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Monetary Aggregation Theory and Statistical Index Numbers

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The aggregate quantities of monetary assets held by consumers, firms, and other economic decisionmakers play important roles in macroeconomics. The Board of Governors of the Federal Reserve System publishes monetary aggregates for the United States that are sums of the dollar amounts of monetary assets held by the nonbank public. These assets include currency, checkable deposits, money market mutual fund shares, and savings and time deposits. Constructing these aggregates by summation implicitly assumes that the owners of the monetary assets regard them as perfect substitutes. Yet, the observation that most economic decisionmakers hold a portfolio of monetary assets that have significantly different opportunity costs, rather than a single asset with the lowest cost, implies that the owners do not regard these assets as perfect substitutes.

The appropriate method of aggregating monetary assets is an important question in macroeconomics. In general, aggregation methods should preserve the information contained in the elasticities of substitution, and, in particular, aggregation methods should not make strong a priori assumptions about elasticities of substitution. After forming four aggregates by simple summation of monetary assets, Friedman and Schwartz (1970) offered the following caution:

The restriction of our attention to these four combinations seems a less serious limitation to us than our acceptance of the common procedure of taking the quantity of money as equal to the aggregate value of the assets it is decided to treat as money. This procedure is a very special case of the more general approach [which] consists of regarding each asset as a joint product having different degrees of "moneyness," and defining the quantity of money as the weighted sum of the aggregate value of all assets, the weights for individual assets varying from zero to unity with a weight of unity assigned to that asset or assets regarded as having the largest quantity of "moneyness" per dollar of aggregate value. The procedure we have followed implies that all weights are either zero or unity.

The more general approach has been suggested frequently but experimented with only occasionally. We conjecture that this approach deserves and will get much more attention than it has so far received. The chief problem with it is how to assign the weights and whether the weights assigned by a particular method will be relatively stable for different periods or places or highly erratic. (pp. 151-2)

Although the microfoundations of money have been widely discussed (see, for example, Pesek and Saving, 1967; Fama, 1980; Samuelson, 1968; and Niehans, 1978), prior to Barnett (1978, 1980, 1981) only a few authors, including Chetty (1969), Friedman and Schwartz (1970), and Hutt (1963), had applied either microeconomic aggregation theory or index number methods to monetary assets.¹ Barnett, Fisher, and Serletis (1992) and Belongia (1995) survey the early literature on the subject.

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¹ Hutt (1963) suggested the index now known as the currency equivalent index.

Barnett (1978, 1980, 1981) developed a method of monetary aggregation based on the assumption that monetary assets are durable goods in a representative consumer's weakly separable utility function. In this model, the representative consumer chooses, subject to a set of intertemporal budget constraints, the optimal quantities of all the decision variables in its utility function: durable goods, nondurable goods and services, leisure time, and monetary assets. The assumption that current-period monetary assets are weakly separable from all other decision variables in the consumer's utility function implies the existence of a monetary aggregate for the consumer.

Barnett (1987, 1990) developed a method of monetary aggregation in the context of a representative, perfectly competitive, non-financial firm. The firm is assumed to maximize profit, subject to a production function that contains monetary assets. The solution to the firm's optimization problem produces demand functions for the firm's factor inputs, including monetary assets. If monetary assets are weakly separable in the firm's production function from all other inputs, then a monetary aggregate will exist for the firm. Other related research has developed methods of monetary aggregation based on models of financial intermediaries as multi-product firms (see Barnett and Zhou, 1994; Barnett, 1987; Barnett, Hinich, and Weber, 1986; and Hancock, 1985, 1986). Barnett (1987) discusses the conditions under which a monetary aggregate will exist for a financial intermediary.

The monetary aggregates that are derived from these microeconomic models do not impose strong a priori assumptions about the substitutability of monetary assets. These methods of monetary aggregation are derived from microeconomic theory, which can therefore be used to analyze the implied monetary aggregates. Precise definitions and statements of these results are given in the section of this article entitled "Monetary Aggregation Theory."

Theoretical monetary aggregates can be approximated by statistical index numbers.² The monetary services indexes that we describe in the next article in this issue of the *Review*, "Building New Monetary Services Indexes: Concepts, Data, and Methods," are such index numbers. Our indexes are based on the same aggregation and statistical index number theory as the Department of Commerce's real quantity and price indexes, which include gross domestic product (GDP) and personal consumption expenditure (PCE), and their dual price indexes, the GDP and PCE deflators.

This article contains four sections. In the first section, we discuss the conditions under which it is valid to aggregate monetary assets. In the second section, we discuss the use of statistical index numbers to track monetary aggregator functions. In the third section, we discuss the role of the consumer's budget constraint and the theory's implied concept of monetary wealth. In the last section of the paper we examine the robustness of the theoretical aggregation results.

MONETARY AGGREGATION THEORY

There are two distinct aggregation problems in economics: aggregation across heterogeneous agents, and aggregation of the various goods purchased by a single agent.³ Although this article focuses on aggregation of the monetary assets held by a single representative consumer, we believe it is relevant to briefly review the issues related to aggregation across consumers and firms.

Aggregation Across Heterogeneous Consumers and Firms

The most common method of developing aggregate demand functions that obey microeconomic decision rules is to assume the existence of a representative agent. A *representative consumer* is one that maximizes utility—subject to market prices, an aggregate (shadow) expenditure variable,

² Diewert (1980, 1981, 1992, and 1993) surveys the theory and application of index numbers in economics.

³ On aggregation theory in economics, see Green (1964), Samuelson and Swamy (1974), Diewert (1980) and Barnett (1981).

and a budget constraint—in such a way that, for each good, the representative consumer's demand function is equal to the sum of the demand functions of all individual consumers. Such a representative consumer will exist if the demand functions for individual consumers satisfy Gorman's (1953) condition that all consumers have parallel and linear Engel curves. In this case, the aggregate expenditure variable is the sum, or mean, of the expenditures of all individual consumers, and redistribution effects may be ignored because different expenditure distributions with the same mean expenditure level will produce the same aggregate demand for each good or service. Consequently, we may construct a model of aggregate consumer demand as if all decisions were being made by a single consumer.

Muellbauer (1976) introduced a somewhat less restrictive set of conditions for existence of a representative consumer that contains Gorman's result as a special case. Muellbauer's conditions allow for nonlinear Engel curves, but the appropriate aggregate expenditure variable is not, in general, the sum, or mean, of the expenditures of the individual consumers. In Muellbauer's generalization, the aggregate expenditure variable is a function of the distribution of expenditures and prices.⁴ In this case, mean-preserving changes in expenditure distributions can have economic effects. Berndt, Darrough, and Diewert (1977) apply this methodology using the income distribution, rather than the expenditure distribution. See Deaton and Muellbauer (1980) for more discussion.

The aggregation conditions of Gorman and Muellbauer are highly restrictive and unlikely to be satisfied exactly in any economy. Nevertheless, it often is desirable to interpret the behavior of aggregate data in terms of microeconomic theory. In such applications, aggregate models commonly assume the existence of a representative consumer and ignore the potential role of heterogeneous preferences and distribution effects in aggregate demand.⁵

The conditions for aggregation across competitive firms are weaker than those for

consumers. Consumers' budget constraints give rise to distribution effects that, to justify aggregation, must be ruled out through relatively stringent aggregation conditions. Because profit-maximizing competitive firms are not subject to such budget constraints, these effects are absent. Debreu (1959) showed that, under perfect competition, a group of optimizing firms can be treated as if it were a single *representative firm* which maximizes profits subject to the sum of the production sets of the individual firms. Barnett (1987) discusses aggregation across firms in greater detail.

