A Simple Model of Price Dispersion and Price Rigidity*

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Abstract

Here are two assertions: First, the law of one price is false – different sellers post different prices for the same good. Second, prices are sticky – some nominal prices fail to respond to changes in the aggregate price level or in real conditions. This note presents a model where money is essential, and these phenomena, price dispersion and price rigidity, emerge endogenously. While other papers make related points, our model is different. For one thing, money is nonneutral, although not because of sticky prices. For another, it is simple enough to permit tractable characterization of both stationary and dynamic equilibria.

Key words: Search, Sticky Prices, Price Dispersion

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1 Introduction

Here are two assertions: First, the law of one price is patently false – different sellers sometimes post different prices for what appears, beyond reasonable doubt, to be the same good. Second, prices are sticky – beyond reasonable doubt, some nominal prices fail to respond to changes in the aggregate price level or in real economic conditions. This note argues that these phenomena, price dispersion and price rigidity, are related, and presents a simple search-based model where they emerge endogenously. The environment builds on Shi (1995) and Trejos and Wright (1995), where money is essential a medium of exchange, except instead of bargaining sellers post prices as in Burdett and Judd (1983). They post prices in dollars because that is what they receive in trade. Equilibrium entails a distribution of prices, all of which yield the same profit. Hence, some sellers can keep nominal prices fixed when the average price increases because, intuitively, their surplus from a sale falls but the probability of a sale rises.

While other papers make related points, an advantage here is tractability: the framework permits simple analyses of stationary and dynamic equilibria, because we have divisible goods but indivisible assets. The indivisible-asset specification is not necessarily realistic, but can be used to succinctly illustrate salient points, and has been employed to good effect not only in monetary economics, but in intermediation theory (Rubinstein and Wolinsky 1987), finance (Duffie et al. 2005) etc. In contrast, recently Head et al. (2012), Liu et al. (2014) and Wang (2014) build divisible-asset models with price dispersion and rigidity, but need devices to avoid the complexity of endogenous asset distributions. Those papers use the alternating-market setup with quasi-linear utility in Lagos and Wright (2005); related papers by Head and Kumar (2005) and Head et al. (2010) use the large-family setup in Shi (1997). We deliver similar results in a different environment, without auxilliary assumptions like alternating markets, special preferences or large families.
Moreover, we avoid several technical problems in previous work. With assets and goods divisible, sellers should in principle post general mechanisms to maximize profit, but Head et al. (2012) and Wang (2014) restrict attention to linear pricing. This is a non-issue with indivisible assets. That is also true with divisible assets but indivisible goods, as in Liu et al. (2014), but monetary models with price posting and indivisible goods display an indeterminacy (continuum of stationary equilibria) without introducing additional complications.\(^1\) Our model has no such problem. Additionally, in those papers money is neutral. Whether or not there is neutrality in reality, their goal was to show how to get it even with sticky prices (see also Caplin and Spulber 1987, Eden 1994 or Golosov and Lucas 2003). We get nonneutrality, but stickiness is not its source. Additionally, we go beyond previous analyses by characterizing not only stationary equilibria, but also discussing multiplicity, dynamics and some implications for asset prices.

Since versions of this model are used in many applications, it also seems worth exploring its structure in detail, as a matter of pure theory. While early papers use symmetric Nash bargaining, others since use generalized Nash, Kalai or strategic bargaining (Coles and Wright 1998; Rupert et al. 2001; Trejos and Wright 2014; Zhu 2014). Still others use posting, auctions or pure mechanism design (Curtis and Wright 2004; Wallace and Zhu 2007; Zhu and Wallace 2007; Julien et al. 2008). There is also work exploring different matching processes (Coles 1999; Corbae et al. 2003; Matsui and Shimizu 2005). Introducing Burdett-Judd matching and posting contributes to our understanding of search theory, in general, as well as nominal price dispersion and rigidity, in particular.\(^2\)

\(^1\)The problem arises in a series of papers following Green and Zhou (1998). Heuristically, suppose cash is used to buy an indivisible good. If sellers think all buyers bring \(m = X\) dollars to the market, they post \(p = X\) as long as \(X\) is not too small; if sellers post \(p = X\), buyers bring \(m = X\) as long as \(X\) is not too big; so \(p = m = X\) is an equilibrium for all \(X\) in some range. Liu et al. (2014) get around this by adding costly credit, but that is not a trivial extension.

\(^2\)In terms of other literatures, research on price dispersion is too voluminous to list, but examples are given below. For an review of sticky price papers, see Section 2 of Liu et al. (2014). For a survey of related monetary models see Lagos et al. (2014).
2 Environment

A $[0, 1]$ set of infinitely-lived agents meet and trade in continuous time. Although we generalize this below, suppose for now they meet bilaterally. There is a set of goods $G$, where each agent $i$ consumes only a subset $G^i$ and produces $g^i \notin G^i$. Any $g \in G^i$ gives $i$ the same utility $u(q)$ from $q$ units, and has production cost $c(q)$. Assume $u(0) = c(0) = 0$, and $\forall q > 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) < 0$ and $c''(q) \geq 0$. Also, there is a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$, and define $q^* \in (0, \hat{q})$ by $u'(q^*) = c'(q^*)$. If $i$ and $j$ meet, $\delta$ is the probability $i$ produces $g^i \in G^j$ and $j$ produces $g^j \in G^i$ – a double coincidence – while $\sigma$ is the probability $i$ produces $g^i \in G^j$ and $j$ produces $g^j \notin G^i$ – a single coincidence. Without affecting the main results, we set $\delta = 0$ to preclude barter.\textsuperscript{3} Still, to make a medium of exchange essential we must rule out credit arrangements, as in Kochelakota (1998), by assuming agents lack commitment and trading histories are private information.

