Inter-generational Redistribution in the Great Recession*

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Abstract

We construct a stochastic overlapping-generations general equilibrium model in which households of different ages are subject to aggregate shocks that affect both wages and asset prices. We use a realistically calibrated version of the model to assess the distributional consequences of severe recessions. Specifically, within the context of this model, we ask whether young people can be better off if they become economically active in the midst of a large and persistent economic downtown. A key determinant of the answer is the size of the decline in asset prices, relative to the decline in wages. If older generations have a strong incentive to sell their assets in the downturn to finance old-age consumption then asset prices decline more strongly than wages in equilibrium. This in turn benefits younger generations that can buy these assets at low prices, potentially more than offsetting the fall in wages these generations experience. Quantitatively, for our prefered calibration the model predicts a percentage decline in asset prices that is more than twice as that of wages. Whereas older cohorts suffer massive welfare losses from the economic downturn, the welfare consequences for young households are adverse as well, but only very mildly so.

Keywords: Recessions, Overlapping Generations, Asset Prices, Aggregate Risk

JEL Codes: E21, D31, D58, D91

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1 Introduction

The current economic downturn is the most severe since the great depression. Labor incomes of households have fallen significantly below trend and prices of real estate and stocks have plummeted. The goal of this paper is to explore the welfare consequences of a severe and long-lasting recession that results in large declines in wages and a collapse in asset prices. Our main objective is to study how these welfare costs vary across different age cohorts.

There is strong reason to believe that the welfare impact of large aggregate shocks is distributed unequally across different generations. Young households have little financial wealth, relative to their labor income, while older households are asset-rich but have little human wealth, measured as the present discounted value of future labor income. Moreover, young households potentially gain from future asset price appreciation following an asset price collapse, while older households have less time to wait for asset prices to recover. Thus it is likely that a steep decline in asset prices has more serious welfare implications for older households.

In the next section, we use data from the Survey of Consumer Finance to document how the ratio of labor income to net worth varies over the life-cycle. We then estimate the magnitude of the declines in net worth associated with the current recession, focussing on how these losses vary with the age of a household. To do so we decompose net worth into different types of assets and debts, and estimate losses by applying asset-class specific price deflators to age-group specific portfolios. We find that the average household experienced a decline in net worth of $177,000 between the middle of 2007 and the trough of the asset price decline in the first quarter of 2009. These losses were heavily concentrated among older age groups: households aged 60-69 lost $312,000 on average. Since then, asset prices and net worth have recovered somewhat, but remain well below their 2007 values.

These empirical facts suggest that the welfare losses from large economic downturns such as the one the U.S. is currently experiencing are unevenly distributed among different age cohorts in the population. However, a more complete welfare analysis requires forecasts for the future evolution of labor income and asset prices, and an understanding of how agents will optimally adjust savings behavior in response to these expected wage and price changes. In the remainder of the paper we therefore construct a stochastic general equilibrium model with overlapping generations in which households of different ages are subject to large aggregate shocks that affect both endogenously determined wages and asset prices. We use a realistically calibrated version of the model to assess the distributional consequences of severe recessions. One question of particular interest that we can ask within the context of this model is whether young people might actually be better off if they become economically active in the midst of a large and persistent economic downtown.

The key to the answer is the size of the decline in asset prices, relative to the decline in wages, in response to a negative aggregate shock. If older generations have a strong incentive to sell their assets in the downturn (e.g., because they strongly value smooth consumption profiles) then asset prices decline more strongly than wages in equilibrium. This in turn benefits younger generations that can buy these assets at low prices, more than compensating for the fall in wages these generations experience. We demonstrate that for realistic parameter values this mechanism can be strong enough to indeed generate welfare gains from recessions for young generations. In our preferred calibration this cohort suffers mild losses, however. Older cohorts, on the other hand, unambiguously face massive declines in welfare from a large economic downturn.

Our paper builds upon and contributes to three broad strands of the literature. First, we use a large scale OLG model with aggregate risk to study the asset pricing and intergenerational consumption implications of a large aggregate shocks. The literature that analyzes asset prices and portfolio choice in stochastic OLG economies includes Huffman (1987), Storesletten, Telmer, and Yaron (2004, 2007), Constantinides, Donaldson, and Mehra (2002) and Kubler and Schmedders (2010). Ríos-Rull (1994, 1996) investigates the properties of business cycles in this class of models whereas Smetters (2006), Krueger and Kubler (2004, 2006) and Miyazaki, Saito, and Yamada (2009) analyze the allocation of aggregate consumption risk across different generations.

Second, a number of papers study the distributional consequences across age cohorts and asset pricing implications of large aggregate shocks. Our analysis is similar in spirit to Doepke and Schneider (2006a,b)'s study of the inflationary episode of the 1970's and to a lesser extent to Meh, Ríos-Rull, and Terajima (2010). A number of studies employs OLG models to investigate the impact of large swings in the demographic structure of the population on factor and asset prices as well as welfare of different age cohorts. Examples include Attanasio, Kitao, and Violante (2007), Krueger and Ludwig (2007), and Ríos-Rull (2001).

Finally, a recent literature estimates empirical models of aggregate consumption that allows for large declines in aggregate consumption (so called disasters) and uses these estimates in consumption based asset pricing models. See e.g Barro (2006, 2009), and Nakamura, Steinsson, Barro, and Ursua (2010).[1]

[1] The latter paper finds that (on average) aggregate consumption falls by 11% during a desaster period, that a

The remainder of this paper is organized as follows. In Section 2 we present life cycle facts on labor income, net worth and portfolio allocations that motivate our quantitative analysis and that we use later to calibrate the model. In section 3 we set up our model and define a recursive competitive equilibrium. In section 4 we analyze a sequence of simple examples that can be characterized analytically and provide crucial insights into the key mechanism of the model. Section 5 is devoted to the calibration of the model and Section 6 reports the findings of our model economies that include environments with with multiple assets. Section 7 concludes. Details about the computational approach as well as additional results from the model are relegated to the appendix.

2 Data

We now document the life cycle pattern of labor income, net worth and portfolio allocations that motivates our focus on heterogeneity along the age dimension, and will serve as inputs for the calibration of the model. The need for detailed data on household portfolios leads us to use the Survey of Consumer Finances (SCF). The SCF is the best source of micro data on the assets and debts of U.S. households. One advantage of the survey is that is over-samples wealthy households, using a list based on IRS data. Because the SCF weighting scheme adjusts for higher non-response rates among wealthier households, it delivers higher estimates for average income than other household surveys, such as the CPS or PSID. The survey is conducted every three years, with the most recent survey conducted in 2007, around the peak in asset prices.

We use the SCF to construct life-cycle profiles for labor income, total income, human wealth (the present discounted value of labor earnings) and net financial and real worth (see Table 1). We construct these profiles by averaging across households partitioned into ten year age groups. We divide total income into an asset-type income component, and a residual non-asset-income component, which we call labor income.[2] Two adjustments relative to the SCF concept of household income are made. First, we add an imputation for implicit rents accruing to home owner-occupiers, and interpret these rents as part of asset income.[3] Second, we subtract interest payments on debts,

large part of these consumption drops are reversed in the long run (i.e. there is only a small effect of the consumption desaster on the long run trend of consumption) and that the average length of a disaster period is 6.5 years. Our calibration will imply a consumption process roughly consistent with these findings.

[2]Asset income is defined as interest or dividend income minus interest payment on debts, income from capital gains and asset sales, one third of business, farm or self-employment income, private retirement income, and imputed rents from owner-occupied housing. Non-asset-income is all other income, which includes wage and salary income, two-thirds of business, farm or self-employment income, social security income, and a variety of public and private transfers.

[3]We set imputed rents equal to the value of primary residence times the rate of return on all other assets. This rate of return is computed as asset income (excluding imputed rents) dividend by aggregate assets (excluding the

thinking of these as a negative component of asset income. We measure net worth as the value of all financial and non-financial assets, less the value of all liabilities. Our SCF-based measure of net worth excludes the present value of future pensions associated with defined benefit private pension plans and social security.

In 2007 average household income was $83,430, while average household net worth was $555,660, for a net worth to income ratio of 6.66.[4] The share of net asset income in total income was 0.16. Average household assets were $659,000, with an average rate of return of 3.1%. Average household debts were $103,300, with an average interest rate on debts of 6.4%. The fact that the average interest rate paid on debts exceeds the rate paid on assets explains why young households have negative net asset income, despite having positive net worth.

