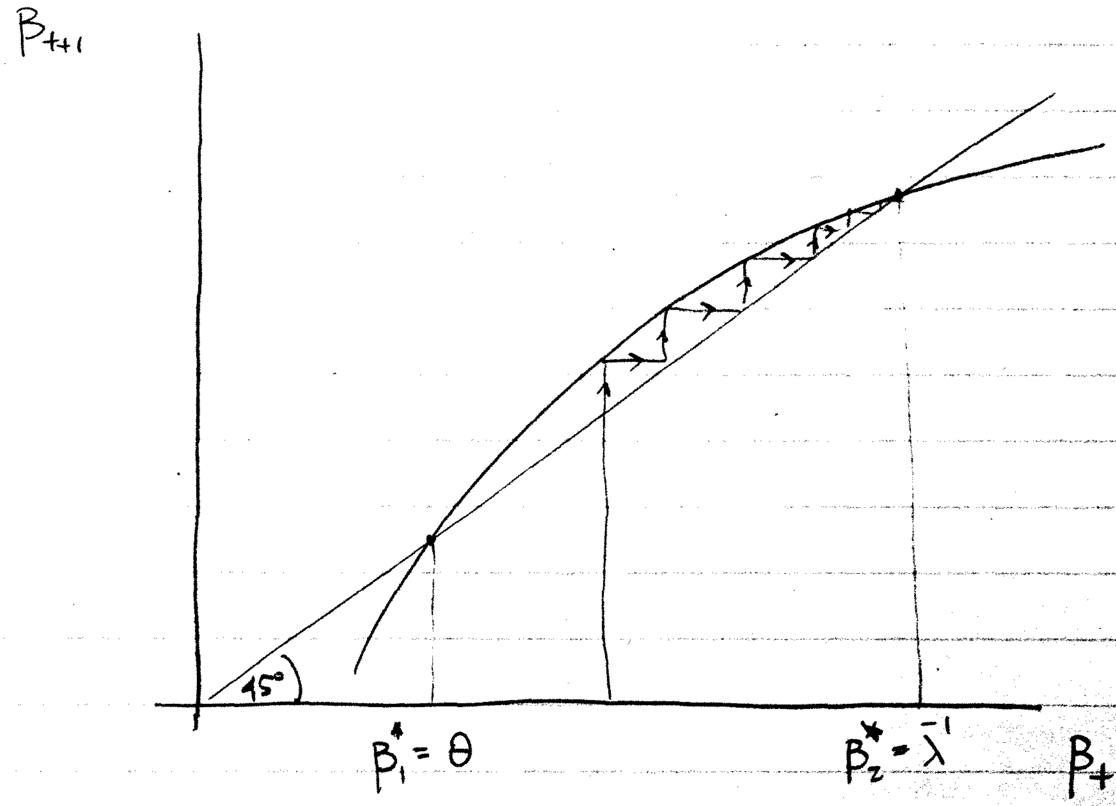
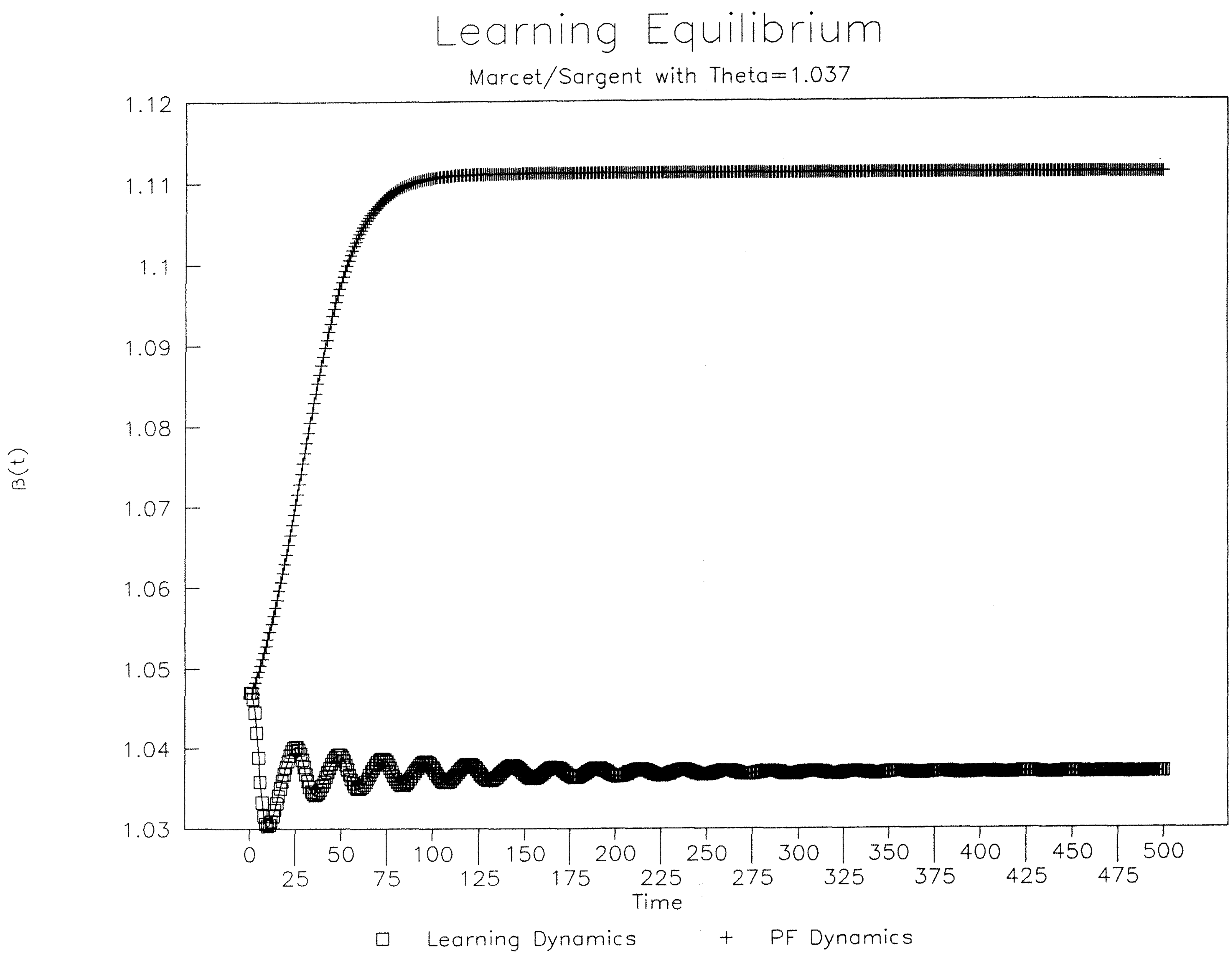
FIGURE 1.

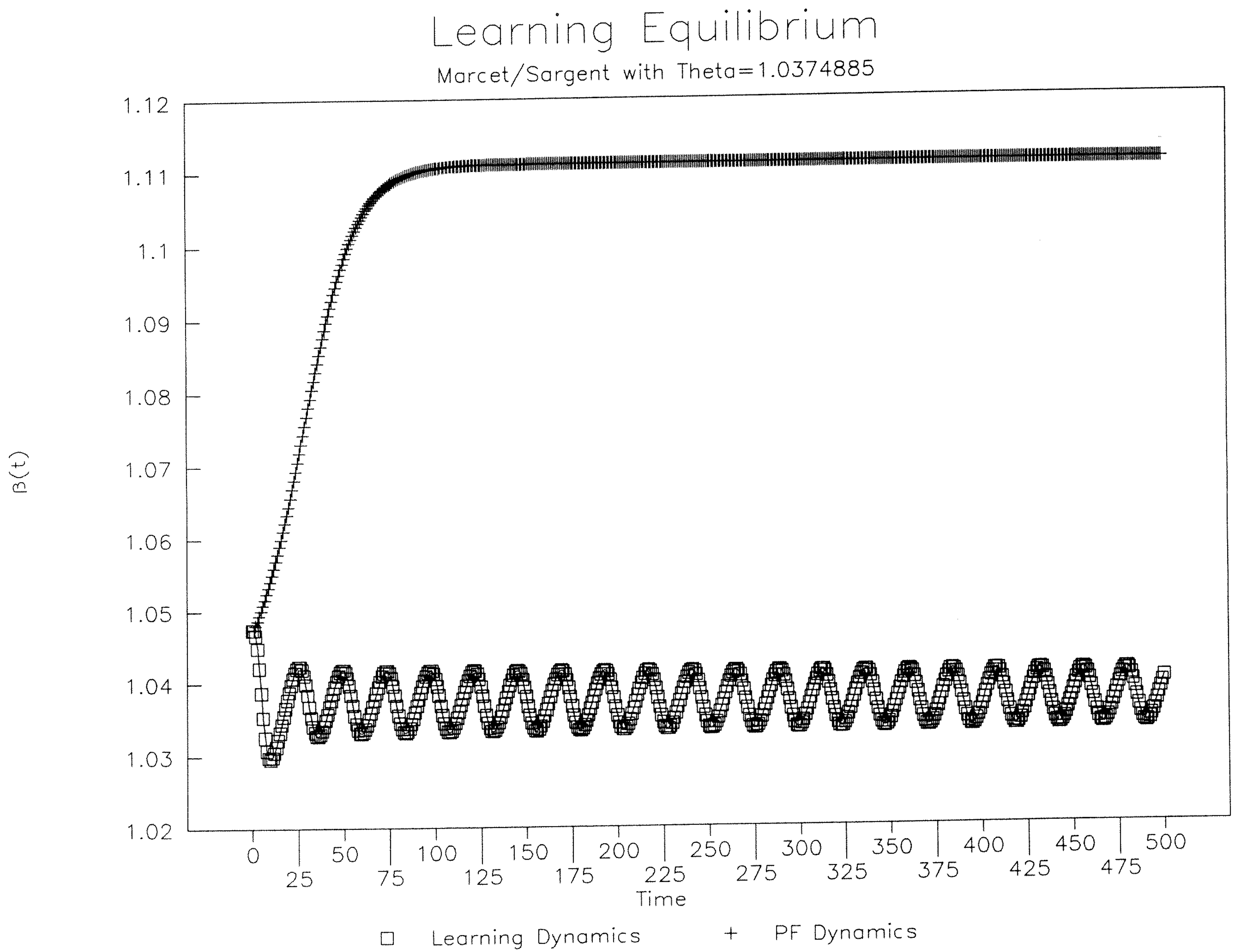


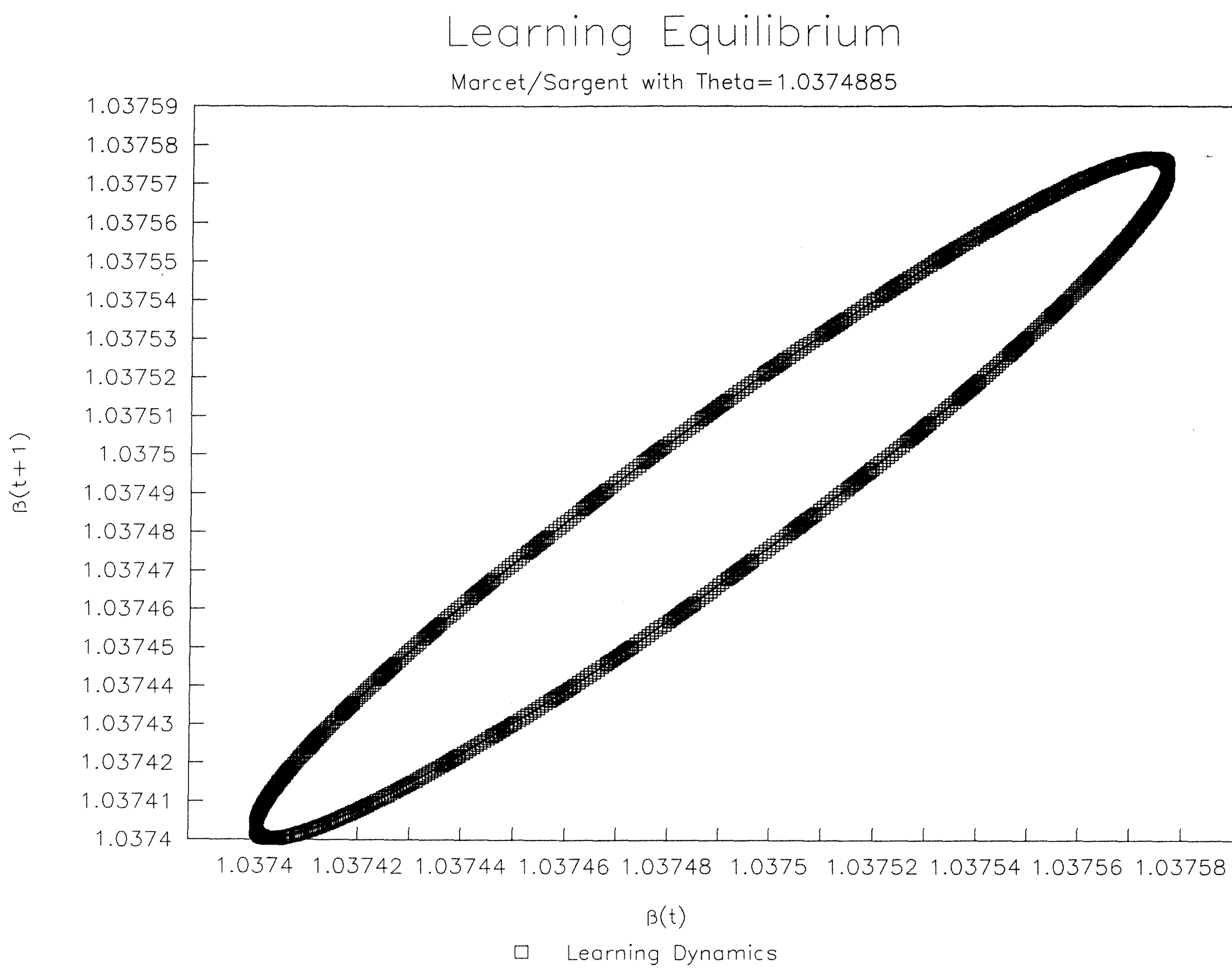
$f^*(\theta), z(\theta)$
$\theta = \bar{\beta}$
$\theta=1$
θ_0
θ
$Z(\theta)$