This implies exchange must be quid pro quo and thus assets have a role as media of exchange. This economy has one asset with flow return $\rho$. If $\rho > 0$ it can be interpreted as a dividend; if $\rho < 0$ it can it be a cost of holding the asset (e.g., as a proxy for inflation); if $\rho = 0$ the asset is usually called fiat money. In this class of models assets are indivisible and can be stored 1 unit at a time, so an individual’s state is his asset holdings $m \in \{0, 1\}$. Given a fixed supply $M \in (0, 1)$, $M$ agents (called buyers) have $m = 1$, while $1 - M$ agents (called sellers) have $m = 0$. After trade a buyer becomes a seller and vice versa. With pure random matching at Poisson rate $\alpha$, the probability per unit time that a buyer $i$ meets a seller producing $g \in G^i$ is $\alpha_1 = \alpha(1 - M)\sigma$, and the rate at which a seller meets an appropriate counterparty is $\alpha_0 = \alpha M \sigma$. As mentioned, some papers explore alternative matching processes, including more-or-less directed search, which can affect the effective arrival rates $\alpha_0$ and $\alpha_1$ but not the key results.\textsuperscript{3}

\textsuperscript{3}A common example has $K$ goods and $K$ types of agents, with type $k$ consuming good $k$ and producing good $k + 1 \mod K$. If $K > 2$ then $\delta = 0$ and $\sigma = 1/K$. 

3
Let $V_m$ be the value function for agents with $m \in \{0, 1\}$. If sellers produce $q$ for buyers in exchange for an asset, in the usual formulation,

$$rV_1 = \alpha_1 [u(q) + V_0 - V_1] + \rho + \dot{V}_1 \tag{1}$$
$$rV_0 = \alpha_0 [V_1 - V_0 - c(q)] + \dot{V}_0 \tag{2}$$

where $r$ is the rate of time preference.\footnote{In words, (1) equates $rV_1$ to the expected surplus from trading away the asset, $\alpha_1 [u(q) + V_0 - V_1]$, plus yield $\rho$ and a pure capital gain $\dot{V}_1$. Similarly, (2) equates $rV_0$ to the expected surplus from trading for the asset plus a pure capital gain.} Let $\Delta = V_1 - V_0$, so the buyer’s and seller’s surpluses are $u(q) - \Delta$ and $\Delta - c(q)$. By way of review of standard results (see Trejos and Wright 2014), suppose first that $q$ is determined by Kalai’s proportional bargaining solution, which gives the buyer and seller shares $\theta_1$ and $\theta_0 = 1 - \theta_1$ of the total surplus: $u(q) - \Delta = \theta_1 [u(q) - c(q)]$ and $\Delta - c(q) = \theta_0 [u(q) - c(q)]$. Subtracting (1)-(2), using $\Delta = \theta_0 u(q) + \theta_1 c(q)$ and $\dot{\Delta} = [\theta_0 u'(q) + \theta_1 c'(q)] \dot{q}$, we get a differential equation in $q$,

$$[\theta_0 u'(q) + \theta_1 c'(q)] \dot{q} = T(q) - \rho, \tag{3}$$

where $T(q) = r (\alpha_0 \theta_0 - \alpha_1 \theta_1) [u(q) - c(q)]$.

Paths solving (3) constitute equilibria as long as $q$ stays in $[0, \hat{q}]$, since $u(q) \geq c(q)$ iff $q \leq \hat{q}$. As is standard, for $\rho = 0$ there is nonmonetary steady state $q = 0$, and there is a unique monetary steady state $q > 0$ iff $\theta_1 > (r + \alpha M \sigma) / (r + \alpha \sigma)$, in which case there are also dynamic equilibria where $\lim_{t \to \infty} q = 0$ as a self-fulfilling inflationary prophecy. For $\rho \neq 0$ there can be multiple monetary steady states, plus dynamics where $\lim_{t \to \infty} q = 0$ or $\lim_{t \to \infty} q > 0$, and there can be sunspot equilibria where $q$ fluctuates stochastically. This is the baseline search-and-bargaining model of money, or more generally, any asset, circulating as a medium of exchange. There is one price, given by $p = 1/q$, in nominal terms, when $M$ is fiat currency, and it is easy to check $\partial p / \partial M > 0$. As is well known, instead of Kalai bargaining, one can use generalized Nash with similar results (e.g., Rupert et al. 2001).
For some reason, perhaps realism, some people (e.g., Prescott 2005) do not like models with bargaining, but seem at least begrudgingly willing to entertain price posting. We are agnostic but certainly sympathetic to the idea that it is interesting to consider alternative pricing mechanisms. However, it not straightforward to embed posting in the above model. Suppose sellers post \( q \), or equivalently \( p = 1/q \). In the baseline specification, that is the same as bargaining with \( \theta_0 = 1 \), and can be regarded as a monetary version of Diamond (1970). In this case, with \( \rho = 0 \) the only equilibrium is \( q = 0 \), since no one would produce at cost \( c(q) > 0 \) to get fiat currency if it yields no surplus when retraded. This can be overturned by having heterogeneity (Curtis and Wright 2004) or directed search (Julien et al. 2008), but while those devices allow monetary equilibria to exist, we want general price dispersion and rigidity, and hence we take a different tack.\(^5\)

