Stability and Equilibrium Selection in Learning Models: A Note of Caution

YiLi Chien, In-Koo Cho, and B. Ravikumar

Relative to rational expectations models, learning models provide a theory of expectation formation where agents use observed data and a learning rule. Given the possibility of multiple equilibria under rational expectations, the learning literature often uses stability as a criterion to select an equilibrium. This article uses a monetary economy to illustrate that equilibrium selection based on stability is sensitive to specifications of the learning rule. The stability criterion selects qualitatively different equilibria even when the differences in learning specifications are small. (JEL C60, D84)

Federal Reserve Bank of St. Louis *Review*, Fourth Quarter 2021, *103*(4), pp. 477-88.
https://doi.org/10.20955/r.103.477-88

1 INTRODUCTION

Under the rational expectations (RE) hypothesis, the expectations of agents are consistent with and always confirmed by equilibrium outcomes. This hypothesis often significantly simplifies the analysis of complicated economic problems. However, the RE hypothesis is silent on how agents form their expectations and, hence, provides no guidance on selecting an equilibrium when multiple RE equilibria occur. In contrast, a learning model specifies the agent's learning rule for forming expectations and is used to select one from multiple RE equilibria. The advantage of using the learning approach for equilibrium selection is noted by Evans and Honkapohja (2001).[1] The standard criterion for equilibrium selection in the learning approach is stability of the learning dynamics. This consideration is quite intuitive: If a learning equilibrium is not stable, then it is unlikely to be the long-run equilibrium outcome.

Examples of using stability of the learning dynamics to eliminate some RE equilibria include Lucas (1986), Marcet and Sargent (1989), Woodford (1990), and Bullard and Mitra (2002). Lucas (1986) argues that adaptive behavior of economic agents may narrow the set of equilibria in some economic models. By using the stability criterion under learning, Marcet

YiLi Chien is a research officer and economist at the Federal Reserve Bank of St. Louis. In-Koo Cho is a professor of economics at Emory University and Hanyang Univerity and a research fellow at the Federal Reserve Bank of St. Louis. B. Ravikumar is a senior vice president, the deputy director of research, and an economist at the Federal Reserve Bank of St. Louis.

and Sargent (1989) eliminate the hyperinflation equilibrium of Sargent and Wallace (1985). In a monetary model with multiple RE equilibria, Woodford (1990) shows that the economy could converge to a stationary sunspot equilibrium under learning. Bullard and Mitra (2002) argue that the stability under learning criterion is necessary for monetary policy evaluation, especially in situations where multiple RE equilibria could be induced by policy.

The focus of our article is on equilibrium selection using the stability criterion under *different specifications of learning*. We conduct our exercise using an example in Bullard (1994), which is a simplified version of the model in Sargent and Wallace (1981). Our choice of the Sargent-Wallace framework is deliberate. The framework is simple and admits two steady states under RE, so we can explore the issue of selection. Furthermore, the key features of the model have been used repeatedly in the learning literature; see, for instance, Marcet and Sargent (1989), Bullard (1994), and Marcet and Nicolini (2003). We employ the model to examine equilibrium selection in different learning specifications via the stability criterion, a criterion that is common in the learning literature.

Our model is an overlapping generations endowment economy where money is the only store of value. The optimal decision of agents depends on their inflation forecasts, and so do the equilibrium outcomes. Under RE, the model has two steady states: high inflation and low inflation. To select one of the two steady states, we consider a learning model where agents forecast inflation using a rule that is a convex combination of past expected inflation and actual inflation. We examine the learning dynamics under two specifications: (i) Agents know only the past prices, and (ii) agents know the current price in addition to past prices. The two specifications imply different values for actual inflation in the learning rule. Under (i), agents use last period's inflation rate, whereas under (ii), they use the current inflation rate. Thus, the only difference between the two specifications is that the value of actual inflation is current in one specification and lagged by just one period in the other.

Our main result is that the stability criterion selects qualitatively different equilibria even when the differences in the learning specifications are minor. In particular, the learning rule using last period's inflation rate implies that the low-inflation RE steady state is the only stable learning equilibrium. Thus, using the stability criterion to select an RE equilibrium implies that the low-inflation steady state would be the long-run outcome. However, under the learning rule using the current inflation rate, both RE steady states are stable. Thus, the stability criterion does not offer useful guidance for equilibrium selection. In other words, our simple model shows that the stability of the learning equilibrium is sensitive to the specification of the learning rule. The learning dynamics may not be robust against seemingly minor differences in the learning rule.