$f^*(\theta_0) = Z(\theta_0)$
$f^*(\theta)$

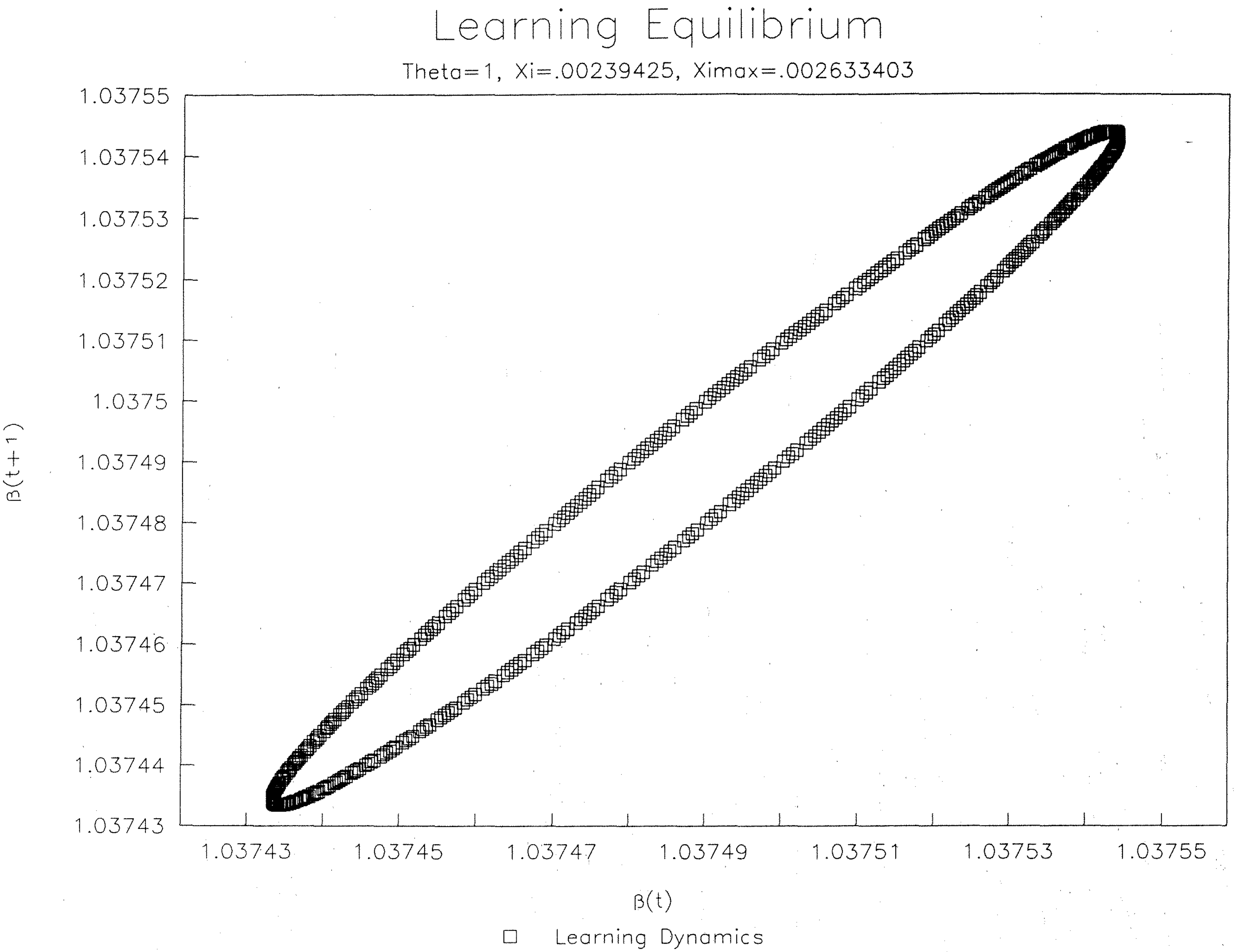
FIGURE 2

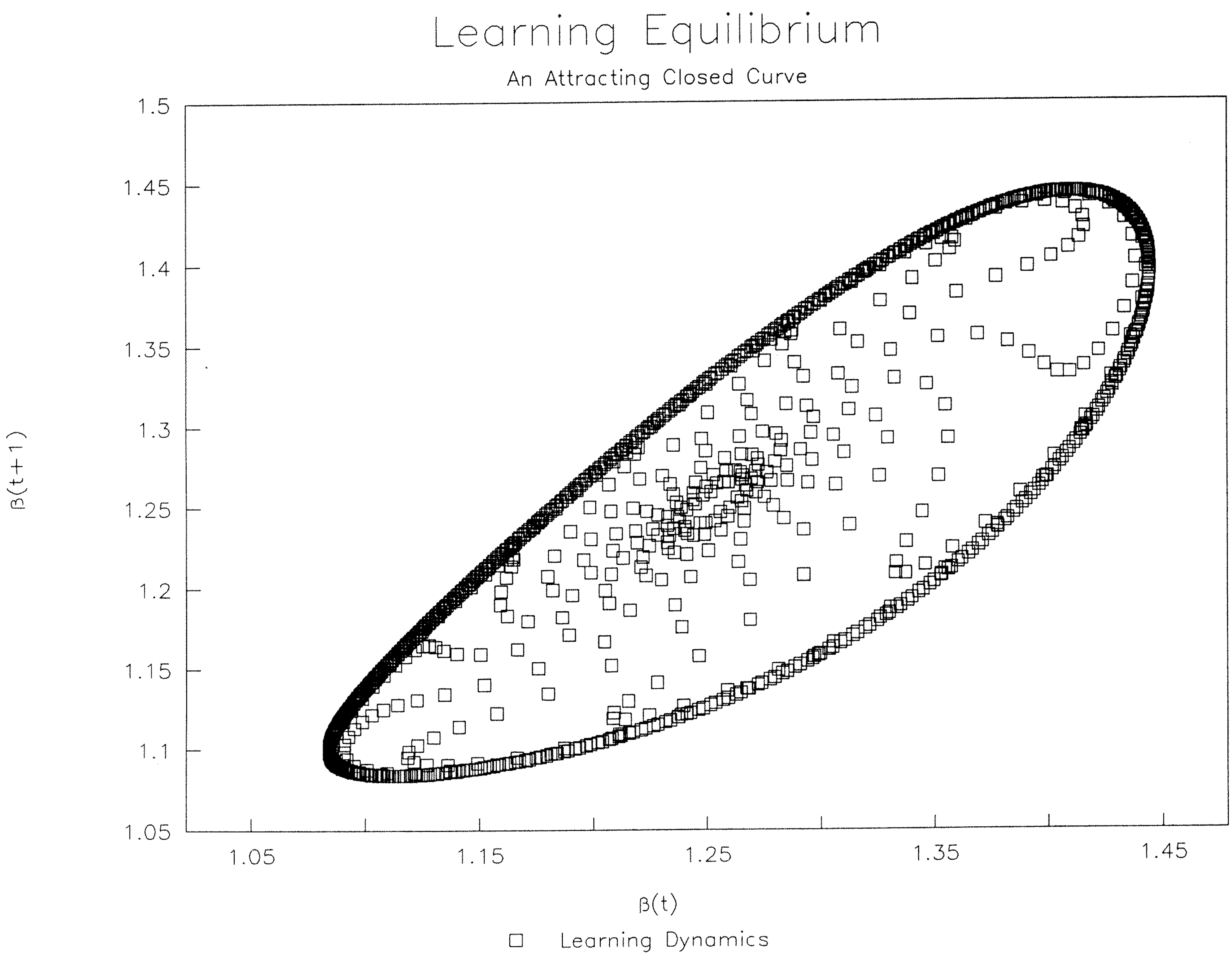



Learning Equilibrium
Marcet/Sargent with Theta=1.037
1.12
1.11
1.1
1.09
1.08
1.07
1.06
1.05
1.04
1.03
ß(t)