The approach in Burdett and Judd (1983) is attractive because it is easy, is arguably realistic, delivers interesting results, and has been successfully deployed in other applications, including a large labor literature following Burdett and Mortensen (1998). The key is a relatively small change in the search process whereby a buyer may sample more than one seller at a time. In general, let \( \alpha_n \) be the rate at which he simultaneously samples \( n \in \{1, 2, \ldots\} \) sellers. For instance, let \( \alpha \) be a Poisson arrival rate of information – say, he gets a ‘catalogue’ containing \( n \) independent quotes of \( q \) on offer by some seller – and let \( \Pr(n) = \pi_n \). We allow \( \pi_0 > 0 \), as a ‘catalogue’ could contain nothing that he likes, but of course \( \pi_0 < 1 \). We also assume \( n \) has a finite mean and variance.

Consistency requires connecting buyer and seller arrival rates. Let \( \beta_n \) be the rate at which a seller gets a customer with \( n \) quotes, his own plus \( n - 1 \) others. At any point in time the measure of buyers with \( n \) quotes is \( M \alpha \pi_n \), and the measure of sellers getting customers with \( n \) quotes is \( (1 - M) \beta_n \). The identity \( (1 - M) \beta_n = 

\(^5\)Curtis and Wright (2004) introduce multiple types, but with any number of types, for generic parameters there are at most 2 prices posted in equilibrium. The auction model in Julien et al. (2008) also has 2 prices.
\(M\alpha \pi_n n\) implies \(\beta_n = b\alpha \pi_n n\), where \(b = M / (1 - M)\) is the buyer/seller ratio, or market tightness. The formulation is flexible. If \(\pi_n = 0 \forall n > 1\), e.g., it reduces to Diamond’s monopoly model, and if \(\pi_1 = 0\) it looks like Bertrand. Formally we have the following (all proofs are in the Appendix):

**Lemma 1** In the limit as \(\pi_1 \to 1\), there is one price and it is the same as bargaining with \(\theta_0 = 1\). In the limit as \(\pi_1 \to 0\), there is one price and it is the same as bargaining with \(\theta_1 = 1\).

Excepting \(\pi_1 = 1\) or \(\pi_1 = 0\), there is a distribution of \(q\) with support given by a nondegenerate interval \(Q = [\underline{q}, \bar{q}]\), or equivalently a distribution of prices \(p = 1/q\) with support \(P = [\underline{p}, \bar{p}]\). Intuitively, suppose all sellers were to post the same terms. Then a buyer contacting more than one seller is indifferent between them, which gives sellers an incentive to shade \(q\) up. Contrary to the law of one price, this theory predicts there must be many prices.

**Lemma 2** If \(\pi_1 > 0\) and \(\pi_n > 1\) for some \(n > 1\), there are no gaps or mass points in the support \(Q\). Moreover, the lower bound \(\underline{q}\) satisfies \(\Delta = u(\underline{q})\).

A buyer with \(n > 0\) quotes obviously picks the highest \(q\) or lowest \(p = 1/q\). Let \(F(q)\) be the CDF of quantity and \(G(p) = 1 - F(1/p)\) the CDF of price. Given \(n\) draws from \(F(q)\), the CDF of the highest \(q\) is \(F(q)^n\), and similarly for the CDF of the lowest \(p\). Then for a buyer, the analog of (1) is

\[
rV_1 = \sum_n \alpha_n \int_{\underline{q}}^{\bar{q}} [u(q) - \Delta] dF(q)^n + \rho + \dot{V}_1.
\]  
(4)

For a seller posting \(q\), the analog of (2) is

\[
rV_0(q) = b\alpha \sum_n \pi_n n F(q)^{n-1} [\Delta - c(q)] + \dot{V}_0(q),
\]  
(5)

since \(\beta_n = b\alpha \pi_n n\) is the rate at which he is in contact with a buyer having \(n\) quotes, whence he gets the sale iff the other \(n - 1\) sellers post less than his \(q\), which occurs
with probability $F(q)^{n-1}$. In particular, since the lowest quote $q$ never beats the
competition, a seller posting $q$ only sells when $n = 1$, and so

$$rV_0(q) = b\alpha \pi_1 [\Delta - c(q)] + \dot{V}_0(q). \tag{6}$$

In equilibrium every posted $q$ entails the same payoff, which means we can
equate (5) and (6) to get

$$\sum_{n=1}^{\infty} \pi_n n F(q)^{n-1} = \pi_1 \frac{u(q) - c(q)}{u(q) - c(q)}, \tag{7}$$

using $\Delta = u(q)$ by Lemma 2. For $V_0(q)$ to be constant on the support $Q$, $F(q)$
must satisfy (7). Moreover, since $F(q) = 1$, (7) implies

$$c(q) = \pi_1 c(q) + (Em - \pi_1)u(q). \tag{8}$$

This gives the upper bound as a function of the lower bound, say $\bar{q} = Q(q)$. Notice
$Q(0) = 0$, $Q(q) = \hat{q}$ and $Q'(q) > 0$, which is useful because it implies $Q \subset [q, \hat{q}]$
as long as $q < \hat{q}$. The next result says $F(q)$ is well defined.