Earlier work by Marcet and Sargent (1989) demonstrated how the stability criterion under RE dynamics selects the equilibrium that validates the "unpleasant monetarist arithmetic" in Sargent and Wallace (1981) but that the same criterion under least-squares learning selects another equilibrium that *invalidates* the unpleasant monetarist arithmetic. Our investigation of the sensitivity issue differs from that in Marcet and Sargent (1989) in an important way: Marcet and Sargent (1989) compare the equilibrium selection under two substantially different specifications—RE and learning. Under RE, the decisionmaker is a forward-looking rational agent, while under the learning dynamics, the decisionmaker is a backward-looking boundedly rational agent. We, on the other hand, compare outcomes under two learning specifications. Under both learning specifications, we maintain the assumption that the decisionmaker is a least-squares learner, but we change the timing of the observation of one variable by just one period.

Our exercise is in the same spirit as the exercises by Hansen and Sargent (2007) that examine robustness of an equilibrium to small changes in specifications. Hansen and Sargent (2007) endow the agents with a set of models instead of just one. The agents have a reference model and entertain a small neighborhood of models around the reference model and respond by choosing the model that performs the best against the worst possible state. Similarly, Cho and Kasa (2017) consider a set of "nearby" learning models close to a benchmark model, but the agent uses model averaging between the benchmark and the nearby models. In Cho and Kasa (2015), the agents choose a model based on a specification test.[2] In contrast to these articles, our agents do not evaluate multiple models simultaneously and do not choose one based on a pre-specified performance criterion. Instead, our agents consider only one model at a time. We examine the equilibrium selected by the stability criterion in each model.

Section 2 sets up an overlapping generations model and characterizes the equilibrium outcome under RE. In Section 3, we investigate the issue of equilibrium selection under learning. In particular, we demonstrate that the stability properties change with minor changes in the learning specifications. Section 4 contains concluding remarks.

2 AN OVERLAPPING GENERATIONS MODEL OF INFLATION

To demonstrate our idea, we adopt a simplified version of the model in Sargent and Wallace (1981). Time is discrete and indexed by $t = 0,1,2,3\ldots$ In each period t, a new generation of households is born and lives for two periods: t and $t+1$. An old generation also exists at $t = 0$. Thus, at any point in time $t \geq 0$: There is a young and an old generation of households. The population size of each generation remains constant over time and is normalized to unity. Each agent in generation t has a logarithmic utility function with no discounting:

$$U_t = \ln c_{1,t} + \ln c_{2,t},$$

where $c_{i,t}$ is consumption of the generation-t agent in $i = 1,2$ period of their life. Each generation-t agent is endowed with 2 and 2λ, $\lambda \in (0,1)$, units of perishable consumption goods when young and old, respectively. The only asset available is fiat money, which is denoted by m_t. The flow budget constraints of a generation-t agent in periods t and $t+1$ are given, respectively, by

$$p_t c_{1,t} + m_t \leq 2 p_t$$

and

$$p_{t+1} c_{2,t} \leq 2\lambda p_{t+1} + m_t,$$

where p_t is the price level in period t.

The government finances its expenses by issuing fiat money. The nominal government budget constraint is given by

$$\xi p_t = M_t - M_{t-1}, \tag{1}$$

where $\xi > 0$ is the *constant* real government expenditure and M_t is the aggregate money supply in period t.

The timing is as follows. In each period, the old agents enter with the nominal balances from the previous period. The young agents make their consumption and saving decisions. The government purchases goods by injecting money. Finally, consumption takes place based on realized prices at the end of the period.

The consumption of the initial old generation is determined by their budget constraint:

$$c_{2,0} = 2\lambda + \frac{m_0}{p_0},$$

where the initial nominal money balance, m_0, is exogenously given.

Note that aggregate gross domestic product (GDP) in each period is pinned down by the endowments and is $2 + 2\lambda$.