In empirical research, statistical tests often reject propositions about the behavior of quantities and prices that are implied by representative agent models. This outcome suggests that either the maintained neoclassical microeconomic demand theory is incorrect (an unpalatable conclusion) or that the hypothesis that aggregate data behave according to the decision rules of a single economic agent is false. In the past, the assertion that macroeconomic models based on aggregate data should embed decision rules obtained from the solution of representative agent optimization problems has been controversial (see Lucas and Sargent, 1978a,b; Friedman, 1978; Ando, 1981; Kirman, 1992). Deaton and Muellbauer (1980) noted the following:

In general, it is neither necessary, nor necessarily desirable, that macroeconomic relations should replicate their microeconomic foundation so that exact aggregation is possible. Indeed, to force them to do so often prevents a satisfactory derivation of market relations at all. (p. 149)

Much theoretical macroeconomic research is, however, conducted with general-equilibrium models that are based on representative agents. The extent to which violations of this assumption weaken our results is discussed in the final section of this article, titled "Limitations and Extensions of Aggregation and Statistical Index Number Theory."

⁴ In some cases, the aggregate expenditure variable can be made independent of prices; this has been called price-independent generalized linearity (PIGL).

⁵ Aggregation is possible without assuming a representative agent. See Barnett (1979, 1981) and Selvanathan (1991) for a stochastic method of aggregation first suggested by Theil (1971, 1975). Diewert (1980) discusses Hicksian aggregation conditions, although Hicksian aggregation is not valid for all feasible movements of prices and quantities. Clements and Izan (1987) discuss stochastic Hicksian aggregation.

General Conditions for Aggregation of Monetary Assets

The general conditions sufficient for the aggregation of a group of economic decision variables are: (1) the existence of a theoretical aggregator function defined over the group variables—that is, the existence of a subfunction, defined over the group of variables, that can be factored out of the economic agent's decision problem; (2) efficient allocation of resources over the group of variables; and, (3) the absence of quantity rationing within the group of variables. If the underlying price or quantity data have been previously aggregated across agents, an additional assumption is required: (4) the existence of a representative agent.⁶ These general conditions are sufficient for theoretically rigorous aggregation of a group of economic variables and are in no way specific to the monetary aggregation case.⁷

To facilitate exposition and provide the reader with the strongest and most elegant linkages between monetary aggregation and microeconomic theory, the balance of this article relies on theoretical assumptions that are significantly less general than those stated above. We focus our discussion on the aggregation of monetary assets held by a price-taking representative consumer. This consumer, who is subject to a set of multi-period budget constraints, is assumed to maximize an intertemporal utility function in which current-period monetary assets are weakly separable from all other decision variables. This model, which is less general but more familiar, is sufficient to allow us to aggregate of current-period monetary assets: (1) The weak separability assumption implies the existence of a theoretical aggregator function that can be defined over current-period monetary assets. (2) The utility maximization implies that the allocation of resources over the weakly separable group will be efficient. (3) Quantity rationing is ruled out.

As we noted above, the microeconomic foundations of monetary aggregation can also be illustrated in models of profit-maxi-

mizing firms and financial intermediaries (Barnett, 1987). Additional generalizations of monetary aggregation theory are also possible. In particular, utility (profit) maximization can be replaced by expenditure (cost) minimization in other versions of these models. Expenditure (cost) minimization will guarantee that allocation of resources over the weakly separable blocks is efficient.

The Consumer's Choice Problem

We begin by describing a representative consumer's intertemporal decision problem in which monetary assets appear in the consumer's utility function; our presentation follows Barnett (1978). Monetary assets have been included in utility functions since, at least, Walras (trans. 1954). Arrow and Hahn (1971) show that, if money has positive value in general equilibrium, then there exists a derived utility function containing money.⁸ Thus any model that does not include money in the utility function but produces a motive for holding money in equilibrium is functionally equivalent to a model that *does* include money in the utility function. Hence, no generality in modeling is lost, or gained, by including money in the utility function.

We assume that, in each period, the representative consumer maximizes intertemporal utility over a finite planning horizon of T periods.⁹ The consumer's intertemporal utility function in any period, t , is

$$U(m_t, m_{t+1}, \dots, m_{t+T}; q_t, q_{t+1}, \dots, q_{t+T}; l_t, \dots, l_{t+T}; A_{t+T}),$$

where, for all s contained in $\{t, t+1, \dots, t+T\}$,

$m_s = (m_{1s}, \dots, m_{ns})$ is a vector of real stocks of n monetary assets,

$q_s = (q_{1s}, \dots, q_{hs})$ is a vector of quantities of h non-monetary goods and services,

l_s is the desired number of hours of leisure, and

⁶ If the conditions for existence of a representative agent do not hold for all individuals, they may nevertheless hold for subgroups of individuals. In this case, separate aggregates may be needed for different groups of individuals.

⁷ Although these conditions are sufficient to permit aggregation, they are not sufficient to allow the behavior of the resulting aggregates to be analyzed with microeconomic theory because the first condition—that a factorable subfunction exists—does not restrict the agent's decision to be a rational, optimizing microeconomic decision. For example, the condition is not sufficient to guarantee that the first-order conditions for the solution to the agent's constrained optimization problem will resemble the familiar conditions of neoclassical demand theory.

⁸ However, the specific reason that money is valued cannot be inferred from the form of the utility function. Duffie (1990) reviews general-equilibrium models in which money has positive value. Feenstra (1986) derives the specific utility functions implied by several popular models of money demand, including the cash-in-advance model. Fischer (1974) derives a production function that contains money balances.

⁹ See Barnett (1987), pp. 116-20. Note that this model assumes perfect certainty, and that all agents are price-takers.

A_{t+T} is the real stock of a benchmark financial asset, held in the final period of the planning horizon, at date $t+T$.

The representative agent is assumed to re-optimize in each period, choosing values of $(m_{t+1}, \dots, m_{t+T}; q_{t+1}, \dots, q_{t+T}; l_{t+1}, \dots, l_{t+T}; A_{t+1}, \dots, A_{t+T})$ that maximize the intertemporal utility function subject to a set of $T+1$ multiperiod budget constraints. The set of multiperiod budget constraints for s contained in $\{t, t+1, \dots, t+T\}$ is the following:

$$\sum_{i=1}^h p_{is} q_{is} = w_s L_s + \sum_{i=1}^n \left[(1 + r_{i,s-1}) p_{s-1}^* m_{i,s-1} - p_s^* m_{i,s} \right] + \left[(1 + R_{s-1}) p_{s-1}^* A_{s-1} - p_s^* A_s \right]$$

where

p_s^* is a true cost of living index,

$p_s = (p_{1s}, \dots, p_{hs})$ is a vector of prices for the h non-monetary goods and services,

$r_s = (r_{1s}, \dots, r_{ns})$ is a vector of nominal holding-period yields on the n monetary assets,

R_s is the nominal holding-period yield on the benchmark asset,

w_s is the wage rate,

A_s is the real quantity of a benchmark asset that appears in the utility function only in the final period of the planning horizon, $t+T$, and

L_s is the number of hours of labor supplied.

Leisure time consumed by the consumer during each period is $l_s = H - L_s$, where H is the total number of hours in a period.

In this model, we assume the existence of the true cost-of-living index, p_s^* ; see Barnett (1987). We also assume that all of the services provided to the

agent by monetary assets, except for the intertemporal transfer of wealth, have been absorbed into the utility function, $U(\cdot)$. Note that, even though the benchmark asset, A_s , appears in each period's budget constraint, it is included in the utility function only during the final period of the planning horizon. This is because the benchmark asset is assumed to furnish no monetary services to the agent, except in the final period. During all other periods, the agent uses the benchmark asset only to transfer wealth from one period to another.