**Lemma 3** $\forall q \in [q, \bar{q}]$ (7) yields a unique $F(q) \in [0, 1]$, and $\forall q \in (q, \hat{q})$, $F(q)$ is
differentiable with

$$F'(q) = \frac{\pi_1 u \left[(q) - c(q)\right] c'(q)}{[u(q) - c(q)]^2 \sum_{n=1}^{\infty} \pi_n n(n - 1)F(q)^{n-2}}. \tag{9}$$

Conditions (7)-(8) describe $F(q)$ and $\bar{q}$ as functions $q$; it remains to determine
$q$. To that end, subtract (4) and (6) to get

$$ru(q) = \alpha \sum_{n=1}^{\infty} \pi_n \int_{q}^{Q(q)} [u(q) - u(q)] dF(q)^n - b\alpha \pi_1 [u(q) - c(q)] + \rho + u'(q)\hat{q}. \tag{10}$$

For comparison with bargaining, write this as $u'(q)\hat{q} = T(q) - \rho$ where now

$$T(q) = \psi u(q) - b\alpha \pi_1 c(q) - \alpha \sum_{n=1}^{\infty} \pi_n \int_{q}^{\bar{q}} u(q)dF(q)^n,$$
and \( \psi = r + \alpha (1 - \pi_0) + b \alpha \pi_1 \). Denote the sum in the last term by \( S(q) \), and interchange summation and integration to get

\[
S(q) = \int_q^{Q(q)} u(q) \sum_{n=1}^{\infty} \pi_n n F(q)^{n-1} F'(q) \, dq.
\]

Then use (7) to get

\[
S(q) = \pi_1 [u(q) - q] \int_q^{Q(q)} u(q) F'(q) \, dq \quad \frac{u(q)}{u(q) - c(q)}.
\]

Thus we arrive at

\[
T(q) = \psi u(q) - b \alpha \pi_1 c(q) - \alpha \pi_1 [u(q) - c(q)] \int_q^{Q(q)} \frac{u(q) F'(q) \, dq}{u(q) - c(q)}.
\]

An equilibrium is a time path \( \langle F(q), q, \hat{q} \rangle \) where: \( q \in (0, \hat{q}] \) satisfies \( u'(q)\hat{q} = T(q) - \rho \) with \( T(q) \) given by (11); the distribution \( F(q) \) solves (7); and \( \hat{q} \) solves (8). Equivalently, one could define it in terms of \( p = 1/q \) and \( G(p) \). A stationary monetary equilibrium, or SME, is one where \( \underline{q} > 0 \) is constant. A dynamic monetary equilibrium, or DME, is one where \( \underline{q} \) is not constant over time. While Lemmata 1-3 are extensions of known results for nonmonetary Burdett-Judd models, the next result is novel, nontrivial, and key to characterizing monetary equilibria:

**Lemma 4** \( T : [0, \hat{q}] \rightarrow \mathbb{R} \) satisfies \( T(0) = 0 \) and \( T(\hat{q}) = [r + \alpha (1 - \pi_0)] \hat{q} > 0 \). If \( u'(0) = \infty \) then \( T'(0) = -\infty \).

The mapping \( T(q) \) is shown in Figure 1, illustrating the properties stated in Lemma 4, and assuming, to ease the presentation, there is at most one inflection point where \( T \) changes from convex to concave.\(^6\) Figure 1 delineates three critical \( \rho \) values: \( \underline{\rho} = \min_{[0, \hat{q}]} T(q) \); \( \hat{\rho} = T(\hat{q}) \); and \( \bar{\rho} = \max_{[0, \hat{q}]} T(q) \). Clearly, \( T'(0) < 0 \) implies \( \underline{\rho} < 0 < \hat{\rho} \), but it can be that \( \bar{\rho} = \hat{\rho} \), as in the left panel, or \( \bar{\rho} > \hat{\rho} \), as in the right panel. Based on all this, the following is obvious:

\(^6\) We cannot prove this, although it was always true in examples. It is not hard to describe what happens when \( T \) wiggles more than shown in Figure 1, but we prefer to avoid the clutter. Also, except for cases of interest like \( \rho = 0 \), we typically ignore nongeneric values like \( \rho = \hat{\rho} \), but they can be understood from Figure 1.
Figure 1: The mapping $T : [0, \hat{q}] \to \mathbb{R}$ given $M$ and $M' > M$

**Proposition 1** Suppose $T(q)$ has at most 1 inflection point, as shown in Figure 1. If $\rho = 0$ there is a unique SME, $q^S > 0$, plus a nonmonetary equilibrium at 0. If $\rho \in (0, \hat{\rho})$ there is a unique SME. If $\rho \in (\hat{\rho}, \bar{\rho})$ or $\rho \in (\bar{\rho}, 0)$ there are two SME, $q^S_L \in (0, \hat{q})$ and $q^S_H \in (q_L, \hat{q})$. Otherwise, there is no trade, with agents hoarding the asset if $\rho > \hat{\rho}$ and disposing of it if $\rho < \bar{\rho}$.