2.1 RE Equilibria

We first demonstrate that there are two steady states under RE. Given the deterministic setup, agents of each generation know the entire sequence of prices. Given the prices in t and $t+1$, using the flow budget constraints of a generation-t agent, we can write the lifetime budget constraint as

$$c_{1,t} + \frac{p_{t+1}}{p_t} c_{2,t} \leq 2 + 2\lambda \frac{p_{t+1}}{p_t}. \tag{2}$$

The problem of a generation-t agent is

$$\max_{\{c_{1,t}, c_{2,t}\}} \ln c_{1,t} + \ln c_{2,t}$$

subject to (2). The flow budget constraints and the choice of consumption determine the real balances of generation-t agents.

An *RE equilibrium* is a sequence of quantities and prices—$c_{1,t}, c_{2,t-1}, \frac{M_t}{p_t}, p_t, \Pi_t$—for $t = 0,1,\ldots,\infty$ such that agents in each generation choose consumption and real balances optimally, and the asset market and goods market clear in every period.

Note that goods market clearing implies that aggregate consumption in each period equals aggregate GDP and is $2 + 2\lambda$. However, as we show below, the distribution of consumption across generations and real balances in each period are affected by the path of inflation.

The first-order conditions for the above maximization problem can be summarized as $\frac{c_{1,t}}{c_{2,t}} = \frac{p_{t+1}}{p_t}$, which together with (2) imply the optimal (interior) choices are

$$c_{1,t} = 1 + \lambda \frac{p_{t+1}}{p_t} \text{ and } c_{2,t} = \frac{p_t}{p_{t+1}} + \lambda.$$

Therefore, the saving of generation t is $\frac{m_t}{p_t} = 2 - c_{1,t} = 1 - \lambda \frac{p_{t+1}}{p_t}$. Since fiat money is the only store of value, that saving must be in the form of real money balances:

$$\frac{M_t}{p_t} = 1 - \lambda \frac{p_{t+1}}{p_t} = 1 - \lambda \Pi_{t+1},$$

where $\Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}$ denotes the *actual* inflation rate between periods t and $t+1$.

If the inflation rate exceeds $\frac{1}{\lambda}$, the real rate of return on money, $\frac{p_t}{p_{t+1}}$, is "too low" and the young agent would like to borrow, not save. However, this is impossible in a two-period overlapping generations setup. Therefore, for inflation rates greater than or equal to $\frac{1}{\lambda}$, the young agent would just consume their endowment. Hence, for money to be held (i.e., for real balances to be positive) the inflation rate must be less than $\frac{1}{\lambda}$. Thus, the demand for money can be written as

$$\frac{M_t}{p_t} = \max\left(0, 1 - \lambda \Pi_{t+1}\right). \tag{3}$$

The asset-market-clearing condition implies that money supplied in each period must equal the money demand in that period. Rewriting equation (1) as

$$\frac{M_t}{p_t} = \frac{M_{t-1}}{p_{t-1}} \frac{1}{\Pi_t} + \xi,$$

we can substitute money demand (3) into the above equation and get

$$\max\left(0, 1 - \lambda \Pi_{t+1}\right) = \frac{\max\left(0, 1 - \lambda \Pi_t\right)}{\Pi_t} + \xi. \tag{4}$$

Consider the case where $\Pi_t < \frac{1}{\lambda}$ for all t. The real balances are positive, so equation (4) can be simplified as

$$\Pi_{t+1} = 1 + \frac{1}{\lambda} - \frac{\xi}{\lambda} - \frac{1}{\lambda} \frac{1}{\Pi_t}.$$

In a steady state, $\Pi_{t+1} = \Pi_t \equiv \Pi$ so the above equation can be rewritten as