Table 1: **Income and Wealth Over the Life Cycle (2007 SCF, $1,000)**

Age of Head	Total Income	Labor Income	Asset Income	Assets	Debts	Net Worth
All	83.43	70.07	13.36	659.00	103.34	555.66
20-29	38.83	39.68	-0.85	130.66	53.30	77.36
30-39	69.83	68.68	1.15	335.87	136.12	199.75
40-49	93.40	84.97	8.43	598.21	132.62	465.59
50-59	117.97	99.56	18.41	959.77	133.24	826.53
60-69	109.06	76.15	32.90	1156.96	104.10	1052.86
70+	57.56	34.46	23.11	756.76	28.48	728.28

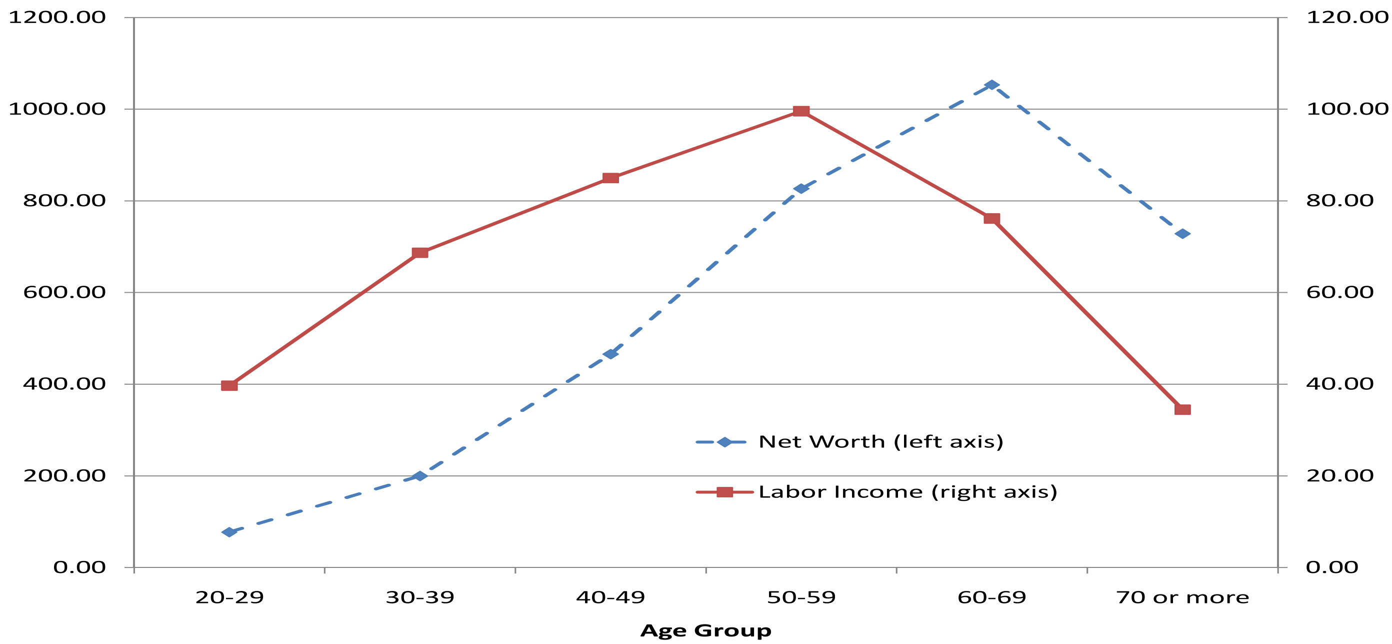
Figure 1 plots the life-cycle profiles for labor income and net worth. Income follows the familiar hump-shape over the life-cycle, while net worth peaks somewhat later. For 20-29 year-olds, average net worth is 1.9 times average labor income, while for household 70 and older, the corresponding ratio is 21.1. Thus the old are much more exposed to fluctuations in asset prices than the young. We will insure, by force of calibration, that the life cycle patterns of labor income and net worth in our structural OLG model are consistent with their empirical counterparts documented here.

While Figure 1 suggests large losses for older households from a slump in asset prices, the risk composition of net worth also varies quite substantially with age. Thus to accurately estimate

value of primary residences and the value of vehicles).

[4]Since income questions refer to the previous calendar year, while questions about wealth are contemporaneous, we adjust income measures for CPI inflation between 2006 and 2007.

Figure 1: **Labor Income and Net Worth by Age, SCF 2007 ($1,000)**



losses by age group, we further decompose portfolios by age group, and examine the patterns for relative price changes across different asset classes. In Table 2 we decompose total net worth into risky net worth and safe net worth, where we define risky net worth as the value of stocks, residential real estate, non-corporate business, and non-residential property. We define safe net worth as the value of all other assets, less all debts.[5] In aggregate, risky net worth is 93.9 of aggregate net worth. However, among 30-39 year-olds, the corresponding ratio is 140.4%, while among those aged 70 or older, it is only 79.2%. These three ratios reflect three facts: (i) in aggregate, net household holdings of safe assets are very small, (ii) younger households are short in safe assets, because they tend to have lots of mortgage debt (which we classify as a riskless liability) and little in the way of financial assets, and (iii) older households tend to have little debt, and lots of assets, a significant part of whuch are riskless financial assets.

Our next task is to estimate price declines for each component of net worth. The 2007 SCF provides a snap-shot of household portfolios in the middle of 2007, roughly when asset prices peaked. The Federal Reserve plans to conduct a special re-survey of some of the same households,

[5] For our purposes, stocks include stocks held directly or indirectly through mutual funds and retirement accounts, and also includes closely-held equity. The category "bonds, cash, CDs" includes bonds (directly or indirectly held), transaction accounts, CDs, and the cash value of life insurance.

Table 2: **Portfolio Shares as a Percentage of Net Worth**

Age of Head	(1) Stocks	(2) Res, real estate	(3) Non-corp bus.	(4) Non-res prop.	(5) RISKY N.W.	(6) Bonds + CDs	(7) Cars	(8) Other assets	(9) Debts	(10) SAFE N.W.
All	30.28	46.99	12.87	3.80	93.95	16.98	3.45	4.23	-18.60	6.05
20-29	13.20	77.67	43.31	1.28	135.46	13.66	15.26	4.51	-68.90	-35.46
30-39	26.27	96.47	12.73	4.97	140.44	13.80	9.73	4.19	-68.15	-40.44
40-49	30.41	57.62	12.55	3.81	104.38	15.17	4.44	4.49	-28.48	-4.38
50-59	32.70	42.40	13.53	3.72	92.35	17.02	2.79	3.96	-16.12	7.65
60-69	32.17	35.62	13.41	4.12	85.31	17.45	2.40	4.73	-9.89	14.69
70+	27.12	39.76	8.98	3.33	79.18	19.26	1.75	3.72	-3.91	20.82

Risky Net Worth (5) is equal to the sum of columns (1)+(2)+(3)+(4). Safe Net Worth (10) is the sum of columns (6)+(7)+(8)+(9). Total Net Worth is the sum of columns (5)+(10).

to study the effects of the financial crisis. In the interim, we can get a fairly good sense of the direct redistributive effects of dramatic changes in asset prices by using aggregate asset-class-specific price series to revalue portfolios, thereby constructing estimates for capital losses across the age distribution.

We assume that SCF portfolios reflect the distribution of household net worth in the second quarter of 2007.[6] We then revalue portfolios for each age group for each successive quarter as follows. For all the components of safe net worth (bonds, vehicles, other assets and debts), we assume no price changes. We price stock wealth using the Wilshire 5000 price index, as of the last trading day in the quarter. We price residential real estate using the Case-Shiller National Home Price Index, which is a quarterly, repeat-sales-based index. We price non-residential property using the Moodys/REAL Commercial Property Price Index, which is a monthly repeat-sales-based index for the prices of apartments, industrial property, commercial property, and retail property. We price non-corporate business wealth using Flow of Funds data. In particular the Flow of Funds reports changes in market values for a variety of asset types by sector. We focus on the asset type "proprietors' investment in unincorporated business" for the household and non-profit sector. Price changes by asset type relative to 2007:2 are reported in Table 3. For stocks and residential property, values reached a low point in the first quarter of 2009, with prices respectively 46.9% and 29.5% below their 2007:2 values. The values of non-corporate business and non-residential property, by contrast, continue to decline through the third quarter of 2009.[7]

[6] Unfortunately the SCF does not provide much information about precisely when households were interviewed.