To characterize DME, just add arrows to Figure 1, pointing left when $T(q) < \rho$ and right when $T(q) > \rho$. It is that easy because, although the aggregate state is a distribution $F(q)$, its path is fully pinned down once we have the path for $q$. In particular, since $q$ the price of the asset in terms of goods, there is asset-price dispersion across trades, these prices are generally above the fundamental value $\rho/r$, and, as the next result says, they can vary over time as a self-fulfilling prophecy.

**Proposition 2** Suppose $T(q)$ at most 1 inflection point. If $\rho = 0$ then, in addition to the steady states 0 and $q^S \in (0, \hat{q})$, $\forall q \in (0, q^S) \ni DME$ with $\lim_{t \to \infty} q = 0$. If $\rho \in (0, \hat{\rho}) \ni DME$. If $\rho \in (\hat{\rho}, \bar{\rho})$ then, in addition to the steady states $q^S_L \in (0, \hat{q})$ and $q^S_H \in (q^S_L, \hat{q})$, $\forall q \in (q^S_L, \hat{q}) \ni DME$ with $\lim_{t \to \infty} q = q^S_H$. If $\rho \in (\bar{\rho}, 0)$ then, in addition to the steady states $q^S_L \in (0, \hat{q})$ and $q^S_H \in (q^S_L, \hat{q})$, $\forall q \in (0, q^S_H) \ni DME$ with $\lim_{t \to \infty} q = q^S_L$. 

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The bargaining version of the model also has belief-based dynamics, but we now suggest that our results are more solidly grounded. While one can impose as equilibrium conditions axiomatic solution concepts like Kalai, Nash or anything else, the strategic foundations are flimsy. Consider a standard extensive-form game where buyer and seller make counteroffers of $q$, until one is accepted. As in Binmore et al. (1986), if the time between offers is $\delta > 0$, there is a unique subgame-perfect equilibrium $q = q^\delta$, and $q^\delta \to q^N$ as $\delta \to 0$, where $q^N$ is the Nash outcome. Coles and Wright (1998) and Coles and Muthoo (2003) show that this applies to environments like the one under study here in, but not out of, steady state. Using Nash bargaining out of steady state is equivalent to using the extensive-form game with myopic agents, who negotiate as if economic conditions were constant even as they change over time.\footnote{To give details, as $\delta \to 0$, the path for $q$ in subgame-perfect equilibrium satisfies a differential equation with $q^N$ as a steady state; but $q \neq q^N$ out of steady state except for special cases, like $u$ and $c$ linear, which is not admissible in this model, or $\theta_1 = 1$, which is admissible but too special. And it matters – e.g., Coles and Wright (1998) show with strategic bargaining the equilibrium set can contain limit cycles that are impossible with Nash (i.e., myopic strategic) bargaining.}

Again, while one can do the dynamics with Nash bargaining, this discussion should give one pause. Similar remarks can be made regarding strategic foundations for Kalai bargaining (Dutta 2012). Now, one can analyze dynamics with strategic bargaining and rational expectations, but it is complicated. In our model, agents are strategic and expectations rational, but the analysis is simple.

### 3 Examples

To economize on notation, with no loss in generality, set $c(q) = q$ and $\alpha = 1$. Now consider first the minimal specification satisfying the conditions in Lemma 2, $\pi_1 > 0$, $\pi_2 > 0$ and $\pi_n = 0 \\forall n > 2$, as is common in applications (e.g., Head et al. 2012). With $\pi_n = 0 \\forall n > 2$, (7) solves immediately for

$$F(q) = \frac{\pi_1}{2\pi_2} \frac{q - q}{u(q) - q}.$$
It is similarly easy to solve for $G(p)$. Also, (8) becomes

$$
\bar{q} = Q(\bar{q}) = \frac{\pi_1 q + 2\pi_2 u(q)}{\pi_1 + 2\pi_2},
$$

while equilibrium condition (11) in this example reduces to

$$
T(\bar{q}) = \psi u(\bar{q}) - b\pi_1 \bar{q} - \frac{\pi_1^2}{2\pi_2} \left[ u(\bar{q}) - \bar{q} \right]^2 \int_{\bar{q}}^{Q(\bar{q})} \frac{u(q) \, dq}{[u(q) - \bar{q}]^3}.
$$

Suppose $u(q) = q^a$ and parameter values are $a = 0.2$, $\pi_1 = 0.48$, $\pi_2 = 0.36$, $M = 0.3$ and $r = 0.04$. In Figure 2, the left panel shows there are two solutions to $T(\bar{q}) = \rho$ when $\rho = 0.067$, at $q^S_H = 0.37$ and $q^S_L = 0.66$. The middle panel shows the $q$ densities for the two equilibria; the right panel shows the $p$ densities. Here the densities in the two equilibria do not overlap, although in general they can.