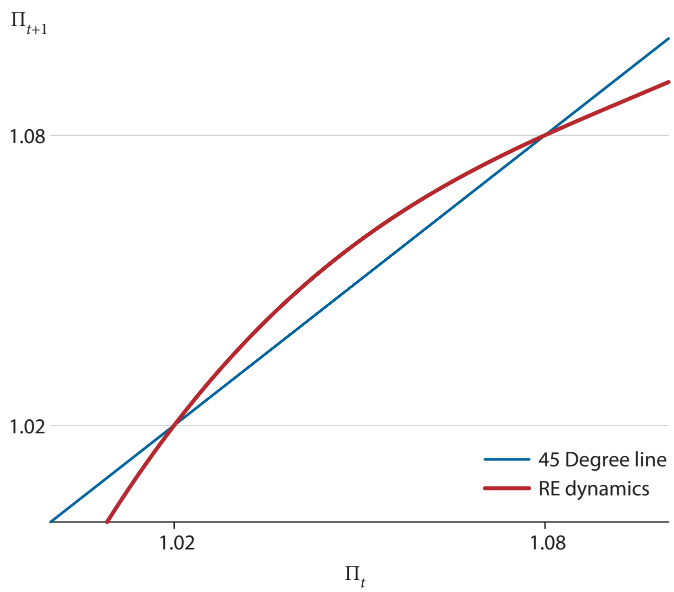
$$\lambda \Pi^2 - \left(\lambda + 1 - \xi\right) \Pi + 1 = 0, \tag{5}$$

which is a quadratic equation in Π. Figure 1 illustrates equation (4).

If $\xi < \lambda - 2\lambda^{\frac{1}{2}} + 1$, then the equation has two real distinct solutions, which represent two steady-state values of Π. Denote them as Π_1^{RE} and Π_2^{RE} and let $\Pi_1^{RE} < \Pi_2^{RE}$ without loss of

Figure 1

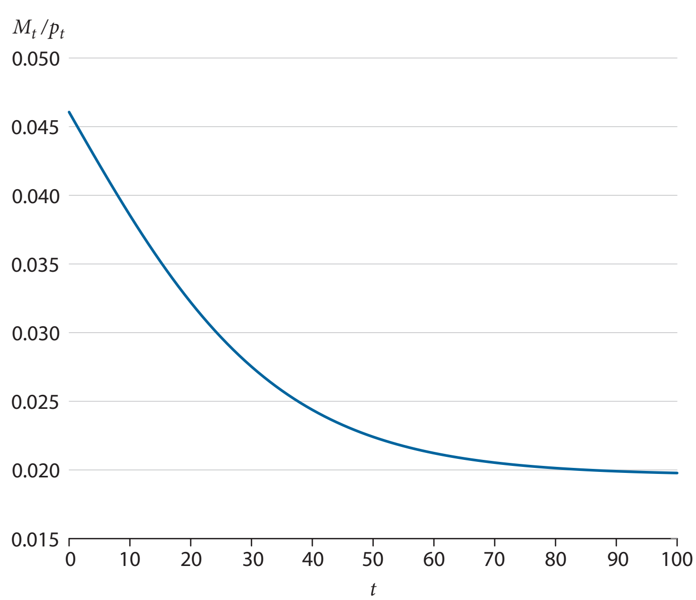
RE Equilibrium



NOTE: The steady-state inflation values are 1.02 and 1.08. The parameter values are $\xi = 0.0015$ and $\lambda = 0.9078$.

Figure 2

Real Balances: Path to the Stable RE Steady State



NOTE: The high-inflation steady state value is 1.08. The parameter values are $\xi = 0.0015$ and $\lambda = 0.9078$. The real money balance is 0.196 in the stable high-inflation steady state. For the simulation path, the initial condition is $\Pi_0 = 1.05$.

generality. It is straightforward to show that $1<\Pi_1^{RE}<\Pi_2^{RE}<\frac{1}{\lambda}$. Clearly, as shown by Figure 1, the low-inflation steady state, Π_1^{RE}, is unstable and the high-inflation steady state Π_2^{RE} is stable.

In Figure 2, we illustrate the path of real balances converging to the stable high-inflation steady state. In the next section, we investigate the issue of equilibrium selection using the stability criterion under learning.

3 EQUILIBRIUM SELECTION UNDER LEARNING

Suppose that agents do not have perfect foresight and they forecast prices based on information available to them. Generation-t agents make their optimal choices based on their expectation of p_{t+1}. Let $\Pi_{t+1}^e \equiv \frac{p_{t+1}^e}{p_t}$ denote the *expected* inflation rate between periods t and $t+1$, where the superscript "e" denotes the expected value. The learning rule we consider is a convex combination of past expected inflation and actual inflation:

$$\Pi_{t+1}^e=\Pi_t^e+\gamma\left[\text{Actual inflation}-\Pi_t^e\right], \tag{6}$$