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
375
400
425
450
475
500
Time
Learning Dynamics
PF Dynamics

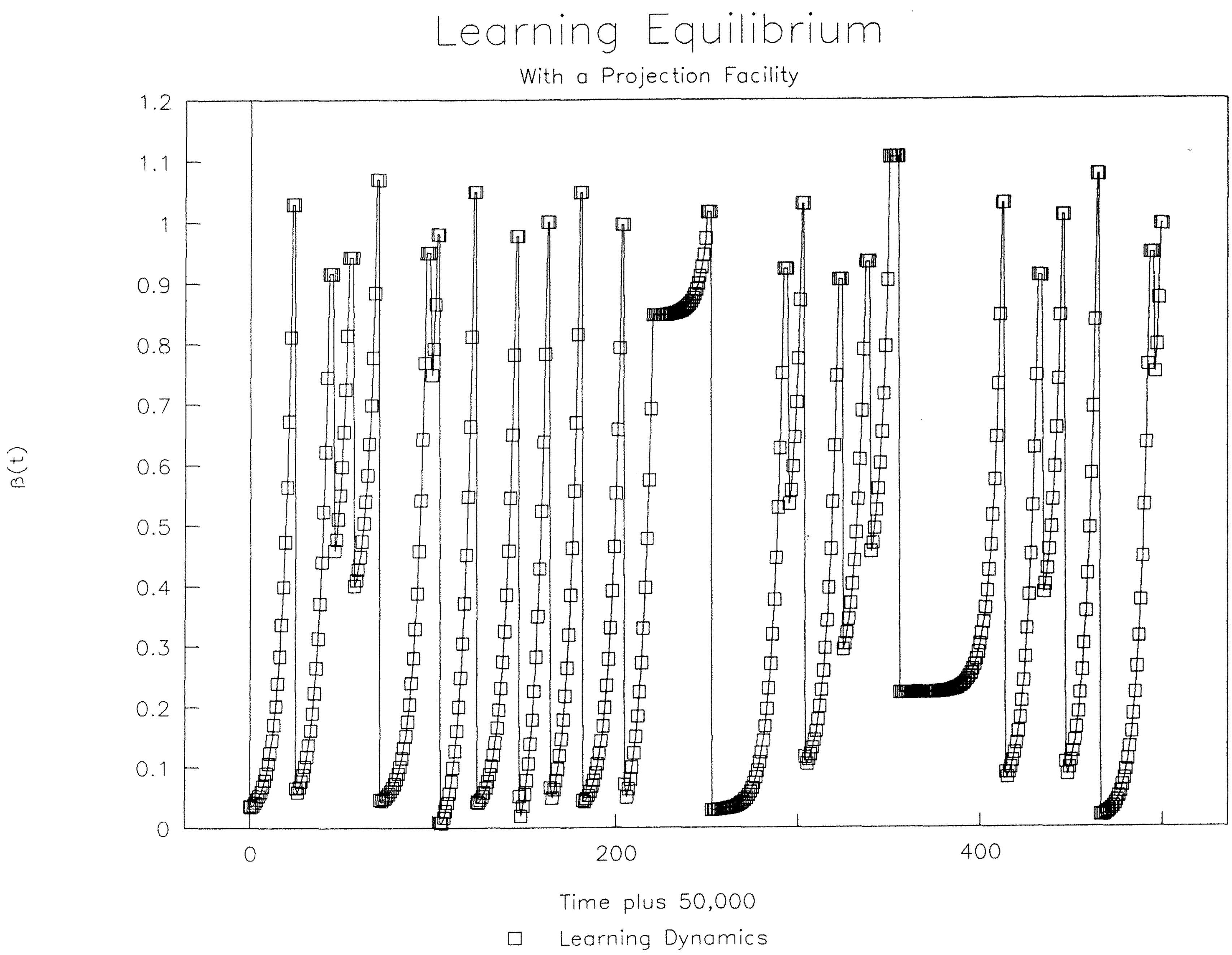

Learning Equilibrium
Marcet/Sargent with Theta=1.0374885
β(t)
1.12
1.11
1.1
1.09
1.08
1.07
1.06
1.05