To simplify notation, let the vector $m_t = (m_{1t}, \dots, m_{nt})$ contain all current-period monetary assets, and let the vector

$$x_t = (m_{t+1}, \dots, m_{t+T}; q_{t+1}, \dots, q_{t+T}; l_{t+1}, \dots, l_{t+T}; A_{t+T})$$

contain all other decision variables in the utility function. Let the vectors

$$m_t^* = (m_{1t}^*, \dots, m_{nt}^*) \text{ and}$$

$$x_t^* = (m_{t+1}^*, \dots, m_{t+T}^*; q_{t+1}^*, \dots, q_{t+T}^*; l_{t+1}^*, \dots, l_{t+T}^*; A_{t+T}^*)$$

denote the solution to the consumer's maximization problem, that is, m_t^* is the consumer's optimal holdings of current-period monetary assets, and x_t^* is the optimal holdings of all other decision variables.

When the utility function

$$U(m_{t+1}, \dots, m_{t+T}; q_{t+1}, \dots, q_{t+T}; l_{t+1}, \dots, l_{t+T}; A_{t+T})$$

is written as $U(m_t, x_t)$, the first-order conditions of this model imply that the marginal rate of substitution between current-period monetary assets m_{it} and m_{jt} , evaluated at the optimum is

$$\frac{\partial U(m_t, x_t)}{\partial m_{it}} \bigg|_{\substack{x_t = x_t^* \\ m_t = m_t^*}} = \frac{\left(p_t^* \frac{R_t - r_{it}}{1 + R_t} \right)}{\left(p_t^* \frac{R_t - r_{jt}}{1 + R_t} \right)}$$

The first-order conditions also imply that the marginal rate of substitution between the current-period monetary asset, m_{it} , and the current-period non-monetary good, q_{kt} , at the optimum is

$$\frac{\frac{\partial U(m_t, x_t)}{\partial m_{it}} \Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}}{\frac{\partial U(m_t, x_t)}{\partial q_{kt}} \Big|_{\substack{x_t = x_t^* \\ m_t = m_t^*}}} = \frac{\left(p_t^* \frac{R_t - r_{it}}{1 + R_t} \right)}{p_{kt}}$$

A general relationship in microeconomic optimization is that, at the optimum, marginal rates of substitution between goods will be equal to the goods' relative prices. In these expressions, the "price" (or opportunity cost) of the current-period monetary asset m_{it} is

$$\frac{p_t^* (R_t - r_{it})}{(1 + R_t)}$$

Barnett (1978) proved this intuition formally, and we discuss this result in more detail in the following section.

The User Costs of Monetary Assets

Monetary assets are treated as durable goods in the model discussed in this article. Similar to other durable goods, monetary assets appear in the utility function, provide services to economic agents, and depreciate (but not fully) during each decision period. To aggregate stocks of these durable monetary assets, we need their equivalent rental prices, or user costs.

Diewert (1974, 1980) discusses the general procedure for constructing the user cost of a durable good (or physical capital asset) from the purchase price of the good, the depreciation rate of the good, and a discount factor.¹⁰ Intuitively, if an agent (consumer or firm) buys one unit of a durable good at the beginning of a decision period and, later, sells the remaining (non-depreciated) part of the unit at the end of the period, the cost to the agent of "renting"

that good for that single period (or, equivalently, the cost of the services that the agent received from the good during the period) is equal to the difference between the initial purchase price of the good and the present value of the amount the agent receives when the non-depreciated part of the unit is sold. If an explicit rental market for the good does not exist, the agent's actions may be interpreted as if he rented the good to himself; in this case, the user cost is usually referred to as an implicit, or equivalent, rental price.

Formally, let p_t and p_{t+1} denote the market prices of a durable good in periods t and $t+1$, respectively; let δ be the depreciation rate; and let D be the discount factor. The *equivalent rental price* of the durable good is

$$\left(p_t - \frac{p_{t+1}(1 - \delta)}{(1 + D)} \right)$$

If the depreciation rate equals unity, as it does for nondurable goods that are fully consumed within a single period, then the rental price equals the purchase price.

Barnett (1978) derived the general form of the equivalent rental price, or user cost, of monetary assets by combining the $T+1$ budget constraints in the general intertemporal consumer model, solving each equation for A_s , and recursively substituting the equations backwards in time, beginning with A_{t+T} . The general form of the *discounted nominal user cost*, π_{is} , of each monetary asset, m_{is} , in each period s contained in $\{t, t+1, \dots, t+T\}$, is

$$\pi_{is} = \left[\frac{p_s^*}{\rho_s} - \frac{p_s^*(1 + r_{is})}{\rho_{s+1}} \right],$$

where r_{is} is the nominal holding period yield and the discount factor, ρ_s , is defined by

$$\rho_s = \begin{cases} 1, & s = t \\ \prod_{u=t}^{s-1} (1 + R_u), & t + 1 \leq s \leq t + T. \end{cases}$$

¹⁰ For earlier treatments in the context of business fixed investment, see Jorgenson (1963) and Jorgenson and Stephenson (1967).

This general form may be specialized to the *current-period nominal user cost*, π_{it} , of monetary asset, m_{it} ,

$$\pi_{it} = \rho_t^* \left(\frac{R_t - r_{it}}{1 + R_t} \right),$$

which may be interpreted as a price of current-period monetary assets (see Barnett, 1978; Donovan, 1978). Note that the current-period nominal user cost is the present value of the interest the agent has foregone by holding the monetary asset (rather than holding the benchmark asset), discounted to reflect the receipt of interest at the end of the period.

The user cost, π_{it} , represents the equivalent rental price of the services provided by a unit of monetary asset, m_{it} . The product, $m_{it}^* \pi_{it}$, represents the optimum expenditure on monetary asset, m_{it} , in the current period. The sum

$$\sum_{i=1}^n m_{it}^* \pi_{it}$$

is total optimal expenditure on (the services provided by) current-period monetary assets. It can be shown that monetary asset, m_{it} , is implicitly assumed to depreciate at the rate of

$$\delta_{it} = 1 - \frac{p_t^* (1 + r_{it})}{p_{t+1}^*}$$

which, for non-interest-bearing monetary assets such as cash, equals the inflation rate (see Fisher, Hudson, and Pradhan, 1993).

For consumers, the user costs of monetary assets are analogous to the user costs for other durable goods; for firms, the user costs of monetary assets are analogous to the user costs for durable physical capital. Barnett (1987, 1990) derived the user cost of monetary assets in the context of a profit-maximizing manufacturing firm with a production function that contains monetary assets and proved that the mathematical expressions for consumers' and

firms' user costs are the same. Because households and firms generally face different market interest rates and prices, the values of their user costs will differ.

Barnett (1987) also derived the user cost of monetary assets for a financial intermediary. When such intermediaries are required by statute to maintain non-interest-bearing reserves against their deposit liabilities, the mathematical form of their user costs will differ from those of consumers and other firms by the size of the implicit reserve-requirement tax. In the absence of statutory reserve requirements, the reserve-requirement tax is zero, and financial firms' user costs for monetary assets have the same form as those of other firms.¹¹ Finally, note that the expressions for both consumers' and firms' user costs can be extended to allow for the taxation of interest income (Barnett, 1980).

Aggregator Functions and Two-Stage Budgeting

Monetary aggregates that are consistent with the solution to the representative agent's decision problem may be derived by imposing additional structure on the model. Assume that the intertemporal utility function is *weakly separable* in the group of current-period monetary assets—that is, that the utility function has the following form:

$$U[u(m_t), m_{t+1}, \dots, m_{t+T}; q_t, q_{t+1}, \dots, q_{t+T}; l_{t+T}, \dots, l_t; A_{t+T}],$$

which may be written as $U[u(m_t), x_t]$, where x_t was defined previously. Note that only current-period monetary assets $m_t = (m_{1t}, \dots, m_{nt})$ are included in the factorable sub-function $u(\cdot)$, which is called the *category subutility function*, defined over current-period monetary assets. Note also that the separability assumption is not symmetric. Weak separability of current-period monetary assets from the other goods and services included in the utility function does not imply that any other combination of decision variables is similarly separable.

¹¹ Many countries, including Canada and Great Britain, do not impose statutory reserve requirements against deposits. Depository institutions in Germany are required to maintain non-interest-bearing deposits at the Bundesbank equal to 2 percent of their transaction deposits. For the United States, Anderson and Rasche (1996) estimate that statutory reserve requirements affected only about 500 depository institutions as of the end of 1995.