Figure 2: Multiple SME with different densities

Now consider $\pi_n = e^{-\lambda \lambda^n / n!} \forall n \geq 0$, a Poisson distribution. Mortensen (2005), e.g., uses this in a version of Head and Kumar (2005), building on tools developed in Mortensen (2003), where he advocates it because (among other things) it makes it easy to endogenize search effort. Substituting $\pi_n$ into (7), we get

$$
\frac{u(q) - q}{u(q) - \bar{q}} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} F(q)^{n-1}}{(n-1)!}.
$$
From a well-known formula, the RHS is $e^{\lambda F(q)}$. Hence,

$$F(q) = \frac{1}{\lambda} \log \left[ \frac{u(q) - q}{u(\bar{q}) - \bar{q}} \right],$$

while (11) reduces to

$$T(q) = \psi u(q) - b\lambda e^{-\lambda} \bar{q} - e^{-\lambda} \left[ u(q) - \bar{q} \right] \int_{q}^{Q(q)} \frac{u(q) dq}{[u(q) - q]^2},$$

with $\psi = r + 1 - e^{-\lambda} + b\pi_1$ and $\bar{q} = e^{-\lambda} q + (1 - e^{-\lambda}) u(q)$.

Finally, consider $\pi_0 = 0$ and $\pi_n = -\omega_n / n \log(1 - \omega) \forall n > 0$, a logarithmic distribution. We have not seen this example in applications, but claim it is related to Caplin and Spulber (1987). It is easy to solve for

$$F(q) = \frac{q - q}{\omega \ [u(q) - q]},$$

and

$$T(q) = \psi u(q) - b\omega q - \int_{q}^{Q(q)} \frac{u(q) dq}{u(q) - q},$$

where $\psi = -\log(1 - \omega)(1 + r) + b\omega$ and $\bar{q} = \omega u(q) + (1 - \omega)q$. Since $F(q)$ is linear, $q$ and $p$ are uniformly distributed. Caplin and Spulber (1987) have an interesting application, related to the discussion below, where they *assume* that $p$ is uniform, with no claim that this is an equilibrium outcome (or that they have microfoundations for money in the first place). Here it is a *result* that $p$ is uniform, which can be considered a rationalization of their assumption.8

4 Discussion

The effects of parameter changes on SME or DME are simple. An increase in $r$, e.g., rotates $T$ around the origin, so $\partial \bar{q} / \partial r < 0$ when the SME is unique, or more generally, at any ‘natural’ SME where $T'(q) > 0$ (as usual, with multiple equilibria

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8We mention as an aside the following: not only do these examples work out nicely, the one difficult part of the general analysis – proving Lemma 4 for an arbitrary $\{\pi_n\}$ distribution – is relatively simple in parametric cases (details on request).
the comparative statics are reversed at alternate solutions). As $\bar{q} = Q(q)$ also falls, the entire distribution shifts left. Similarly, an increase in $M$ raises $b$, which shifts $T(q)$ so that $\partial q / \partial M < 0$ in any ‘natural’ SME. Again the distribution shifts to the left, and note that $\bar{q}$ falls by less than $q$, as it is easy to check $qQ'(q)/Q(q) < 1$. Hence, higher $M$ spreads the support and makes $F'(q)$ lower at the edges. With fiat money, $\rho = 0$, injecting $M$ makes it harder to find sellers, so agents produce less $q$ for cash. This nonneutrality has something to do with the restriction $m \in \{0, 1\}$, but that seems appropriate, corresponding to a long-standing notion that the real effects of monetary injections depend on changing the distribution of liquidity.\(^9\)

![Figure 3: Sticky prices after an increase in $M$](image)

To discuss the implications for pricing behavior, consider Figure 3, where the quantity and price densities are $f(q, M)$ and $g(p, M)$, before and after $M$ rises to $M'$. After $M$ rises, all sellers that were formerly pricing between $\underline{p}$ and $\underline{p}'$ must adjust $p$ because it is no longer in the equal-payoff support $\mathcal{P}$. However, any seller pricing between $\underline{p}'$ and $\bar{p}$ has no incentive to change – his $p$ can fall relative to the aggregate $\mathbb{E}p$, and by construction the equilibrium probability of a sale increases.

\(^9\)As Francis Bacon put it “Money is like muck, not good except it be spread.” Still, as usual, a neutrality result lurks in the wings: changing the denomination on the asset for all agents holding it is irrelevant, even if changing the measure of agents holding it (spreading the muck) is not. This kind of nonneutrality is one reason Wallace (2014) champions this model, and it generalizations to $m = \{0, 1, 2\}$, although then numerical methods are necessary (Molico 2006).
by exactly enough to compensate. So prices are sticky in the sense used in the Introduction: some sellers do not reprice when $E\pi$ increases, although of course others do, since how else would $E\pi$ change? Collectively, sellers respond so as to achieve the new SME, but it is no puzzle if many individuals stick to their old $p$ when $M$ changes. Obviously, nominal prices can be sticky in this sense after changes in utility or technology, too.