where $\gamma\in(0,1)$ is the gain parameter. The value of actual inflation used in the learning rule depends on the information available to the agents. We examine two specifications in the next two subsections. Given the learning rule, the lifetime budget constraint of a generation-t agent is

$$c_{1,t}+\Pi_{t+1}^e c_{2,t}\le 2+2\lambda\Pi_{t+1}^e.$$

A *learning equilibrium* is a sequence of quantities, prices, and forecasts—$c_{1,t},c_{2,t-1},\frac{M_t}{p_t},p_t,\Pi_t,\Pi_t^e$—for $t=0,1,\ldots,\infty$ such that agents in each generation choose consumption and real balances optimally based on their forecast of inflation, the asset market and goods market clear in every period, and the dynamics of expected inflation satisfy equation (6).

Aggregate GDP and consumption in each period in a learning equilibrium are the same as in an RE equilibrium: $2 + 2\lambda$.

Similar to (3), the demand for real balances under learning is

$$\frac{M_t}{p_t}=\max\left(0,1-\lambda\Pi_{t+1}^e\right). \tag{7}$$

Again, the real money demand has a zero lower bound: If expected inflation exceeds $\frac{1}{\lambda}$, then the expected real return on money is too low and the young agents will not hold any money.

As in Section 2.1, using equation (1), we get $\frac{M_t}{p_t}=\frac{M_{t-1}}{p_{t-1}}\frac{1}{\Pi_t}+\xi$. Substituting money demand into this equation, we get the asset-market-clearing condition:

$$\max\left(0,1-\lambda\Pi_{t+1}^{e}\right)=\max\left(0,1-\lambda\Pi_{t}^{e}\right)\frac{1}{\Pi_{t}}+\xi.$$

Under the assumption of $\Pi_t^e < \frac{1}{\lambda}$ for all t, the above relationship yields the law of motion for inflation under learning:

$$\Pi_t = \frac{1-\lambda\Pi_t^e}{1-\lambda\Pi_{t+1}^e-\xi}. \tag{8}$$

3.1 Actual Inflation Using Past Prices

Assume that the agents do not know the price p_t, so they do not know the actual inflation Π_t between periods $t-1$ and t. When the agents forecast the inflation between periods t and $t+1$, the actual inflation used in forecasting Π_{t+1}^e is Π_{t-1}, which is the most recent data available to the agents. The learning rule then is as follows:

$$\Pi_{t+1}^e = \Pi_t^e + \gamma\left[\Pi_{t-1} - \Pi_t^e\right]. \tag{9}$$

Note that lack of knowledge of p_t does not affect the demand for real balances, since equation (7) implies that the demand depends on the *ratio* of prices.

Using equation (8) to substitute for Π_{t-1} in the learning rule above, we get the law of motion for expected inflation Π_{t+1}^e under learning:

$$\Pi_{t+1}^e = \Pi_t^e + \gamma\left[\frac{1-\lambda\Pi_{t-1}^e}{1-\lambda\Pi_t^e-\xi} - \Pi_t^e\right]. \tag{10}$$

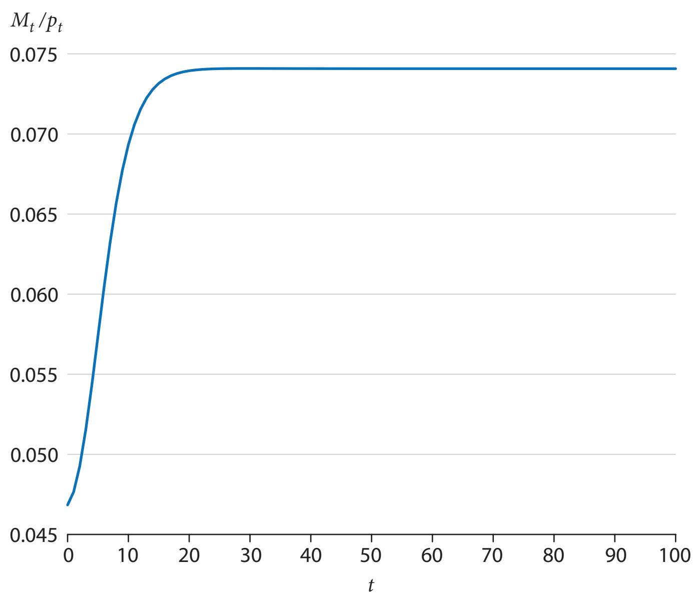
Three features of equation (10) are worth noting. First, the steady-state version of (10) is identical to the RE steady-state equation (5). Hence, the learning dynamics imply the same two steady states as under RE: Π_1^{RE} and Π_2^{RE}. Second, equation (10) is not tractable since it is a second-order nonlinear difference equation, which does not have analytical solutions. Finally, we can show that, under this learning rule and conditional on a small enough γ, the low-inflation steady state, Π_1^{RE}, becomes stable and the high-inflation steady state, Π_2^{RE}, becomes unstable (see Chien, Cho, and Ravikumar, 2021).[3]