1.04
1.03
1.02
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
375
400
425
450
475
500
Time
Learning Dynamics
PF Dynamics


Learning Equilibrium
Marcet/Sargent with Theta=1.0374885
1.03759
1.03758
1.03757
1.03756
1.03755
1.03754
1.03753
1.03752
1.03751
1.0375
1.03749
1.03748
1.03747
1.03746
1.03745
1.03744
1.03743
1.03742
1.03741
1.0374
β(t+1)
1.0374
1.03742
1.03744
1.03746
1.03748
1.0375
1.03752
1.03754
1.03756
1.03758
β(t)
Learning Dynamics


Learning Equilibrium
Theta=1, Xi=.00239425, Ximax=.002633403
1.03755
1.03754
1.03753
1.03752
1.03751
1.0375
1.03749
1.03748
1.03747
1.03746
1.03745
1.03744
1.03743
β(t+1)
1.03743
1.03745
1.03747
1.03749
1.03751
1.03753
1.03755
β(t)
Learning Dynamics



Learning Equilibrium
An Attracting Closed Curve
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
β(t+1)
1.05
1.15
1.25
1.35
1.45
β(t)
Learning Dynamics


Learning Equilibrium
With a Projection Facility
β(t)
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
200
400
Time plus 50,000
Learning Dynamics