[7] For comparison, Table 3 also reports some alternative price series. The Flow of Funds reports price changes

Table 3: **Price Declines Relative to 2007:2 by Risky Asset Class**

PRICE SERIES USED	2007 Q3	2007 Q4	2008 Q1	2008 Q2	2008 Q3	2008 Q4	2009 Q1	2009 Q2	2009 Q3
Stocks	1.0	-2.7	-12.4	-14.2	-22.0	-40.3	-46.9	-38.4	-28.6
Res. Real Estate	-1.7	-6.9	-13.0	-14.8	-17.8	-23.9	-29.5	-27.4	-25.1
Non-corp. Bus.	0.2	-1.6	-4.5	-8.2	-10.9	-17.9	-23.5	-25.5	-27.1
Non-Res. Property	0.9	0.7	-0.2	-9.6	-7.1	-14.3	-20.9	-33.9	-41.5
ALTERNATIVE SERIES									
Stocks (Flow of Funds)	0.6	-3.5	-10.3	-13.6	-23.6	-39.0	-47.1	-36.2	-26.0
Real Estate (OFHEO)	-1.5	-4.0	-4.9	-5.4	-8.5	-12.2	-11.8	-10.2	-11.6
Real Estate (Flow of Funds)	-2.8	-5.8	-10.3	-14.5	-18.6	-24.0	-30.1	-28.7	-27.7

We now turn to investigating how these price changes have reduced household net worth by applying the price changes in Table 3 to the life-cycle profiles for aggregate net worth and its decomposition as outlined in Tables 1 and 2.[8] Table 4 reports changes in net worth from 2007:2 by quarter, in the aggregate and by age group. In the first set of columns, we simply report dollar losses across our risky asset types. Second, we divide total dollar losses by age-group specific average net worth and age-group specific annual income. Third, we report dollar losses relative to aggregate average total income. In our discussion we place special emphasis on the cumulative price changes as of the first quarter of 2009, since this was the quarter in which portfolio-weighted asset values attained their nadir.

The average household saw a decline in price-change-induced decline in net worth of $177,310 between 2007:2 and 2009:1, which amounted to 32% of 2007:2 net worth, and more than twice average 2007 annual income. Almost half of this total decline was driven by a decline in stock prices, and almost half by a decline in house prices.

Losses vary widely by age. Younger households lost much less, while those in the 60-69 age group lost the most: $311,940 on average, or nearly four times average annual income for this age group. At the same time, differences in portfolio composition were large enough to generate substantial age variation in returns. In particular, because younger households were more leveraged, they lost more as a percentage of net worth: 30-39 year-olds lost 45% of net worth, while households older than 70 lost only 27%. In other words, absent age variation in portfolios,

for directly-held corporate equities: this series aligns closely with the Wilshire 5000 index. The Flow of Funds also reports a price series for residential real estate, based on the LoanPerformance Index from First American Corelogic. This series closely tracks the Case-Shiller series. By contrast, the house price series published by OFHEO (based on data from Fannie Mae and Freddie Mac) shows significantly smaller declines in house values.

[8] Of course, this exercise ignores the endogenous response of household portfolios to changing asset prices between 2007 and 2009.

but given the empirical age profile for net worth, the losses experienced by younger households would have been smaller, and those experienced by older households would have been even larger.

Table 4: **Estimated Losses by Age Group as of 2009:1**

	Dollar Losses ($1,000)					Losses as Percentage of		
Age of Head	Stocks	Res. real estate	Non-corp. bus.	Non-res. property	Total Losses	Net Worth	Income	Avg. Income
All	78.99	77.13	16.78	4.42	177.31	31.9	212.5	212.5
20-29	4.79	17.75	7.86	0.21	30.60	39.6	78.8	36.7
30-39	24.64	56.92	5.96	2.08	89.59	44.9	128.3	107.4
40-49	66.47	79.23	13.70	3.71	163.12	35.0	174.7	195.5
50-59	126.91	103.50	26.23	6.43	263.07	31.8	223.0	315.3
60-69	159.01	110.75	33.10	9.08	311.94	29.6	286.0	373.9
70+	92.73	85.52	15.34	5.07	198.65	27.3	345.1	238.1

Figure 2 presents the losses as a percentage of net worth for additional dates. By the third quarter of 2009, asset prices had partially recovered to around their values in 2008:4, but were still 25.5 percent below their peak values.

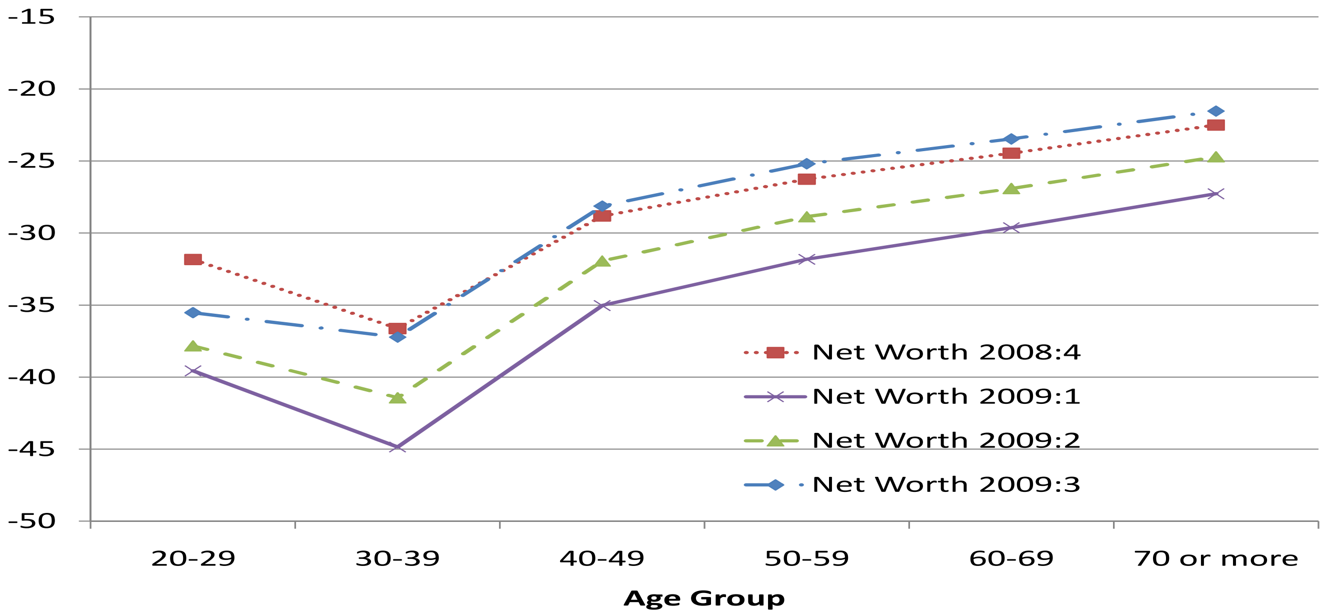
Our empirical analysis reveals two key facts that guide our choice of the model. First, there is substantial heterogeneity by household age in labor income, human wealth and net worth, and thus in the exposure to aggregate wage and asset price risk. Thus, at the very minimum an OLG model with one financial asset whose price varies with aggregate economic conditions is necessary. Our benchmark model contains exactly (and only) these features. Second, in the data portfolio allocations between risky and riskless assets display significant age heterogeneity as well, and therefore we will consider extensions of the benchmark in which such age variation in asset portfolios is permitted explicitly in our model. We now turn to its formal description.

3 The Model

The economy is populated by overlapping generations that are subject to aggregate shocks impacting both total factor productivity (and thus aggregate wages and dividends) as well as age-specific labor efficiency endowments and thus the age-wage distribution.

The Stochastic Structure The current aggregate shock is denoted by z, has finite support Z, and evolves over time according to a Markov chain with transition matrix $\Gamma_{z,z'}$.

Figure 2: **Percentage decline in net worth by age group relative to 2007:2**



Technology A representative firm owns a fixed factor K and operates a Cobb-Douglas technology that owns and uses a fixed factor K (interpreted as land, or the fixed stock of capital) and hires labor L as inputs and produces a quantity Y of nonstorable consumption goods as output. Its total factor productivity (TFP) is affected by the aggregate productivity shock z. Therefore

$$Y = z\, K^{\theta}\, L^{1-\theta}$$

where $\theta \in (0,1)$ is capital share of (net) output.

We normalize the total amount of the fixed factor to $K = 1$. The representative firm is publicly traded and the number of outstanding shares is constant and normalized to 1 as well.

Endowments and Preferences Households live for I periods and then die with certainty. Therefore the economy is populated by I distinct age cohorts at any given point in time. Each age cohort is composed of identical households. In each period of their life these households are endowed with one unit of time which they supply to the market inelastically. Their age-dependent labor productivity profile is given by $\{\varepsilon_i\}_{i=1}^{N}$. In this way we can capture heterogeneity in the impact of economic downturns on labor incomes of different age cohorts. We normalize units so that $\sum_{i=1}^{N} \varepsilon_i = 1$. Thus the aggregate supply of labor is constant and equal to $L = 1$ at all times. We also assume that newborn households start with zero asset holdings.

Households have standard time-separable preferences over stochastic consumption streams $\{c_i\}_{i=1}^N$ that can be represented by

$$E \sum_{i=1}^{N} \prod_{j=1}^{i} \beta_j \, u(c_i)$$

where β_i is the time discount factor between age $i-1$ and i, and varies with age.[9] This age variation stands in for unmodeled changes in family size and composition, age-specific mortality risk and other age-specific shifts in the marginal utility of consumption. We will use the heterogeneity, by age, in time discount factors in calibration to insure that equilibria in our economy display the same life cycle pattern of wealth that we documented empirically in SCF data in Section 2.