Hence, selection using the stability criterion implies that the low-inflation steady state should be the long-run equilibrium outcome of this model. This result is also true in Marcet and Sargent (1989). Furthermore, since the stable steady state under RE is different from that under learning rule (9), the paths of real balances converging to the respective stable steady states would be different. Figure 3 illustrates the path of real balances converging to the low-inflation steady state under learning rule (9). Starting from "similar" initial conditions, the real balances increase monotonically to the stable state under learning, but decline monotonically under RE.

In the next subsection, we show that the equilibrium selection using stability is sensitive to the learning specification.

Figure 3

Real Balances: Path to the Stable Steady State Under Learning Using Past Inflation



NOTE: The low-inflation steady-state value is 1.02. The parameter values are $\xi = 0.0015$, $\lambda = 0.9078$, and $\gamma = 0.05$. The value of the low-inflation steady state is the same as in the RE equilibrium. The real money balance is 0.741 in the stable low-inflation steady state. For the simulation path, the initial conditions are $\Pi_0 = \Pi_1 = 1.05$.

3.2 Actual Inflation Using Current and Past Prices

Now suppose that when agents forecast inflation Π_{t+1}^e they do know the price p_t and, hence, are able to use the actual inflation Π_t to update their forecast. Hence,

$$\Pi_{t+1}^e = \Pi_t^e + \gamma\left[\Pi_t - \Pi_t^e\right]. \tag{11}$$

Note that the only difference between equations (9) and (11) is the actual inflation used in the formula: Π_{t-1} in (9) and Π_t in (11).

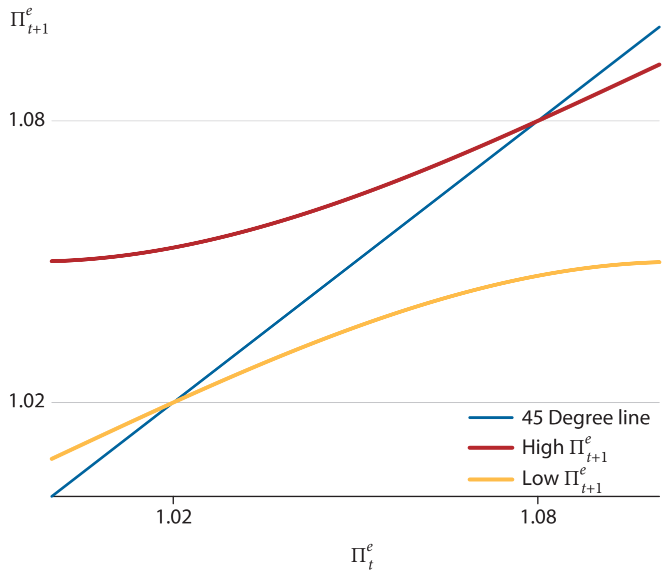
Using equation (8) to substitute for Π_t in (11), we get an alternative law of motion for expected inflation Π_{t+1}^e:

$$\Pi_{t+1}^e = \Pi_t^e + \gamma\left[\frac{1-\lambda\Pi_t^e}{1-\lambda\Pi_{t+1}^e - \xi} - \Pi_t^e\right]. \tag{12}$$

In the steady state, equation (12) is reduced to RE steady-state equation (5). Hence, Π_1^{RE} and Π_2^{RE} remain the two possible steady states under learning rule (11). However, other features of equation (12) are quite different from those of equation (10). First, equation (12) is a first-order nonlinear equation and its solution is analytically tractable. Second, despite having the same set of steady states, both steady states are stable. To see this, Figure 4 illustrates learning dynamics (12). For a given Π_t^e, equation (12) is a quadratic equation in Π_{t+1}^e and, hence,