Weak separability of the utility function for the representative consumer is an important assumption because it permits formulating the consumer's decision as a two-stage budgeting problem. It implies that the marginal rates of substitution among the variables within the weakly separable group are independent of the quantities of the decision variables *outside* the group (Goldman and Uzawa, 1964). In our context, weak separability of current-period monetary assets from the other decision variables in the consumer's utility function implies that the marginal rates of substitution between current-period monetary assets reduces to

$$\frac{\partial u(m_t) / \partial m_{it}}{\partial u(m_t) / \partial m_{jt}},$$

which, evaluated at the optimum, equals

$$\frac{\left. \frac{\partial u(m_t)}{\partial m_{it}} \right|_{m_t = m_t^*}}{\left. \frac{\partial u(m_t)}{\partial m_{jt}} \right|_{m_t = m_t^*}} = \frac{\pi_{it}}{\pi_{jt}}.$$

Barnett (1980, 1981, 1987) used this result to show that the vector of current-period monetary assets which solves the consumer's (weakly separable) intertemporal utility-maximization problem,

$$m_t^* = (m_{1t}^*, \dots, m_{nt}^*),$$

is exactly the same vector that would be chosen by a consumer in solving the following simpler problem, which involves only current-period variables,

$$\begin{aligned} & \text{Max}_{m=(m_1, \dots, m_n)} u(m) \\ & \text{subject to } \sum_{i=1}^n m_{it} \pi_{it} = y_t, \end{aligned}$$

$$\text{where } y_t = \sum_{i=1}^n m_{it}^* \pi_{it}$$

is the optimal total expenditure on monetary assets implied by the solution to the agent's intertemporal decision problem.

Barnett's result establishes that, under this type of weak separability, the representative consumer's general, intertemporal decision problem is formally equivalent to a two-stage budgeting problem. In the first stage, the consumer chooses the optimal total expenditure, y_t , on current-period monetary assets, and the optimal quantities of the other monetary assets, goods, services, and leisure that appear in the utility function. In the second stage, the consumer chooses the optimal quantities of the individual current-period monetary assets,

$$m_t^* = (m_{1t}^*, \dots, m_{nt}^*),$$

subject to the optimal total expenditure on current-period monetary assets, y_t , chosen in the first stage. Interpreted as a two-stage budgeting problem, the second stage of the problem corresponds to maximizing the subutility function, $u(\cdot)$, subject to the expenditure constraint implied by the first stage (Green, 1964).

If $u(\cdot)$ is first-degree homogeneous, it is a *monetary quantity aggregator function*.¹² The representative consumer will view the *monetary quantity aggregate*, $M_t = u(m_t^*)$, as if it were the optimum quantity of a single, elementary good, which we call *current-period monetary services*. This allows the first-stage decision to be interpreted as the simultaneous choice, given market prices and the consumer's budget constraint, of the optimal quantities of (1) current-period monetary services, and (2) all other decision variables. It also justifies the use of microeconomic demand theory to study the behavior of monetary aggregates.

In general, quantity and price aggregates are said to be *dual* if the price aggregate, multiplied by the quantity aggregate, equals the total expenditure (price times quantity) on all individual goods in the aggregate. Dual to the monetary quantity aggregate, M_t , is the *dual user cost aggregate*, Π_t , defined as the unit expenditure function

¹² If the category subutility function $u(\cdot)$, is homothetic, simply choose a first-degree homogeneous cardinalization of the subutility function.

$$\begin{aligned} \Pi_t &= E(\pi_t, 1) \\ &= \min_{m=(m_1, \dots, m_n)} \left\{ \sum_{i=1}^n m_i \pi_{it} : u(m) = 1 \right\} \end{aligned}$$

where $\pi_t = (\pi_{1t}, \dots, \pi_{nt})$ is the vector of nominal user costs of current-period monetary assets, as defined above.

The consumer is assumed to maximize $U[u(m_t), x_t]$ subject to the $T+1$ multi-period budget constraints discussed above. The first-order conditions for this problem imply that the marginal rates of substitution between current-period monetary assets, m_{it} , and current-period non-monetary goods and services, q_{kt} , can be written as

$$\frac{\frac{\partial U[u(m_t), x_t]}{\partial m_{it}}}{\frac{\partial U[u(m_t), x_t]}{\partial q_{kt}}} \bigg|_{\substack{x_t = x_t^* \\ m_t = m_t^*}} = \frac{\pi_{it}}{p_{kt}}$$

Weak separability of current-period monetary assets from other decision variables, and first-degree homogeneity of the category subutility function, $u(\cdot)$, imply that these expressions can be combined into fewer expressions of the form

$$\frac{\frac{\partial U(u, x_t)}{\partial u}}{\frac{\partial U(u, x_t)}{\partial q_{kt}}} \bigg|_{\substack{x_t = x_t^* \\ u = M_t = u(m_t^*)}} = \frac{\Pi_{it}}{p_{kt}}$$

for all current-period non-monetary goods and services, q_{kt} . These expressions are the first-order conditions for the first-stage decision, and have the interpretation that M_t is the optimal quantity of an elementary good, monetary services, whose price is

Π_t . This result can be generalized, so that all of the first-order conditions involving current-period monetary assets can be rewritten as first-order conditions involving only the aggregates M_t and Π_t .¹³ These new first-order conditions have standard microeconomic interpretations.

The final step in the argument is to show that the budget constraint can be rewritten in terms of the aggregates, M_t and Π_t . As we noted earlier, Barnett (1978) showed that the $T+1$ multi-period budget constraints could be combined into a single budget constraint. It can be shown that current-period monetary assets enter this single budget constraint as total expenditure on current-period monetary assets,

$$\sum_{i=1}^n m_{it} \pi_{it}$$

First-degree homogeneity of the function $u(\cdot)$ implies that the following property, which is called *factor reversal*, holds as an identity:

$$\Pi_t M_t = \sum_{i=1}^n \pi_{it} m_{it}^* = y_t$$

The product of the optimal quantity of monetary services, M_t , and its dual user cost Π_t , equals the optimal total expenditure on current-period monetary assets. The budget constraint, at the optimum, can therefore be rewritten in terms of the aggregates, M_t and Π_t . Because these aggregates satisfy the conditions for factor reversal, the dual user cost aggregate is implicitly defined by

$$\Pi_t = \left(\sum_{i=1}^n m_{it}^* \pi_{it} \right) / M_t$$

The above discussion demonstrates formally that the first-stage decision can be interpreted as the simultaneous choice of optimal quantities of monetary services, M_t , and all other decision variables outside the weakly separable block of current-period monetary assets, subject to prices and a budget constraint, where the price of

¹³The exceptions are the first-order conditions that involve only current-period monetary assets. These are the first-order conditions for the second-stage allocation decision.

monetary services is given by the dual opportunity cost, Π_t . The first-stage decision produces $y_t = M_t \Pi_t$, the optimal total expenditure on current-period monetary services, and this optimal expenditure is allocated among the current-period monetary assets in a second-stage decision. Any current-period monetary portfolio substitution that does not change the level of the monetary aggregate is irrelevant to other decision variables in the model. The monetary aggregates, M_t and Π_t , therefore contain all the information about the consumer's portfolio of current-period monetary assets that is relevant to other aspects of the representative consumer's decision problem.

STATISTICAL INDEX NUMBERS

In the previous section of this article, microeconomic theory was used to identify monetary quantity aggregates and their dual user cost aggregates, for current-period monetary assets. In empirical research, neither the functional forms of the aggregator functions nor the values of their parameters are known. If we sought to estimate the aggregator functions directly, we would be forced to make specific assumptions about the functional forms of the utility (production) or expenditure (cost) functions.