Expectations $E(.)$ are taken with respect to the underlying stochastic process governing aggregate risk. The period utility function is of the constant relative risk aversion variety

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

where the parameter $\frac{1}{\sigma}$ measures the intertemporal elasticity of substitution, and $\sigma = 1$ corresponds to log-utility.

3.1 Market Structure

Labor is traded in frictionless spot markets. In all versions of our model households can transfer resources across time by trading shares of ownership in the representative firm, or equivalently, ownership of the fixed factor of production. Because of the TFP shock this asset bears aggregate risk.

We consider three alternative market structures that differ in the set of additional assets that can be traded. In the benchmark model (Section 3.1.1) households can *only* save by purchasing shares in the risky asset. This model allows us to highlight the main mechanisms at work in the most transparent way, and admits close form solutions for special cases.

The second market structure (Section 3.1.2) encompasses a risky and a riskless asset. However,

[9] We normalize $\beta_1 = 1$.

households' portfolio allocations are set exogenously so the age-specific asset composition matches those in the SCF data. This version of the model permits us to evaluate quantitatively how important the significant age differences in portfolio shares are for the asset pricing and welfare predictions of the model.

The drawback of the previous model is that, given equilibrium prices, those most adversely affected by aggregate asset price declines (the elderly) are not permitted to trade away from it, potentially biasing the calculations for the age distribution of the welfare consequences of negative aggregate shocks. Therefore, in section 3.1.3 we describe a version of our model in which the portfolio composition is endogenously chosen by households and asset markets are sequentially complete. In contrast to the previous version now prices and portfolio allocations are both endogenous.

3.1.1 One Asset Economy

We now state the definition of a recursive competitive equilibrium when only shares of the risky firm are traded. The aggregate state of the economy is described by the current aggregate shock z and the aggregate distribution of wealth (shares) $S = (S_1, \ldots, S_I)$ across age cohorts. Individual state variables include a household's age i and its individual share of wealth, denoted by s.

Therefore the dynamic programming problem of the household reads as

$$\begin{aligned} v_i(z, S, s) &= \max_{c \geq 0, s'} \left\{ u(c) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \; v_{i+1}(z', S', s') \right\} && (1) \\ c + s' \; p(z, S) &= \varepsilon_i \; w(z) + s \;\; [p(z, S) + d(z)] && (2) \\ S' &= G(z, S) && (3) \end{aligned}$$

where $p(z, S)$ is the price of one share of the asset and $d(z)$ is the dividend payed by that asset (notice that competitive factor markets make the dividend depend only on the aggregate shock). G is the law of motion for the distribution of shares. Let $c_i(z, S, s)$ and $g_i(z, S, s)$ denote the optimal policy functions for consumption and share holdings.

Definition 1. *A recursive competitive equilibrium are value and policy functions for each age* $\{v_i, c_i, g_i\}$, *pricing functions* w, d, p *and an aggregate law of motion* G *such that*

1. *Given the pricing functions and the aggregate law of motion the value functions* $\{v_i\}$ *solve*

the recursive problem of the households and $\{c_i, g_i\}$ are the associated policy functions.

2. *Wages and dividends satisfy*

$$w(z) = (1-\theta)\, z \qquad \text{and} \qquad d(z) = \theta z \tag{4}$$

3. *Markets clear*[10]

$$\sum_{i=1}^{I} g_i(z, S, S_i) = 1 \tag{5}$$

$$\sum_{i=1}^{I} c_i(z, S, S_i) = z \tag{6}$$

4. *Consistency (each agent is representative of her age group)*

$$S'_1 = 0 \tag{7}$$

$$S'_{i+1} = G_i(z, S) = g_i(z, S, S_i) \qquad \forall i = 1, \dots, I-1. \tag{8}$$

3.1.2 Two Asset Economy with Exogenous Portfolios

In section 2 we documented substantial heterogeneity in the allocation of portfolios by age. To assess the importance of this heterogeneity we now propose an environment where the representative firm issues one period risk-free bonds, each of which is a promise to pay one unit of the consumption good. We denote the number of outstanding bonds by $\widehat{B}$ and treat it as an exogenous parameter of the model. All objects in the economy with exogenous portfolios are denoted with a hat, and it is understood that all equilibrium objects are functions of the exogenously given amount of bonds $\widehat{B}$. Therefore in this version of the model the firm is leveraged, and a fraction of total wealth is held in the form of (very) risky shares and the rest is held in the form of riskless bonds. Consequently $\widehat{B}$ is also a measure of leverage in this model.[11]

In this economy is easier to use as aggregate state variable the age distribution of shares of *total wealth.* We denote this distribution by $A = (A_1, \dots, A_I)$, where beginning of the period total

[10] Only one of these two conditions is required as the other one is guaranteed by Walras law.

[11] It is perhaps helpful to imagine the following sequence of events in every period: first the shock gets realized, then production ensues, then wages and bond holders are paid, then new bonds are issued and finally the remainder of the value of production is distributed to shareholders as dividends.

wealth includes the value of stocks and dividends (as in the one-asset economy) plus the value of outstanding bonds.

As before, total capital income equals θz. The gross amount that bonds pay out is $\widehat{B}$, the amount collected by issuing new bonds is $\widehat{B}\,\widehat{q}(z,A)$, where $\widehat{q}(z,A)$ is the inverse of the gross risk free interest rate. Therefore, in each period the net interest paid on bonds is $[1-\widehat{q}(z,A)]\,\widehat{B}$. The rest of capital income θz is distributed as dividends and denoted by $\widehat{d}(z,A)$. Formally,

$$\theta z \;=\; \underbrace{[1-\widehat{q}(z,A)]\;\widehat{B}}_{\text{Interest Income}}+\underbrace{\theta\, z-[1-\widehat{q}(z,A)]\;\widehat{B}}_{\text{Dividends}}=[1-\widehat{q}(z,A)]\,\widehat{B}+\widehat{d}(z,A) \tag{9}$$

The value of the leveraged representative firm (with outstanding debt in an amount $\widehat{B}$ payable in the next period), after the dividends are distributed, is denoted by $\widehat{p}(z,A)$. Consequently, we have

$$\begin{aligned}\text{Total wealth} \;&=\; \widehat{p}(z,A)+\widehat{d}(z,A)+\widehat{B}\\ &=\; \widehat{p}(z,A)+\theta\, z+\widehat{q}(z,A)\;\widehat{B}.\end{aligned}$$

In this economy the fractions of household savings that go to stocks and to bonds *by assumption are determined exogenously*. Let $c_i(z,A,a)$ and $y_i(z,A,a)$ denote the optimal policy functions for consumption and total savings. Let λ_i be the exogenous age-dependent share of the savings that goes to stocks. Thus the number of shares purchased by the household today is given by $\frac{\lambda_i\, y}{\widehat{p}(z,A)}$ and the number of bonds is given by $\frac{(1-\lambda_i)\, y}{\widehat{q}(z,A)}$. Tomorrow the bonds pay out one unit of consumption per bond purchased, and stocks pay out $\left[\widehat{p}(z',A')+\widehat{d}(z',A')\right]$ per share. Thus, a household of age i that saves y units of the good will get a share $a_i'', A', y)$ of total wealth in the next period defined by equation (10):

$$a_i'(z',A',y)\left[\widehat{p}(z',A')+\widehat{d}(z'')+\widehat{B}\right]=\frac{\lambda_i\;\left[\widehat{p}(z',A')+\widehat{d}(z',A')\right]}{\widehat{p}(z,A)}\;y+\frac{1-\lambda_i}{\widehat{q}(z,A)}\;y. \tag{10}$$

Consequently the dynamic programming problem of the households reads as:

$$v_i(z,A,a)=\max_{c\geq 0,y,a_i'}\left\{u\,(c)+\beta_{i+1}\sum_{z'\in Z}\Gamma_{z,z'}\;v_{i+1}\,[z',A''),a_i'\,(z',A'(z'),y)]\right\}\qquad \text{s.t.} \tag{11}$$

$$
\begin{aligned}
c+y &= \varepsilon_i \; w(z) + \left[\widehat{p}(z,A)+\widehat{d}(z,A)+\widehat{B}\right] a && (12)\\
A'(z') &= \widehat{G}(z,A,z') && (13)\\
a_i'(z',A',y) & \quad \text{satisfies (10)} && (14)
\end{aligned}
$$

Definition 2. *A recursive competitive equilibrium with exogenous portfolio shares,* $\{\lambda_i\}$*, are value, policy and share laws of motion functions* $\{v_i, c_i, y_i, a_i'\}$*, pricing functions* $w, \widehat{d}, \widehat{p}, \widehat{q}$ *and an aggregate law of motion* $\widehat{G}$ *such that*