Figure 4

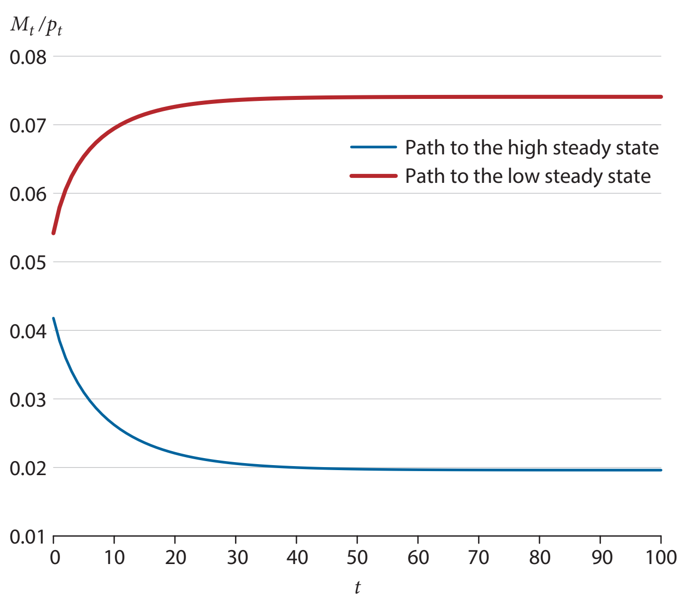
Learning Dynamics Using Current Inflation



NOTE: The steady-state inflation values are 1.02 and 1.08. The parameter values are $\xi = 0.0015$, $\lambda = 0.9078$, and $\gamma = 0.05$. The values of steady-state inflation are identical to those in the RE equilibrium.

Figure 5

Real Balances: Path to the Stable Steady States Under Learning Using Current Inflation



NOTE: The steady-state inflation values are 1.02 and 1.08. The parameter values are $\xi = 0.0015$, $\lambda = 0.9078$, and $\gamma = 0.05$. The values of steady-state inflation are identical to those in the RE equilibrium. The steady-state values for real balances are 0.0741 and 0.0196. For the simulation path, the initial condition is $\Pi_0 = 1.05$.

has two possible solutions. For any Π_t^e, the red and yellow lines of Figure 4 illustrate the solutions of high and low expected inflation rates, Π_{t+1}^e, respectively. As indicated by the figure, the slopes of both the red and yellow lines are less than 1 around their corresponding steady-state values. Therefore, both steady states are stable and the stability criterion under learning does not help select an equilibrium outcome.

Figure 5 illustrates the paths of real balances to the two steady states. These outcomes are qualitatively different from the path in Figure 3. While the stability criterion implies a unique path for real balances under learning specification (9), the same criterion implies multiple paths under specification (11).

4 CONCLUDING REMARKS

Stability under learning is often considered an important criterion for equilibrium selection, especially when there are multiple RE equilibria. In a simple overlapping generations model with high and low RE steady-state inflation rates, we use stability of the learning equilibrium as a criterion for equilibrium selection. Our results show that the stability of learning dynamics is sensitive to the specifications of the learning rule. When agents use last period's inflation in the learning rule, the low-inflation RE steady state is stable, while the high-inflation steady state is not. When agents use current inflation in the learning rule, both steady states are stable. The notion that such a seemingly minor variation in the specification could lead to qualitatively different learning equilibria questions the validity of using stability as a criterion for equilibrium selection. ■

NOTES

[1] According to Evans and Honkapohja (2001, pg. 13-14), "Another advantage of the learning approach arises in connection with the issue of multiple equilibria...Throughout the book the multiplicity issue will recur frequently, and we will pay full attention to this role of adaptive learning as a selection criterion."

[2] Esponda and Pouzo (2016) formally extend the notion of misspecification from exogenous to endogenous data-generating environments.

[3] For equation (10) to describe a learning equilibrium, we have to impose an additional restriction that $\Pi_{t+1}^e < \frac{1}{\lambda}$. If any element in the sequence of Π_t^e's exceeds $\frac{1}{\lambda}$, then (10) cannot be used to recover future expected inflation.

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