Statistical index numbers are specification- and estimation-free functions of the prices and optimal quantities observed in two time periods.¹⁴ Unlike aggregator functions, statistical index numbers contain no unknown parameters. A statistical index number is said to be *exact* for an aggregator function if the index number tracks the aggregator function, evaluated at the optimum, without error.

The Continuous-Time Case

We begin our discussion of index-number theory with the result that the Divisia quantity index (first suggested as an index number by Divisia, 1925) is exact for the monetary quantity aggregate, M_t . The *continuous-time Divisia quantity index*,

M_t^D is defined for monetary assets by the differential equation

$$\frac{d \log(M_t^D)}{dt} = \sum_{i=1}^n w_{it} \frac{d[\log(m_{it}^*)]}{dt}$$

where, for $i=1, \dots, n$,

$$w_{it} = \frac{m_{it}^* \pi_{it}}{\sum_{j=1}^n m_{jt}^* \pi_{jt}}$$

represents the monetary assets' expenditure shares.¹⁵

Similarly, the *continuous time Divisia user cost index*, Π_t^D , is defined by

$$\frac{d \log(\Pi_t^D)}{dt} = \sum_{i=1}^n w_{it} \frac{d[\log(\pi_{it})]}{dt}$$

Note that the continuous-time Divisia quantity and user cost indexes satisfy factor reversal; that is, the product of the Divisia quantity and price indexes equals the total expenditure on the assets included in the index, thus,

$$M_t^D \Pi_t^D = \sum_{i=1}^n m_{it}^* \pi_{it} = y_t,$$

(see Leontief, 1936).

It is possible to describe the path of the monetary quantity aggregate, M_t , in continuous time using only the first-order conditions for utility maximization and the first-degree homogeneity of the category subutility function, even though the category subutility function, $u(\cdot)$, is unknown. As discussed previously, $M_t = u(m_t^*)$ is the solution to the representative consumer's optimization problem,

$$\text{Max}_{m=(m_1, \dots, m_n)} u(m) \text{ subject to } \sum_{i=1}^n m_i \pi_{it} = y_t,$$

which includes only current-period monetary assets and their user costs. The first-order conditions are

¹⁴ If consumers and firms are not price-takers, then it may be necessary to use shadow, rather than observed, prices in the indexes (Diewert, 1980). An additional problem is that the existence of a representative firm in Debreu's (1959) proof depends on the assumption of perfectly competitive markets.

¹⁵ All logs in this article are natural, or base e , logarithms.

$$\left. \frac{\partial u(m)}{\partial m_i} \right|_{m=m_i^*} = \lambda \pi_{it} \text{ for each } i,$$

where λ is the Lagrange multiplier for the budget constraint. The category subutility function is first-degree homogeneous by assumption, and therefore satisfies Euler's law: thus,

$$\begin{aligned} M_t = u(m_t^*) &= \sum_{j=1}^n \left[\left(\left. \frac{\partial u(m)}{\partial m_j} \right|_{m=m_j^*} \right) (m_{jt}^*) \right] \\ &= \sum_{j=1}^n \lambda \pi_{jt} m_{jt}^* \\ &= \lambda \sum_{j=1}^n \pi_{jt} m_{jt}^*. \end{aligned}$$

To study the time path of the quantity aggregate, we take its derivative with respect to time:

$$\begin{aligned} \frac{dM_t}{dt} &= \frac{du(m_t^*)}{dt} \\ &= \sum_{i=1}^n \left[\left(\left. \frac{\partial u(m)}{\partial m_i} \right|_{m=m_i^*} \right) \left(\frac{dm_{it}^*}{dt} \right) \right] \\ &= \sum_{i=1}^n \lambda \pi_{it} \frac{dm_{it}^*}{dt} \\ &= \lambda \sum_{i=1}^n \pi_{it} m_{it}^* \frac{d \log(m_{it}^*)}{dt}. \end{aligned}$$

Dividing this expression by the previous one, we obtain

$$\begin{aligned} \frac{d \log(M_t)}{dt} &= \sum_{i=1}^n w_{it} \frac{d \log(m_{it}^*)}{dt} \\ &= \frac{d \log(M_t^D)}{dt}. \end{aligned}$$

The last equation above expresses the growth rate of the continuous-time Divisia quantity index. The solution to this differential equation is a line integral that is path independent under the maintained assumption of weak separability (Hulten, 1973).

We conclude that, in continuous time, the Divisia quantity index is exact for the unknown monetary quantity aggregate, M_t . We emphasize that the exact tracking ability of the Divisia index is an implication of economic theory, not an approximation. The arguments in this section are valid for any first-degree homogeneous quantity aggregator function, and, therefore, the result is not specific to monetary aggregation.

The Discrete-Time Case

In discrete time, the situation is quite different; there is no statistical index number that is exact for an arbitrary aggregator function. Discrete time index-number theory is based on two facts: (1) mathematical functions exist that can provide second-order approximations to unknown aggregator functions; and, (2) statistical index numbers exist that are exact for some of these functions. An important class of mathematical functions, *locally-flexible functional forms*, are able to provide local second-order approximations to arbitrary discrete-time aggregator functions, in the sense that they can attain arbitrary elasticities of substitution at a single point (Diewert, 1971). This property is equivalent to the usual mathematical definition of second-order approximation (Barnett, 1983). Diewert (1976) showed that there exists a class of statistical index numbers, which he called *superlative*, that are exact for certain, specific flexible functional forms. Thus, superlative index numbers are able to provide second-order approximations to unknown, arbitrary economic aggregator functions, in discrete time.

A statistical index number is said to be *chained* if the prices and quantities used in the index number formula are the prices and quantities of adjacent periods, such as m_{it} and m_{it+1} , and said to be *fixed base* if the prices and quantities used in the index number formula are those of the current and a fixed base period, such as m_{it} and m_{i0} . Chained superlative index numbers have a distinct advantage over fixed-base

superlative indexes. For a chained superlative index, the center of the second-order approximation moves in such a way that the remainder term in the approximation relates to the changes between successive periods, rather than to the change from the base period to the current period. As a result, because changes in prices and quantities between adjacent periods are typically smaller than changes in prices and quantities between a fixed base period and the current period, the chained index number is likely to provide a better approximation to the unknown aggregator function than a fixed-base index number (Diewert, 1978). Further, Diewert's (1978, 1980) theorems prove that if the changes in prices and quantities are typically small between adjacent periods, chaining will tend to minimize the differences among alternative superlative index numbers. These latter theorems are based on numerical analysis and do not require economic optimization.

Many familiar index numbers are superlative. Two of the most important are the Fisher ideal index and the Törnqvist-Theil discrete-time approximation to Divisia's (1925) continuous-time quantity index.¹⁶ Diewert (1976) showed that the Fisher ideal index is exact for a homogeneous quadratic functional form (see also Konüs and Byushgens, 1926). For monetary aggregation, the *Fisher ideal quantity index*, M_t^F , is defined by

$$M_t^F = M_{t-1}^F \sqrt{\frac{\frac{\sum_{i=1}^n m_{it}^* \pi_{it}}{\sum_{i=1}^n m_{i,t-1}^* \pi_{it}} \cdot \frac{\sum_{i=1}^n m_{it}^* \pi_{i,t-1}}{\sum_{i=1}^n m_{i,t-1}^* \pi_{i,t-1}}}{\left(\frac{M_t^{TT}}{M_{t-1}^{TT}} \right)}}$$

The growth rate of the Fisher ideal quantity index is the geometric mean of the growth rates of the well-known Paasche and Laspeyres quantity indexes. The Paasche and Laspeyres indexes provide only first-order approximations to the underlying aggregate and are not superlative. Until recently, Paasche and Laspeyres indexes were the basis for most

Department of Commerce measures of economic activity, although these measures are now based on the chained Fisher Ideal index (Young, 1992, 1993; Triplett, 1992). The Laspeyres price index still underlies the Bureau of Labor Statistics' Consumer Price Index (CPI).¹⁷

For monetary aggregation, the *Törnqvist-Theil monetary quantity index*, M_t^{TT} , is defined as follows:

$$M_t^{TT} = M_{t-1}^{TT} \prod_{i=1}^n \left(\frac{m_{it}^*}{m_{i,t-1}^*} \right)^{\frac{1}{2}(w_{it} + w_{i,t-1})}$$

Diewert (1976) demonstrated that the Törnqvist-Theil quantity index is exact for the translog flexible functional form, and thus is superlative.