1. *Given the pricing functions and the aggregate law of motion, the value functions* $\{v_i\}$ *solve the recursive problem of the households and* $\{c_i, y_i, a_i'\}$ *are the associated policy functions.*

2. *Wages and dividends satisfy*

$$
w(z) = (1-\theta)z \qquad \text{and} \qquad \widehat{d}(z,A) = \theta\; z - [1-\widehat{q}(z,A)]\,\widehat{B}
$$

3. *Markets clear (we now omit the clearing of the goods market)*

$$
\begin{aligned}
\sum_{i=1}^{I} \;\lambda_i \;\; y_i(z,A,A_i) &= \widehat{p}(z,A)\\
\sum_{i=1}^{N} \;(1-\lambda_i)\;\; y_i(z,A,A_i) &= \widehat{B}\;\;\widehat{q}(z,A)
\end{aligned}
$$

4. *Consistency (each agent is representative of her age group)*

$$
\begin{aligned}
A_1' &= 0\\
A_{i+1}' &= \widehat{G}_i(z,A,z') = a_i'\left[z', \widehat{G}(z,A,z'), \widehat{y}_i(z,A,A_i)\right] \qquad \forall i = 1,\dots,I-1 && (15)
\end{aligned}
$$

In Appendix A we relate asset prices in the this economy to the price of the stock in the one asset economy, and argue that by choice of portfolio shares $\{\lambda_i\}$ and/or outstanding bonds $\widehat{B}$ in the model *with exogenous portfolios* one can generate equity premia of arbitrary size.

3.1.3 Economy with Endogenous Portfolios and Complete Markets

Finally, we let households freely choose their portfolios. Since in our applications the number of values the aggregate state can take is 2, markets are sequentially complete when households can trade a bond and a stock. For computational reasons[12] it is easier to first solve an economy with a complete set of contingent claims, and then to construct from the equilibrium the prices of stocks and bonds.

We use tildes "$\widetilde{\ }$" to denote complete market objects. Let $\widetilde{s}_i(z')$ be an asset purchased by a household of age i that delivers one share next period (i.e. entitles to the price and dividend of one share), but *only* if aggregate state z' is realized. The state of the economy is now the distribution of shares of stocks, and is given by S, as in the one-asset economy. Denote the state contingent prices by $\widetilde{p}(z, S, z')$. The price of the firm today, after it has paid out dividends $d(z) = \theta z$, is denoted by

$$\widetilde{p}(z, S) = \sum_{z' \in Z} \widetilde{p}(z, S, z') \tag{16}$$

With this asset market structure the maximization problem of the households now reads as

$$\widetilde{v}_i(z, S, s) = \max_{c \geq 0, s'(z')} \left\{ u(c) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \ \ \widetilde{v}_{i+1} \left[z''(z'), s'(z') \right] \right\} \tag{17}$$

$$\text{s.t.} \qquad c + \sum_{z'} s'') \, \widetilde{p}(z, S, z') = \varepsilon_i w(z) + [\widetilde{p}(z, S) + \theta \, z] \, s \tag{18}$$

$$S'(z') = \widetilde{G}(z, S, z') \tag{19}$$

with solution $\widetilde{c}_i(z, S, s)$, $\widetilde{s}_i'(z, S, s, z')$.

Definition 3. *A recursive competitive equilibrium with complete markets, are value, policy and shares laws of motion functions* $\{\widetilde{v}_i, \widetilde{c}_i, \widetilde{s}_i'(z')\}$, *pricing functions* $w, \widetilde{p}$ *and an aggregate law of motion* $\widetilde{G}$ *such that*

1. *Given the pricing functions and the aggregate law of motion the value functions* $\{\widetilde{v}_i\}$ *solve*

[12] Complete markets allow us to construct securities that only pay in some states of the world yielding for well-behaved household maximization problems. Solving the portfolio choice problem between bonds and stocks is much harder because the returns on both assets are very co-linear.

the recursive problem of the households and $\{\widetilde{c}_i, \widetilde{s}_i'\}$ are the associated policy functions.

2. *Wages and dividends satisfy*

$$w(z) = (1-\theta)z \qquad \text{and} \qquad d(z) = \theta z \tag{20}$$

3. *Markets clear*

$$\sum_{i=1}^{I} \widetilde{G}_i(z, S, z') = 1, \qquad \forall z' \in Z. \tag{21}$$

4. *Consistency (representative agent)*

$$\begin{aligned} S_1' &= 0 \\ S_{i+1}' &= \widetilde{G}_i(z, S, z') = \widetilde{s}_i(z, S, S_i, z') = \qquad \forall\, z', i \in Z \times I - 1. \end{aligned} \tag{22}$$

Note that the no arbitrage condition (that shares can be bought directly or by means of state contingent contracts) is built into the definition of $\widetilde{p}(z, S)$ in equation (16). Also note that from state contingent prices of shares $\widetilde{p}$, and the law of motion of the the wealth distribution we can construct the gross risk free interest rate as:

$$1 + \widetilde{r}^f(z, S) = \frac{1}{\sum_{z'} \frac{\widetilde{p}(z,S,z')}{\widetilde{p}(z', G(z,S,z')) + z'\theta}} \tag{23}$$

while the realized gross real returns on risky stocks depends on the shocks of two consecutive periods and is given by

$$1 + \widetilde{r}(z, S, z') = \frac{\widetilde{p}[z', \widetilde{G}(z, S, z')] + z'\theta}{\widetilde{p}(z, S)}. \tag{24}$$

In order to talk about portfolio shares in risky vs safe assets, we introduce redundant assets representing a risk free bond as well as a non state contingent share in a levered firm. Let $\bar{B}$ be the number of bonds issued by the firm in every period and state of the world and $\tilde{p}^b(z, S)$ be the price of such a bond. There are two ways of ensuring one unit of good unconditionally in the next period. One could either buy the risk free bond or instead by $\tilde{b}(z, S, z')$ shares of the state contingent share, where:

$$\tilde{b}(z, S, z') = \frac{1}{\tilde{p}(z', G(z, S, z')) + \theta z'} \tag{25}$$

A no arbitrage argument yields the price of the risk free bond

$$\tilde{p}^b(z, S) = \sum_{z'} p(z, S, z')\tilde{b}(z, S, z') \tag{26}$$

To price the non-contingent share of the levered firm, note that the dividend is now:

$$\tilde{d}(z, S) = \theta z + \tilde{p}^b(z, S)\bar{B} - \bar{B} \tag{27}$$

One can buy a non-contingent claim to this dividend by either purchasing a share or by purchasing $\tilde{e}(z, S, z')$ shares of each state contingent share, where:

$$\tilde{e}(z, S, z') = \frac{d(z', G(z, S, z')) + \tilde{p}^s(z''))}{\tilde{p}(z', G(z, S, z')) + \theta z'} \tag{28}$$

A recursive equation for $\tilde{p}^s(z, S)$ is therefore attained as:

$$\tilde{p}^s(z, S) = \sum_{z'} \tilde{p}(z, S, z')\tilde{e}(z, S, z') \tag{29}$$

Since buying one share for each state contingency is the same as buying the entire non-contingent levered firm and all outstanding debt we have that $\tilde{p}(z, S) = \tilde{p}^s(z, S) + \tilde{p}^b(z, S)\bar{B}$, which allows us to simplify (29) to:

$$\tilde{p}^s(z, S) = \tilde{p}(z, S) - \tilde{p}^b(z, S)\bar{B}. \tag{30}$$

4 Developing Intuition: Three Examples

In order to understand the key mechanisms at work in our model we now study three simple economies: a representative agent version of our environment, and two special cases of our general OLG economy. In each of these examples, the only asset traded is unlevered equity. These examples are designed to highlight a) what determines the magnitude of asset price collapses, relative to the decline in output, and b) how a given wage and asset price decline can translate into welfare effects that vary across different generations.

>From now on assume that the aggregate shock takes only two values $Z = \{z_l, z_h\}$, where z_l stands for a severe recession. We will measure the magnitude of the decline of asset prices, relative to output (which is given by z) by

$$\xi = \frac{d \ln(p_h/p_l)}{d \ln(z_h/z_l)}$$

where it is understood that, in general, prices and thus the elasticity ξ, are functions of the aggregate state of the economy. An elasticity of $\xi = 3$, for example, indicates that the percentage decline of stock prices in recessions is three times as large as that of output.