Similar remarks apply to DME. With $\rho = 0$ there are inflationary equilibria where $q \to 0$ and $E\pi \to \infty$, but many sellers stick to posted prices for extended periods, only changing when $p$ falls out of the shrinking $P$. With $\rho \neq 0$ there are deflationary equilibria where some sellers stick to their prices while $E\pi$ falls, thus reducing the probability they make a sale, but getting a higher surplus when they do. In the interest of space, we only sketch some additional dynamic implications. First, as in Trejos and Wright (1993), assume $M$ follows a stochastic process and agents have rational expectations. Then $E\pi$ rises and falls when $M$ realizations are high or low, but many sellers can stick to the same $p$ after shocks as long as the supports overlap. Second, as in Shi (1995) or Ennis (2001), there are sunspot equilibria where endogenous variables follow stochastic processes as self-fulfilling prophecies, and again many sellers can stick to the same $p$ as $E\pi$ rises and falls.

In sum, there are various reasons for changes in price distributions, including one-time unanticipated movements or different realizations of stochastic processes for $M$ as well as real factors, and including self-fulfilling prophecies that give rise to deterministic or sunspot dynamics. Prices can be sticky in every case. Once one understands how frictions lead to dispersion, it is immediate how to generate stickiness, with no restrictions on timing or costs of adjustment.\textsuperscript{10} For evidence

\textsuperscript{10}The result is robust to various perturbations in the environment. If sellers have heterogenous costs, e.g., dispersion still obtains, but a seller is only indifferent across $p$ in a subset of the support. Thus, low-cost sellers prefer low $p$ and a high probability of a sale, but there is an interval $P_L$ in which they are indifferent, and similarly an interval $P_H$ for high-cost sellers. Still, $P = P_L \cup P_H$ looks like the baseline model, and $p$ can still be sticky in the conditional supports. Heterogeneity does not upset the general argument about stickiness.
that this is neither trivial nor generally understood, consider Golosov and Lucas (2003): “Menu costs are really there: The fact that many individual goods prices remain fixed for weeks or months in the face of continuously changing demand and supply conditions testifies conclusively to the existence of a fixed cost of repricing.” That is incorrect. We are not the first to mention this, and give full credit to the papers mentioned in the Introduction, Caplin and Spulber (1987), Eden (1994) and others. Yet our model is different, and has the virtue of simplicity, as well as relatively solid microfoundations for money, dispersion and rigidity.

5 Conclusion

This paper takes a different position than much of macroeconomics, where nominal rigidity is a primitive and price dispersion emerges as an outcome due to inflation (e.g., Woodford 2003). We take real frictions as a primitive, and derive dispersion and sticky prices as outcomes. Importantly, dispersion emerges even without inflation, as seems true in the data (Campbell and Eden 2014). We deliver closed-form solutions for special cases used in different models in the literature. For monetary dynamics, we avoid some pitfalls in earlier bargaining models. Money is nonneutral here, although not because of stickiness. Putting the results together, sticky prices are neither necessary not sufficient for nonneutrality, and the fact that some sellers do not change $p$ over extended periods does not testify to the existence of menu costs or arbitrary restrictions on changing prices.
Appendix: Proofs

In what follows we set $c(q) = q$, $\alpha = 1$ and $\pi_0 = 0$ to reduce notation; this is without loss of generality given we renormalize objects like $r$ and $\rho$.

**Lemma 1**: With bargaining, based on (3), we have

$$\theta_1 = 1 \implies \dot{q} = r_q - \alpha_1 [u(q) - q] - \rho \tag{12}$$

$$\theta_1 = 0 \implies u'(q) \dot{q} = r u(q) - \alpha_0 [u(q) - q] - \rho \tag{13}$$

where $\alpha_1$ and $\alpha_0$ are the effective arrival rates for buyers and sellers. With posting, consider first $\pi_1 \to 1$. Then (8) implies $\bar{q} = Q(\underline{q}) = \underline{q}$, so there is a single price. Also, (11) reduces to

$$T(q) = (r + \alpha_0) u(q) - \alpha_0 \bar{q}$$

where $\alpha_0$ is the effective arrival rate for a seller. Consequently $u'(q) \dot{q} = T(q) - \rho$, which is identical to (13).

With posting and $\pi_1 \to 0$, the seller with the lowest $q$ gets a payoff satisfying $r V_0(q) = \dot{V}_0$, because there are no buyers with $n = 1$, so he never beats the competition. Hence, (5) implies $c(q) = q = \Delta \forall q$, and all sellers get 0 surplus. Moreover, since the LHS of (7) must be strictly positive, $q \to u(q)$ as $\pi_1 \to 0$, so again there is a single price. Then (4) implies

$$r V_1 = \alpha_1 [u(q) - \bar{q}] + \rho + \dot{V}_1,$$

where $\alpha_1$ is the effective arrival rate for a buyer. This combined with $r V_0(q) = \rho_0 + \dot{V}_0$ and $q = \Delta$ leads to

$$\dot{q} = r q - \alpha_1 [u(q) - q] - \rho,$$

which is identical to (12).