The dual user cost index, Π_t^{Dual} , defined by the recursive formula

$$\Pi_t^{Dual} = \Pi_{t-1}^{Dual} \left(\frac{\sum_{i=1}^n \pi_{it} m_{it}^*}{\sum_{i=1}^n \pi_{i,t-1} m_{i,t-1}^*} \right) \left/ \left(\frac{M_t^{TT}}{M_{t-1}^{TT}} \right) \right.$$

is based on a weak form of factor reversal, and is dual to the Törnqvist-Theil monetary quantity index.

The monetary services indexes and their dual user-cost indexes, which are described in the next article in this issue, "Building New Monetary Services Indexes," are chained superlative index numbers, employing the Törnqvist-Theil discrete-time quantity index-number formula and its dual-user cost index formula.

MONETARY SERVICE FLOWS AND MONETARY WEALTH

In the preceding section, the Törnqvist-Theil index was shown to measure the flow of monetary services produced by the representative consumer's monetary assets. In this section, we derive an expression for the stock of the consumer's monetary wealth. As with all goods and services, changes in the price of monetary services will affect

¹⁶Mathematically, the Törnqvist-Theil discrete-time index is the Simpson's rule approximation of the continuous-time Divisia index.

¹⁷The Advisory Commission to Study the Consumer Price Index (1996) recommended that the Bureau of Labor Statistics switch to using superlative index numbers.

both the demand for, and supply of, monetary services and other non-monetary goods and services. In addition to substitution and income effects, there are wealth effects associated with monetary assets.

One measure of monetary wealth is the discounted present value of the representative consumer's expected expenditure on monetary services. The measure follows immediately from Barnett's (1978, 1987) result that the multi-period budget constraints for the intertemporal decision, indexed by s contained in $\{t, t+1, \dots, t+T\}$,

$$\sum_{i=1}^h p_{is} q_{is} = w_s L_s + \sum_{i=1}^n \left[(1+r_{i,s-1}) p_{s-1}^* m_{i,s-1} - p_s^* m_{i,s} \right] + \left[(1+R_{s-1}) p_{s-1}^* A_{i,s-1} - p_s^* A_s \right],$$

can be combined into a single budget constraint. Monetary assets enter this single budget constraint through the term

$$V_t = \sum_{s=t}^T \sum_{i=1}^n \left[\frac{p_s^*}{\rho_s} - \frac{p_s^* (1+r_{is})}{\rho_{s+1}} \right] m_{is} = \sum_{s=t}^T \sum_{i=1}^n \pi_{is} m_{is},$$

where the discount factor,

$$\rho_s = \begin{cases} 1, & s = t \\ \prod_{u=1}^{s-1} (1+R_u), & t+1 \leq s \leq t+T \end{cases}$$

and the discounted nominal user costs, π_{is} , were discussed previously. Letting T go to infinity and evaluating V_t at the optimum yields,

$$V_t = \sum_{s=t}^{\infty} \sum_{i=1}^n \pi_{is} m_{is}^* = \sum_{s=t}^{\infty} y_s,$$

where y_s is the discounted expected optimal total expenditures on monetary assets in period s .

The economic interpretation of V_t is relatively straightforward, although its

measurement may be difficult. V_t is the discounted present value of all current and future expenditures on monetary services; in other words, V_t is the stock of monetary wealth. Unfortunately, V_t is an infinite forward sum of discounted expenditures and, as such, cannot be directly computed. To measure this definition of the stock of monetary wealth, we assume that the representative consumer forms *static expectations*. Specifically, we assume that the consumer's mathematical expectations of all future prices and rates, including the benchmark rate, equals the current values: for all s contained in $\{t+1, t+2, \dots\}$, $E_t[r_{is}] = r_{it}$ and $E_t[R_s] = R_t$, where $E_t[\]$ is the mathematical expectation operator based on the time- t information set. As a result, the consumer's expected optimal holdings of monetary assets in all future periods are equal to current holdings; that is, $E_t[m_{is}^*] = m_{it}^*$ for all s contained in $\{t+1, t+2, \dots\}$. Under this assumption, Barnett (1991) has shown that this measure of the stock of monetary wealth is equal to the Rotemberg *currency-equivalent (CE) index*, CE_t , which is defined as follows:

$$CE_t = p_t^* \sum_{i=1}^n \frac{R_t - r_{it}}{R_t} m_{it}^*$$

(see Rotemberg, 1991; and Rotemberg, Driscoll, and Poterba 1995). As a measure of monetary wealth, in this special case, the CE index can be used to study the response of consumer behavior to changes in the stock of monetary wealth.

With this interpretation of the CE index, it is possible for both the Törnqvist-Theil and CE indexes to be contained in the same model, because they are measures of different concepts. The Törnqvist-Theil index measures the current-period flow of monetary services. In contrast, the CE index, under static expectations, measures the discounted present value of current and future expenditures on monetary services, equal to the stock of monetary wealth. Equivalently, the Törnqvist-Theil index may be seen as a measure of the demand for a flow of monetary services, and the CE index

as a measure of a term in the consumer's budget constraint (Barnett, 1991). The CE index is able to measure the flow of monetary services in one special case. When, in addition to the assumptions that underlie the Törnqvist-Theil index, the category subutility function, $u(\cdot)$, is quasi-linear in a monetary asset whose own rate is always zero, the CE index will measure the flow of monetary services. Thus, the CE index is statistically inferior to the Törnqvist-Theil index as a measure of the flow of monetary services (Barnett, 1991).

The *simple sum index*, SS_t , defined as

$$SS_t = \sum_{i=1}^n p_t^* m_{it}^*$$

measures the flow of monetary services only if the representative agent's indifference curves for monetary assets are parallel lines and, hence, the agent regards all monetary assets as perfect substitutes. If the assets, in fact, have different user costs, then this agent would choose a corner solution and hold only one monetary asset in equilibrium—an implication that is both counterintuitive and counterfactual.¹⁸

In the context of the static-expectations model introduced above, the simple sum index may be interpreted as a "stock" variable; it is not, however, a measure of the stock of monetary wealth. The following relationship is useful in describing the stock concept that the simple sum index measures:

$$\begin{aligned} SS_t &= \sum_{i=1}^n p_t^* m_{it}^* \\ &= p_t^* \sum_{i=1}^n \frac{(R_t - r_{it}) m_{it}^*}{R_t} + \\ &\quad p_t^* \sum_{i=1}^n \frac{r_{it} m_{it}^*}{1 + R_t} \left[1 + \frac{1}{(1 + R_t)} + \frac{1}{(1 + R_t)^2} + \dots \right]. \end{aligned}$$

The index can, with this expression, be decomposed into two terms. The first term,

$$p_t^* \sum_{i=1}^n \frac{(R_t - r_{it}) m_{it}^*}{R_t}$$

is the CE index. The second term,

$$p_t^* \sum_{i=1}^n \frac{r_{it} m_{it}^*}{1 + R_t} \left[1 + \frac{1}{(1 + R_t)} + \frac{1}{(1 + R_t)^2} + \dots \right],$$

is the discounted present value of all current and future interest received on monetary assets, under the assumption of static expectations. Thus, in the context of the static-expectations model, the simple sum index may be interpreted as the sum of the discounted present value of expenditures on monetary services, plus the discounted present value of interest income from monetary assets; it cannot be interpreted, however, as a measure of the stock of monetary wealth (Barnett, 1991).