4.1 Example I: Representative Agent Model

Our first simple economy is the standard infinitely-lived representative agent Lucas asset pricing model (translated into our physical environment). [13] Given a representative agent, the distribution of wealth is degenerate, and given Markov shocks, the only state variable is current productivity z. Furthermore, if aggregate shocks are i.i.d. (which we will assume in our baseline calibrations for the quantitative models described above) then

$$\left(\frac{p_l}{p_h}\right) = \left(\frac{z_l}{z_h}\right)^{\sigma}$$

and thus $\xi^{RA} = \sigma$. The same result obtains in two other cases, even when aggregate shocks are not i.i.d. These cases are $\sigma = 1$ (unitary inter-temporal elasticity of substitution) or $\beta = 1$ (no discounting). The logic for why prices become more sensitive to output as the inter-temporal elasticity of substitution $1/\sigma$ is reduced is familiar and straightforward. For the stock market to clear, stock prices (and expected returns) must adjust to output fluctuations such that (identical) households never want to buy nor sell stocks. The less willing are households to substitute consumption over time, the more prices must fall (and expected returns increase) in order to induce households to maintain constant stock holdings in response to a decline in output and thus consumption.

In the recent great recession in the United States, asset prices fell roughly three times as much as output. The representative agent economy generates $\xi^{RA} = 3$ when $\sigma = 3$. Of course, while this economy serves as a useful benchmark, it has nothing to say about differential welfare effects

[13] The detailed analysis is contained in Appendix B.

across age-groups, our main object of interest in this paper.

4.2 Example 2: Two-Period OLG Model

We use our first and simplest OLG example to discuss how the presence of finite lives in the OLG model affects the elasticity ξ of price changes to output changes. Let $I = 2$, $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$, that is, households live for two periods, but only earn labor income $(1-\theta)z$ in the first period of their lives. Since young households start with zero assets, all wealth is held by old agents. Thus the wealth distribution is time invariant and degenerate in this economy. As in the representative agent model, the only state variable is the exogenous shock $z \in \{z_l, z_h\}$.

Consumption of young and old households is given by

$$\begin{aligned} c_1(z) &= (1-\theta)z - p(z) \\ c_2(z) &= \theta z + p(z) \end{aligned}$$

and the price of shares are determined by the intertemporal Euler equation

$$p(z)\left[(1-\theta)z - p(z)\right]^{-\sigma} = \beta \sum_{z' \in \{z_L, z_H\}} \Gamma_{z,z'} \left[\theta z' + p(z')\right]^{-\sigma} \left[\theta z' + p(z')\right] \tag{31}$$

In general, no closed form solution is available for this functional equation that determines $p(.)$. However, in Appendix C we derive an approximate expression for the elasticity ξ^{2p} for this two period ($2p$) OLG model, assuming i.i.d. shocks:[14]

$$\begin{aligned} \xi^{2p} &\approx \frac{\sigma(1-\theta)}{1-\theta\frac{(R-\sigma)}{(R-1)}} \\ &= \xi^{RA} \times \frac{1-\theta}{1-\theta\frac{(R-\sigma)}{(R-1)}} \end{aligned}$$

where $R = \frac{\theta+p}{p} > 1$ is the steady state gross return on the stock.[15]

Note that for $\sigma = 1$ (log-utility) this formula is exact. We make the following observations. First, for $\sigma = 1$ we have $\xi^{2p} = \xi^{RA} = 1$ and prices fall by exactly as much as output in a downturn. Second, for $\sigma > 1$ we have $\xi^{2p} > 1$ and $\xi^{2p} < \xi^{RA}$. That is, as long as the intertemporal elasticity

[14] This approximation involves taking linear approximations to pricing equations around the point $z_l/z_h = 1$.

[15] For θ such that $R = \beta^{-1}$ the expression simplifies to $\xi^{2p} \approx \frac{\sigma(\beta+1)}{\sigma\beta+1}$.

of substitution $1/\sigma$ is smaller than one, asset prices fall by more than output. However, the fall is smaller in the life cycle OLG economy than in the representative agent economy with infinitely-lived households. For example, suppose we think of a period as 30 years, take $\sigma = 3$, and set β and θ such that the model generates the same interest rate and the same wealth to income ratio as our baseline six period model (see section 6). These choices imply $\beta = 0.311$, and $\theta = 0.3008$. Then, using the expression above, $\xi^{2p} = 1.97$ in the OLG economy, compared to $\xi^{RA} = 3$ in the representatitive agent economy.[16] The finding that stock prices are less volatile, relative to output, in OLG economies compared to the infinitely lived representative agent economy will reappear consistently in the various economies we study.

The current old generation clearly suffers from the recession since the price of the asset, the only source for old-age consumption, is lower in the bad than in the good aggregate state of the world. Moreover, for $\sigma > 1$, consumption of the old is more sensitive to aggregate shocks than consumption of the young:

$$\frac{c_1(z_h)}{c_1(z_l)} < \frac{z_h}{z_l} < \frac{c_2(z_h)}{c_2(z_l)}$$

This second inequality reflects the fact that $c_2(z_h)/c_2(z_l) = p_h/p_l > z_h/z_l$, (since $\xi^{2p} > 1$), while the first inequality follows from market clearing: $(c_1(z_h) + c_2(z_h)) / (c_1(z_l) + c_2(z_l)) = z_h/z_l$. The fact that aggregate risk is disproportionately born by the old explains why stock prices are less volatile in this economy than in the analogous representative agent economy. Recall that stocks are effectively priced by younger agents, because the supply of stocks by the old is inelastic at any positive price. Because the old bear a disproportionate share of aggregate risk, the young are relatively insulated, their consumption fluctuates less than output, and thus smaller price changes (relative to the representative agent economy) are required to induce them to purchase the aggregate supply of equity at each date.

One might wonder whether it is possible that $c_1(z_h)/c_1(z_l) < 1$, so that newborn households may indeed prefer to enter the economy during a recession rather than during a boom. The answer turns out to be no: while stock prices fall by more than output in the event of a recession, they never fall by enough to compensate the young for the decline in their labor earnings they experience. The logic for this result is straightforward. In a two-period OLG economy, stock prices are defined by the inter-temporal first-order condition for young households (eq. 31). With i.i.d. shocks, the

[16] For the calibration just described, assuming $z_l/z_h = 0.9$ and $pr(z = z_h) = 0.85$ the true elasticity is $\xi^{2p} = 1.99$, compared to the value of 1.97 from the approximate expression.

right hand side of this condition is independent of the current value for z. Taking the ratio of the two pricing equations across states, the ratio of stock prices across states is given by:

$$\frac{p_h}{p_l} = \left(\frac{c_1(z_h)}{c_1(z_l)}\right)^{\sigma}$$

Now the only advantage to the young from entering the economy during a recession is that they buy stocks cheaply, $p_h/p_l > 1$. But then the optimality restriction above implies that $c_1(z_h)/c_1(z_l) > 1$, so the young must suffer relatively low consumption if they enter during a recession. Intuitively, the force that pushes stock prices down in a recession is that low prices are needed to induce the young to buy stocks when the marginal utility of current consumption is high. But a high marginal utility of consumption requires low consumption.

This example teaches us that for the young to potentially gain from a recession, we need people to live for at least three periods (at least given the specification of intertemporal preferences we assume). We therefore conclude this section with a three period example designed to highlight the model elements required for the young to indeed benefit from a recession.

4.3 Example 3: Three-Period OLG Model

Now households live for three periods, $I = 3$. Households do not value consumption when young, and discount the future at a constant factor $\beta_2 = \beta_3 = \beta$. They are only productive in the first period of their lives, i.e. $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = 0$.

By construction, young households buy as many stocks as they can afford, while the old sell all the stocks they own. Only the middle-aged make an interesting stock purchase decision, trading off current versus future consumption. In a recession, falling stock prices will have countervailing effects on the middle-aged's stock trade decision. On the one hand, low current stock prices offer an incentive to reduce stock sales to exploit higher expected stock returns (the substitution effect). On the other, consumption smoothing calls for larger stock sales, since stock sales are the only source of income for this group (the income effect).

Given that young households start their lives with zero asset holdings and that the total number of wealth shares has to sum to one, the only endogenous aggregate state variable in this simple economy is the share of asset held by old households S_3 which we for simplicity denote by $S_3 = S$. Consequently the share of assets owned by middle-age households is given by $S_2 = 1 - S$.

The first order condition for middle-aged household can then be written as

$$p(z,S)\ u'\left[(1-S)(p(z,S)+\theta z)-G(z,S)\ p(z,S)\right]=\\ \beta\sum_{z'}\Gamma_{z,z'}\ \left[p(z',G(z,S))+\theta z'\right]\ u'\left[G(z,S)\left(p(z'')\right]\right. \quad (32)$$

where consistency requires that tomorrow's asset share of the old is equal to the number of shares purchased by the current middle-aged households: $S'=G(z,S)$. In this expression marginal utility from consumption when middle aged, $c_2=(1-S)(p(z,S)+\theta z)-G(z,S)p(z,S)$ is equated to expected discounted marginal utility from old age consumption $c_3=G(z,S)(p(z',S')+\theta z')$, adjusted by the gross return on assets $(p(z'')\,/p(z,S)$. Given a pricing function $p(z,S)$, equation (32) defines the optimal policy function $G(z,S)$.