**Lemma 2**: If there were a mass point at $q_1$, a seller posting $q_1$ could profitably deviate to $q_1 + \varepsilon$ for some $\varepsilon > 0$, because he would increase the probability of a sale discretely with a small increase in cost. If there were a gap between $q_1$ and $q_2 > q_1$, a seller posting $q_2$ could profitably deviate to $q_3 \in (q_1, q_2)$, since he lowers cost while losing no sales. Finally, if the lowest $q$ does not take the entire surplus from buyers, a seller posting $\underline{q}$ can profitably deviate to $\underline{q} - \varepsilon$. ■
Lemma 3: Let \( L(y) = \sum_{n=1}^{\infty} \pi_n y^{n-1} \). For a given \( q \), (7) says that \( F(q) = y \) where \( y \) is the solution to

\[
L(y) = \pi_1 \frac{u(q) - c(q)}{u(q) - c(q)}.
\]

Notice \( L(0) = \pi_1 \) and \( L(1) = \mathbb{E} \), and that \( L(y) \) is differentiable with \( L'(y) = \sum_{n=1}^{\infty} \pi_n n (n-1) y^{n-2} > 0 \). Setting \( y = 1 \), (14) reduces to \( q = \tilde{q} \). Setting \( y = 0 \), (7) reduces to \( q = \hat{q} \). These results imply that \( \forall q \in [\hat{q}, \tilde{q}] \) there is a unique \( y \in [0, 1] \) solving (14), and hence a unique number \( F(q) \) in \([0, 1]\) solving (7). The formula (9) comes from differentiating (7) and rearranging, where the sum in the denominator is well defined because \( n \) has a finite mean and variance. ■

Lemma 4: It is obvious that \( T(0) = 0 \) and \( T(\hat{q}) = [r + \alpha (1 - \pi_0)] \hat{q} > 0 \), because the integral vanishes at \( \hat{q} = 0 \) or \( \hat{q} = \hat{q} \). For the limit, note from (11) that

\[
T'(\hat{q}) = \psi u'(\hat{q}) - b \pi_1 - u \left[ Q(\hat{q}) \right] F' \left[ Q(\hat{q}) \right] \left[ \pi_1 + (\mathbb{E} n - \pi_1) u'(\hat{q}) \right] + \alpha \pi_1 u(\hat{q}) F'(\hat{q}) + \pi_1 \int_{\hat{q}}^{Q(\hat{q})} u(q) \frac{[u(q) - \hat{q}] u'(q) - [u'(q) - 1] [u(q) - \hat{q}]}{[u(q) - \hat{q}]^2} F'(q) dq.
\]

We know there are no mass points in the distribution and \( F(q) \) is differentiable with \( \lim_{\hat{q} \to 0} F'(\hat{q}) = \Omega \) for some \( \Omega > 0 \). Hence

\[
\lim_{\hat{q} \to 0} T'(\hat{q}) = \psi \lim_{\hat{q} \to 0} u'(\hat{q}) - b \pi_1 - \Omega \pi_1 \lim_{\hat{q} \to 0} u \left[ Q(\hat{q}) \right] - \Omega (\mathbb{E} n - \pi_1) \lim_{\hat{q} \to 0} u \left[ Q(\hat{q}) \right] u'(x) + \pi_1 \Omega \lim_{\hat{q} \to 0} u(q)
\]

\[
+ \pi_1 \lim_{\hat{q} \to 0} \int_{\hat{q}}^{Q(\hat{q})} u(q) \frac{[u(q) - \hat{q}] u'(q) - [u'(q) - 1] [u(q) - \hat{q}]}{[u(q) - \hat{q}]^2} F'(q) dq.
\]

The second and fourth limits on the RHS are trivially 0. The fifth limit on the RHS is 0, because the integral vanishes when the measure of the range goes to 0, even if the integrand goes to \( \infty \) or is indeterminate, by Theorem 5.25 in Zygmund and Wheeden (1977). Using l’Hopital’s rule on the third limit, we get

\[
\lim_{\hat{q} \to 0} u \left[ Q(\hat{q}) \right] u'(x) = \lim_{\hat{q} \to 0} u' \left[ Q(\hat{q}) \right] Q'(\hat{q}) u''(\hat{q})
\]

\[
= \lim_{\hat{q} \to 0} u'(\hat{q}) \lim_{\hat{q} \to 0} u''(\hat{q}) \frac{\pi_1 + (\mathbb{E} n - \pi_1) u'(\hat{q})}{\mathbb{E} n}
\]
Putting things together we have

\[ \lim_{q \rightarrow 0} T'(q) = \lim_{q \rightarrow 0} u'(q) \left[ \psi + \Omega (E_n - \pi_1) \lim_{q \rightarrow 0} u''(q) \frac{\pi_1 + (E_n - \pi_1) u'(q)}{E_n} \right] - b \pi_1. \]

This implies

\[ \lim_{q \rightarrow 0} \frac{T'(q)}{u'(q)} = \psi + \frac{\Omega (E_n - \pi_1) \pi_1}{E_n} \lim_{q \rightarrow 0} u''(q) + \frac{\Omega (E_n - \pi_1) (E_n - \pi_1)}{E_n} \lim_{q \rightarrow 0} u'(q)u''(q) \]

The last two terms are negative, and at least the last one is \(-\infty\). Therefore, it must be the case that \(\lim_{q \rightarrow 0} T'(q) = -\infty\). \[ \blacksquare \]
References