LIMITATIONS AND EXTENSIONS

The discussion in the previous sections of this paper has been based on very strong microeconomic assumptions. In particular, we have assumed (1) the existence of a representative agent (consumer or firm), (2) blockwise weak separability of current-period monetary assets, (3) homotheticity of the category subutility function, and (4) perfect certainty.¹⁹ In this section, we will discuss violations of these assumptions, and recent theoretical advances that attempt to address such problems.

Representative Agents, Weak Separability, and Divisia Second Moments

Assumptions (1) through (4) imply that the chained Törnqvist-Theil discrete-time approximation to the continuous-time Divisia index provides a second-order approximation to the unknown monetary aggregate, M_t . If the conditions that are necessary for the existence of a representative agent are not satisfied, or if the weak

¹⁸The existence of a corner solution identifies the price index that is dual to the simple sum index as Leontief, that is, the smallest user cost among the weakly separable block of monetary assets. For arguments against the use of simple sum indexes, see Fisher (1922).

¹⁹We also have assumed that the agent's portfolio is always in equilibrium; see Spencer (1994).

separability of current-period monetary assets from an agent's other decision variables is violated during some sample periods, then the growth rates of the optimal quantities of monetary assets, $m_{1t}^*, \dots, m_{nt}^*$, user costs, $\pi_{1t}, \dots, \pi_{nt}$, or expenditure shares, w_{1t}, \dots, w_{nt} , may contain information that is not contained in the Törnqvist-Theil index. Barnett and Serletis (1990) proposed the *dispersion dependency test* of the aggregation assumptions based on the Divisia second moments (variances) of the growth rates of quantities, user costs, and expenditure shares, as defined subsequently.²⁰

The log change (growth rate) of the Törnqvist-Theil quantity index is

$$\Delta \log(M_t^{TT}) = \sum_{i=1}^n \bar{w}_{it} \Delta \log(m_{it}^*),$$

where for $i=1, \dots, n$,

$$\bar{w}_{it} = \frac{1}{2}(w_{it} + w_{i,t-1})$$

are the monetary assets' between-period average expenditure shares. This expression is in the form of an average share-weighted mean of the log change of component quantities (component growth rates). Theil (1967) pointed out that the growth rate of the Törnqvist-Theil quantity index has a natural interpretation as the mean of the component quantity growth rates, where the average shares induce a valid measure of probability. Thus, the growth rate of the Törnqvist-Theil quantity index may be interpreted as the first moment of a distribution. Similarly, the growth rate of the *Törnqvist-Theil user-cost index*, P_t^{TT} , is in the form of an average share-weighted mean of component user-cost growth rates,

$$\Delta \log(P_t^{TT}) = \sum_{i=1}^n \bar{w}_{it} \Delta \log(\pi_{it}).^{21}$$

Finally, the growth rate of the *Törnqvist-Theil expenditure-share index*, S_t^{TT} , is in the form of an average share-weighted mean of component expenditure-share growth rates,

$$\Delta \log(S_t^{TT}) = \sum_{i=1}^n \bar{w}_{it} \Delta \log(w_{it})$$

Thus, the growth rates of the Törnqvist-Theil user-cost and expenditure-share indexes also can be interpreted as the first moments of an underlying probability distribution.²² Theil (1967) showed that the growth rates of the three indexes are related by the identity

$$\begin{aligned} \sum_{i=1}^n \bar{w}_{it} \Delta \log(w_{it}) + \Delta \log(y_t) \\ = \sum_{i=1}^n \bar{w}_{it} \Delta \log(m_{it}^*) + \sum_{i=1}^n \bar{w}_{it} \Delta \log(\pi_{it}) \end{aligned}$$

The stochastic interpretation of the Törnqvist-Theil indexes as first moments can be generalized to higher moments of the underlying distributions, which are usually called *Divisia higher moments*.²³ The *Divisia quantity growth-rate variance*,

$$K_t = \sum_{i=1}^n \bar{w}_{it} \left[\Delta \log(m_{it}^*) - \Delta \log(M_t^{TT}) \right]^2,$$

is the variance of the growth rates of the individual quantities. The *Divisia user-cost variance*,

$$J_t = \sum_{i=1}^n \bar{w}_{it} \left[\Delta \log(\pi_{it}) - \Delta \log(P_t^{TT}) \right]^2,$$

is the variance of the growth rates of the component user costs. The *Divisia expenditure-share growth-rate variance*,

$$\Psi_t = \sum_{i=1}^n \bar{w}_{it} \left[\Delta \log(w_{it}) - \Delta \log(S_t^{TT}) \right]^2$$

is the variance of the growth rates of the component expenditure shares. The *covariance* of the growth rates of the component user costs and quantities is

²⁰ Barnett and Serletis (1990) show that aggregation error introduces an additive remainder term in the second-order discrete-time approximation. Some econometric tests for specification error in linear regression models exploit a similar correlation between regression residuals and nonlinear functions of the independent variables (Ramsey, 1969; Thursby, 1979; Thursby and Schmidt, 1977).

²¹ Although the continuous-time Divisia price index is dual to the continuous-time Divisia quantity index, the discrete-time Törnqvist-Theil price index is not dual to discrete-time Törnqvist-Theil quantity index (Theil, 1967). In applications, the Törnqvist-Theil quantity index should be used with its dual user cost index, not the Törnqvist-Theil user cost index.

²² Clements and Izan (1987) developed an alternative interpretation of the Divisia index. In a model of statistical Hicksian aggregation, the Divisia price index is interpreted as the generalized least squares estimate of the common trend in a price formation function, provided that the variances of the individual component price estimates are inversely proportional to their expenditure share. For a critique and extensions, see Diewert (1995).

²³ Diewert (1995) suggests a class of measures of functional form error based on the dispersion of component growth rates of statistical price indexes. If Diewert's approach is generalized to quantity and share indexes, the Divisia second moments are the squares of a member of this class, so the second moments can also be interpreted as providing measures of the reliability of an index.

$$\Gamma_t = \sum_{i=1}^n \left\{ \bar{w}_{it} \left[\Delta \log(\pi_{it}) - \Delta \log(P_t^{TT}) \right] \right. \\ \left. \times \left[\Delta \log(m_{it}^*) - \Delta \log(M_t^{TT}) \right] \right\}.$$

Theil (1967) showed that these four second moments of are related by the following identity:

$$\psi_t = K_t + J_t + 2\Gamma_t.$$

Dispersion-dependency tests based on Divisia second moments for United States monetary data are presented in Barnett and Serletis (1990) and Barnett, Jones, and Nesmith (1996). The evidence in these studies suggests that Divisia second moments do contain economic information not contained in the growth rates of the Törnqvist-Theil quantity, price, and expenditure share indexes. In other words, for at least some time periods, movements in the observed quantities and prices of monetary assets are not consistent with the movements that would be implied by the actions of a representative agent with a weakly separable utility function. In such cases, Barnett and Serletis (1990) suggest that including Divisia second moments in macroeconomic models might provide at least a partial correction for the aggregation error.²⁴

Homothetic Preferences

The theoretical results presented in this paper have been derived under the maintained hypothesis that the category subutility function for current-period monetary assets, $u(\cdot)$, is homothetic. If homotheticity is violated, then the quantity and price aggregator functions are not the subutility and the unit-expenditure functions, respectively, and the Divisia index will not track the utility function in continuous time.

In this section, we extend our previous discussion to include economic indexes that are correct in general, even if homotheticity is violated. Assuming that the representative agent's utility function is

weakly separable in current-period monetary assets, we can define quantity and user-cost aggregator functions—namely, the Konüs and Malmquist indexes—that are correct in the absence of homotheticity.