The second functional equation determining the pricing and optimal policy functions states that the equilibrium demand for shares of the young, $1-G(z,S)$ equals the number of shares that can be purchased with total labor income of the young, which is $w(z)/p(z)=(1-\theta)z/p(z)$. Thus

$$[1-G(z,S)]p(z,S)=(1-\theta)z \quad (33)$$

Equations (32) and (33) form a pair of functional equations that jointly determine the unknown pricing and policy functions $p(z,S)$ and $G(z,S)$. Consumption $\{c_2(z,S),c_3(z,S)\}$ and welfare $\{v_1(z,S),v_2(z,S),v_3(z,S)\}$ at all ages can easily be calculated from these equilibrium functions.[17]

4.3.1 Log-Utility

As in the previous example a closed form solution of the model is not available in general, but as well known since Huffman (1987) if the period utility function is logarithmic ($\sigma=1$) we can fully

[17] These are given explicitly as

$$\begin{aligned} c_o(z,S) &= S(p(z,S)+\theta z) & (34)\\ c_m(z,S) &= (1-S)(p(z,S)+\theta z)-G(z,S)p(z,S) & (35)\\ v_o(z,S) &= u(c_o(z,S)) & (36)\\ v_m(z,S) &= u(c_m(z,S))+\beta\sum_{z'}\Gamma_{z,z'}\ u[c_o(z',G(z,S))] & (37)\\ v_y(z,S) &= \beta\sum_{z'}\Gamma_{z,z'}\ v_m(z',G(z,S)) & (38)\end{aligned}$$

characterize the equilibrium.[18] It is given by

$$
\begin{aligned}
p(z,S) &= z\, p(S) \\
G(z,s) &= G(S) \\
c_3(z,S) &= z\, c_3(S) \\
c_2(z,S) &= z\, c_2(S) \\
v_3(z,S) &= \log(z) + \Psi_3(S) \\
v_2(z,S) &= \log(z) + \beta E_z \log(z') + \Psi_2(S) \\
v_1(z,S) &= \beta E_z \log(z'^2 E_z E_{z'} \log(z'') + \Psi_1(S)
\end{aligned}
$$

where Ψ_i are known functions. Crucially, given log-utility, the optimal number of shares carried into old age is independent of the aggregate productivity shock z, and thus $G(z,S) = G(S)$. Note that this result also implies non-stochastic wealth dynamics in the model: $S' = G(S)$. Thus from any initial condition $S_0 \in (0,1)$ the wealth share of the elderly converges monotonically to its unique positive steady state value. Furthermore the ratio of wages to asset prices $w(z)/p(z)$ is independent of z. As in the previous model, with log utility $\xi^{3p} = 1$.

Given the previous result the welfare consequences of a recession for the different generations can be easily calculated as:

$$
\begin{aligned}
v_3(z_l,S) - v_3(z_h,S) &= \log\left(\frac{z_l}{z_h}\right) < 0 \\
v_2(z_l,S) - v_2(z_h,S) &= \log\left(\frac{z_l}{z_h}\right) + \beta\left(E_{z_l} - E_{z_h}\right)\log(z') \\
v_1(z_l,S) - v_1(z_h,S) &= \beta\left(E_{z_l} - E_{z_h}\right)\log(z') \\
&\quad + \beta^2\left(E_{z_l} - E_{z_h}\right) E_{z'} \log(z'')
\end{aligned} \tag{39}
$$

Thus old households always lose from a recession, not surprisingly since they simply consume the value of their assets which are worth less in a recession. For the middle aged and young generations the welfare consequences of a recession depend on the properties of the stochastic process driving aggregate risk. Middle-aged generations are also unambiguous losers of recessions unless $\beta > 1$ and aggregate productivity is strongly negatively correlated so that $E_{z_l}\log(z') \gg E_{z_h}\log(z')$. For young generations, on the other hand, the model with log-utility hints at the possibility of benefits

[18] In fact, this is true for an arbitrary lifetime horizon I as long as households own only labor income in the first period of their lives.

from recessions. If aggregate shocks are i.i.d, then from equation (39) young generations are exactly indifferent between being born into good or bad aggregate circumstances. On the other hand, if aggregate shocks display sufficiently negative serial correlation young households may gain from an economic downturn.

The crucial property of the model with log-utility is that the number of shares middle-aged households sell to finance consumption is independent of the aggregate shock, due to income and substitution effect cancelling out: $G(z, S) = G(S)$. In equilibrium asset prices are proportional to aggregate shocks and thus fall to the same extent as wages. This result suggests that with an intertemporal elasticity lower than one, middle-aged households will have a stronger motive to sell shares to smooth consumption in response to temporary declines in asset payoffs (the income effect will dominate the intertemporal substitution effect). This is turn should lead to a decline in asset prices that is larger than the corresponding fall in wages, generating welfare gains from economic downturns for young households. We document next that this is indeed the case.

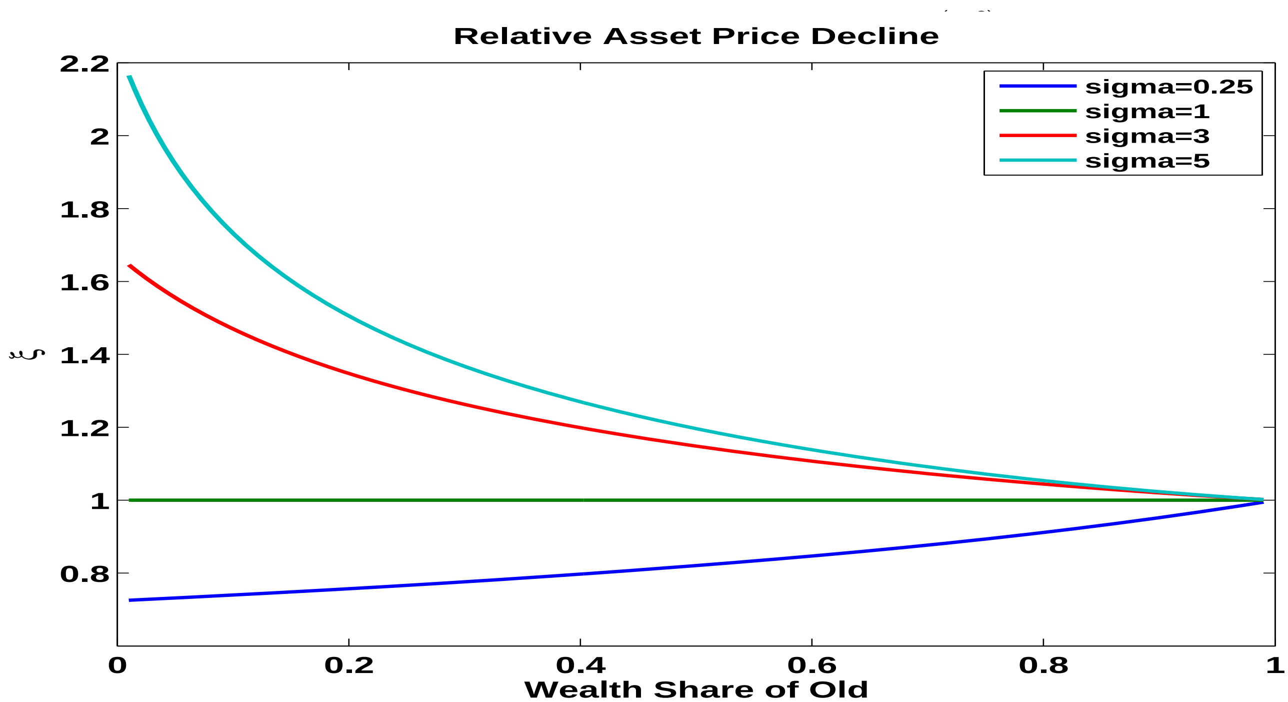
4.3.2 General Intertemporal Elasticity of Substitution

For $\sigma \neq 1$ the recursive competitive equilibrium of the model needs to be solved numerically, but this is straightforward to do with only one continuous state variable S. We choose the same parameter values as in the previous example. The capital share remains at $\theta = 0.3008$ and the time discount factor equals $\beta = (0.311)^{\frac{20}{30}} = 0.459$, resulting in the same annualized discount factor as in the two period model. [19] The aggregate shock takes two values with $z_l/z_h = 0.9$ Thus a fall in aggregate technology leads to a decline of aggregate output in the order of 10%, a value we will also use below in the calibrated version of the full model. We assume that aggregate shocks are uncorrelated over time in this model in which a period lasts for 20 years.

Figure 3 plots the elasticity of asset prices to output, ξ^{3p}, as a function of the share of wealth held by the old generation, for various values of the IES $1/\sigma$. Note that, as demonstrated above, for the logarithmic case $\sigma = 1$ we have $\xi^{3p} = 1$, independent of the wealth distribution S.