Let $u(\cdot)$ be the agent's category subutility function, which is not necessarily homothetic. The monetary quantity aggregate may be defined as the *distance function* $d(m_t^*, \tilde{u})$, which is defined implicitly by

$$u\left(\frac{m}{d(m, \tilde{u})}\right) = \tilde{u}$$

The user-cost aggregate, dual to the distance function, is the expenditure function evaluated at the reference utility level, \tilde{u} , defined by

$$e(\pi_t, \tilde{u}) = \min_{m_t} \left\{ \sum_{i=1}^n \pi_{it} m_{it}; u(m_t) = \tilde{u} \right\}$$

(Barnett, 1987; Deaton and Muellbauer, 1980). Normalizing these quantity and price aggregates to equal one in a base period produces *exact economic indexes*.

The *Malmquist quantity index* is defined by

$$M(m_t^*, m_0^*, \tilde{u}) = \frac{d(m_t^*, \tilde{u})}{d(m_0^*, \tilde{u})}$$

(Malmquist, 1953), and the *Konüs user-cost index* is defined by

$$K(\pi_t, \pi_0, \tilde{u}) = \frac{e(\pi_t, \tilde{u})}{e(\pi_0, \tilde{u})}$$

(Konüs, 1939). Both indexes have been normalized to equal unity in period $t=0$, and have been defined in terms of monetary assets and user costs.

Although these general indexes are correct whether or not the category subutility function is homothetic, a shortcoming is that both depend on the reference utility level, \tilde{u} . Konüs (1939) showed that the value of the user-cost index, in any period, can be bounded above and below, albeit at different reference utility levels. The upper bound in the case of monetary assets is a *Laspeyres price index*, given by

²⁴ If aggregation conditions do not hold, additional higher-order moments of the index's distribution could contain additional information. Dispersion dependency testing could therefore be extended to testing other moments.

$$\frac{\sum_{i=1}^n \pi_{it} m_{i0}^*}{\sum_{i=1}^n \pi_{i0} m_{i0}^*} \geq K[\pi_t, \pi_0, u(m_0^*)],$$

where m_{i0}^* is the optimal quantity of monetary asset i in the base period $t=0$. The lower bound is a *Paasche price index* given by

$$\frac{\sum_{i=1}^n \pi_{it} m_{it}^*}{\sum_{i=1}^n \pi_{i0} m_{it}^*} \leq K[\pi_t, \pi_0, u(m_t^*)],$$

where m_{it}^* is the optimal quantity of monetary asset i in period t . See Frisch (1936) and Leontief (1936) for early discussion of these issues. Diewert (1993) reviews the early history of index-number theory.

It can now be seen why homotheticity is such a valuable property. The Konüs and Malmquist indexes do not require homotheticity, but they depend on a specific reference utility level. It is easily shown that this dependence vanishes when the category subutility function is first-degree homogeneous. With first-degree homogeneity, the distance function is proportional to the utility function, with the proportionality factor equal to the reference utility level. The Malmquist index may thus be written as

$$\begin{aligned} M(m_t^*, m_0^*, \tilde{u}) &= \frac{d(m_t^*, \tilde{u})}{d(m_0^*, \tilde{u})} = \frac{u(m_t^*) \cdot \tilde{u}}{u(m_0^*) \cdot \tilde{u}} \\ &= \frac{u(m_t^*)}{u(m_0^*)} \end{aligned}$$

First-degree homogeneity also implies that

$$e(\pi, \tilde{u}) = e(\pi, 1) \tilde{u},$$

so the Konüs index may be written as

$$K(\pi_t, \pi_0, \tilde{u}) = \frac{e(\pi_t, 1)}{e(\pi_0, 1)},$$

which is independent of the reference utility level. Further, with first-degree

homogeneity, the Konüs index is bounded by the Paasche and Laspeyres indexes

$$\frac{\sum_{i=1}^n \pi_{it} m_{i0}^*}{\sum_{i=1}^n \pi_{i0} m_{i0}^*} \geq K(\pi_t, \pi_0, \tilde{u}) \geq \frac{\sum_{i=1}^n \pi_{it} m_{it}^*}{\sum_{i=1}^n \pi_{i0} m_{it}^*},$$

for any reference utility level, \tilde{u} . This is a formal statement, in the monetary aggregation case, of the often-quoted result that base-period-weighted (Laspeyres) price indexes overstate the true increase in prices, and current-period-weighted (Paasche) price indexes understate the true change in prices. Note that this result is critically dependent on the assumption of homotheticity. In general, if homotheticity is violated, the Paasche price index may actually exceed the Laspeyres price index (see Deaton and Muellbauer, 1980). The potential upward, or downward, bias resulting from the use of Laspeyres and Paasche price indexes is discussed by Triplett (1992).

Although homotheticity produces attractive simplifications of aggregation theory, it is implausible that any population is well characterized by an assumption of identical homothetic utility functions—Samuelson and Swamy (1974) label this a “Santa Claus” assumption. When homotheticity is violated, the Törnqvist-Theil discrete-time approximation to the continuous-time Divisia index has been shown to track the distance function. Caves, Christensen, and Diewert (1982) proved that the Törnqvist-Theil index is superlative in the broader sense that it can provide a second-order approximation to the Malmquist quantity index, even when homotheticity is violated. No other statistical index number is known to have this important property. The monetary services indexes presented in the next article in this issue of the *Review*, “Building New Monetary Services Indexes,” which are based on the Törnqvist-Theil index, should be robust to violations of homotheticity.

Perfect Certainty

Recently, economists have addressed the problem of constructing monetary aggregates that include assets with risky returns (see Collins and Edwards, 1994; Orphanides, Reid, and Small, 1994; and Barnett, 1994). Prior to this section, this article has included only perfect-certainty models, even though some of the monetary services indexes presented in the next article in this issue include capital-uncertain monetary assets, such as U.S. Treasury bills, with variable returns. Aggregation theory needs to be extended to include risky assets.

Extending the theory to include risk-neutral consumers is straightforward. Barnett (1994) has shown that, under risk neutrality, the Törnqvist-Theil discrete-time approximation to the continuous-time Divisia index provides a second-order approximation to the unknown aggregator function, where the user costs are defined as the expected value of the nominal current-period perfect-certainty user costs,

$$\pi_{it} = E_t \left\{ p_t^* \frac{R_t - r_{it}}{1 + R_t} \right\}.$$

Extending the theory further to include risk-averse consumers is more difficult. Rotemberg (1991) noted that, under risk aversion, Barnett's (1994) result does not hold because the asset's user costs are correlated with the representative agent's marginal utility. In related work, Barnett and Zhou (1994) derived a supply-side model of monetary aggregation under risk aversion, based on the profit-maximizing behavior of depository financial institutions. Barnett and Liu (1994) and Barnett, Liu, and Jensen (1997) discuss a generalized Divisia quantity index in which the user cost is adjusted to account for risk. The size of their adjustment depends on the agent's degree of risk aversion and on the covariance between the asset's rate of return and the agent's consumption stream. Empirically, they find that there is negligible difference between their generalized Divisia

index and a more standard index, for the set of monetary assets included in the Federal Reserve Board's monetary aggregates. Hence, we believe that the omission of risk adjustment in the monetary services indexes constructed in this research project is unlikely to be empirically relevant.

CONCLUSION

This paper has surveyed the microeconomic theory of monetary aggregation. This theory is built from the aggregator functions of representative agents. Applied to a representative consumer's utility function, it generally requires that current-period quantities of monetary assets be weakly separable from other assets, goods, services, and leisure. Applied to a representative firm's production function, the theory requires that current-period monetary assets be weakly separable from other inputs.

The aggregation theory and methods surveyed in this article are based on models of the optimizing behavior of representative economic agents. These methods underlie the construction of our monetary services indexes and related variables, presented in the following article in this issue, as well as the Department of Commerce's measures of real economic activity, such as GDP. Because these methods assume an optimizing representative agent, they have a common foundation with modern general-equilibrium business-cycle models. Including our quantity and dual user cost indexes in empirical models of economic activity does not require any stronger assumptions than those already embedded in the Department of Commerce's measures of aggregate economic activity.

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