This figure displays two key findings. First, the lower is the willingness of households to intertemporally substitute consumption (the higher is σ), the larger is the fall in asset prices, relative to output, in a recession. Second, the extent of asset price movements are strongly

[19] We are assuming 30 year periods in the two period OLG model and 20 year periods in the 3 period model. Note also that it is impossible to match the empirical wealth-to-earnings ratio by appropriate choice of β in this simple model since households only earn labor income in the first period of their life and save all of it.



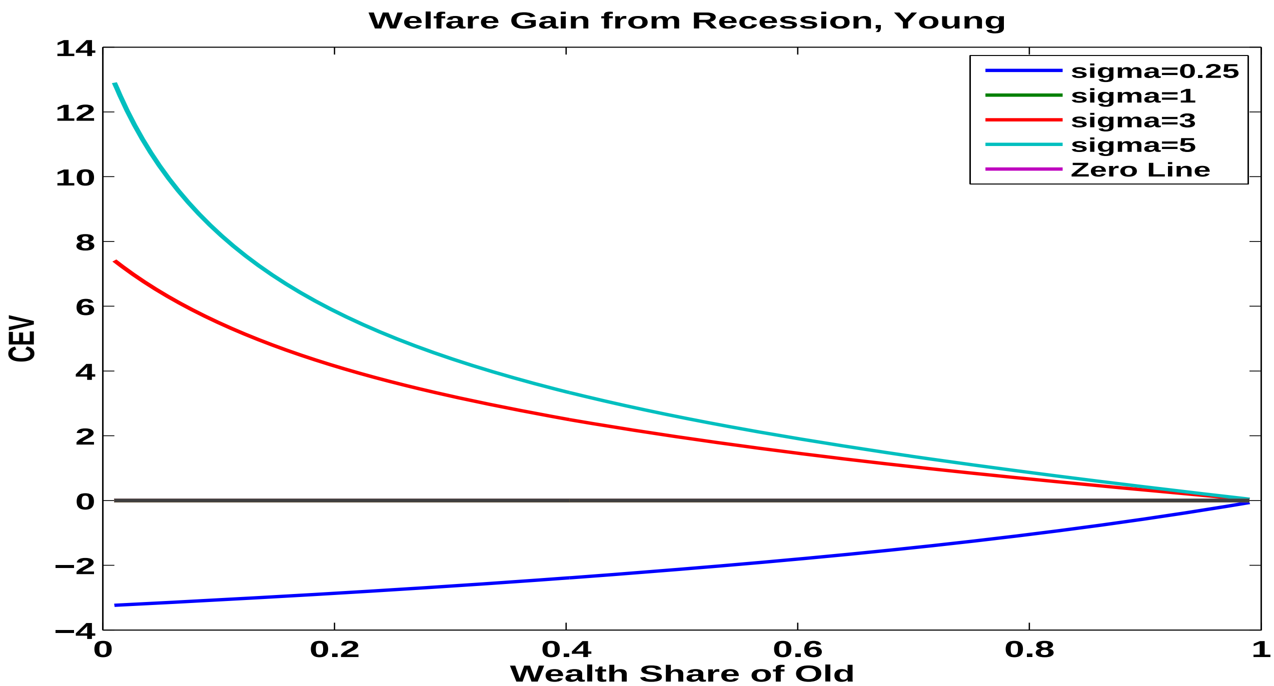
affected by the wealth distribution S if preferences are not logarithmic, something that could not occur in the 2 period model. The larger is the share of wealth held by the middle aged (the smaller is S), the larger is ξ^{3p}. This is because, among the agents selling assets, the old always sell all their stocks, irrespective of z, while the middle-aged will sell more stocks in a recession than in a boom, as long as the inter-temporal elasticity of substitution is less than one. Thus the larger the share of wealth in the hands of middle aged relative to the old, the larger is the downward pressure on prices in response to a negative shock, since the young must buy more stocks with the same amount of earnings.[20]

As comparison to the other examples for $\sigma = 3$ (and the quantitative exercise below) the wealth share of the old generation converges to $S = 34.2\%$ if the economy experiences a long sequence of good shocks, and for $S = 0.342$ we find a price elasticity of $\xi^{3p} = 1.234$.

The welfare consequences for young generations of starting their economic lives in a recession, relative to an expansion, are displayed in Figure 4. We measure welfare consequences as the percentage increase in consumption in all periods of a household's life, under all state contingencies,

[20]In the economies for which $\sigma \neq 1$ the wealth holdings of the old at the start of a recession need not be the same as at the start of an expansion, on average. The figure displays results from a hypothetical thought experiment that traces out the differences between expansions and recessions, conditional on the *same* wealth distribution in the economy.

Figure 4: **Welfare Consequences of Recessions for Young Households**



that a household born in an expansion would require to be as well off as being born in a recession, with positive numbers thus reflecting welfare gains from a recession. Again we plot these numbers as a function of the wealth distribution S, and for different σ. We observe that the welfare consequences from recessions for the young follow the elasticity ξ of asset prices, relative to wages output rather closely. confirming that this statistic is a crucial determinant of the welfare impact of recessions in our class of OLG economies.

The purpose of our simple examples was to develop the intuition for the magnitude of asset price declines, relative to output and the resulting implications for lifetime welfare of different generations. Our last example in this section displayed welfare gains from a recession for the young as long as $\sigma > 1$. However, note that in this example we stacked the deck in several ways in favor of obtaining this result. First, young households do not value consumption and thus are not affected directly by a decline in current aggregate consumption. Second, the middle-aged and the old have no source of income other than selling shares, which means that they bear a disproportionate share of the burden of recession. Third, aggregate shocks are purely temporary, so young households can expect asset prices to recover before they need to sell stocks later in their life cycle.

The remainder of the paper now documents the size of the asset price decline and the distribution of welfare consequences from a severe and long-lasting recession in a realistically calibrated OLG economy in which life cycle labor income and wealth profiles match those observed in the 2007 SCF.

5 Calibration

To describe the calibration process, we start with the *two-asset economy with exogenous portfolios*. The one asset economy is a special case in which portfolio shares do not vary by age. We then retain the calibrated parameters for the economy in which households choose portfolios freely.

We assume agents enter the economy as adults, and live for $I = 6$ periods, where a period is 10 years. The preference parameters to calibrate are the coefficient of relative risk aversion σ, and the life-cycle profile for discount factors $\{\beta_i\}_{i=2}^I$. The technology parameters are capital's share θ, the life-cycle profile for earnings $\{\varepsilon_i\}_{i=1}^I$, the supply of bonds $\widehat{B}$, the life-cycle profile for portfolio shares allocated to stocks, $\{\lambda_i\}_{i=1}^I$, and the support and transition probability matrix Γ for the aggregate productivity shock.

We first calibrate the preference and life cycle parameters using a non-stochastic version of the economy, in which the productivity shock is set to its average value $\bar{z} = 1$. The calibration of the stochastic process for productivity comes afterwards.

Let r_e denote the net return on equity in the non-stochastic economy, and let r_b be the net return on bonds. Our calibration strategy can be summarized as follows:

1. Fix risk aversion, σ, to a benchmark value of 3.
2. Set the life-cycle profile $\{\varepsilon_i\}_{i=1}^N$ equal to the empirical life-cycle profile for labor income, and the portfolio shares $\{\lambda_i\}_{i=1}^N$ equal to age-group specific shares of risky assets in net worth.
3. Set capital's share θ and the supply of bonds $\widehat{B}$ so that the model generates realistic returns to risky and safe assets, r_e and r_f.
4. Set the life cycle profile $\{\beta_i\}_{i=2}^N$ so that the model generates the SCF life-cycle profile for net worth documented in Section 2, given the other determinants of life-cycle saving: risk aversion σ, the profile for earnings $\{\varepsilon_i\}_{i=1}^N$, the exogenous portfolio shares $\{\lambda_i\}_{i=1}^N$, and the returns to risky and safe assets, r_e and r_b.

We now describe this calibration procedure in more detail. It delivers a realistic (as measured by the SCF 2007) joint life-cycle distribution for earnings, net worth, portfolio composition, and consumption. This is necessary for our calibrated OLG model to serve as a suitable laboratory for exploring the distributional impact of aggregate shocks.

Returns Following Piazzesi, Schneider, and Tuzel (2007) we target annual returns on safe and risky assets of 0.75% and 4.75% per annum, where the latter is the return on an equally-weighted portfolio of stocks returning 6.94% and housing returning 2.52%. Given our period length is 10 years, this implies

$$1 + r_b = \frac{1}{q} = 1.0075^{10} \tag{40}$$

and

$$1 + r_e = \frac{\theta - (1-q)\widehat{B} + p}{p} = 1.0475^{10} \tag{41}$$

We now describe in more detail how we calibrate the life-cycle profiles, before moving to discuss how to compute the pair $\left(\theta, \widehat{B}\right)$ that delivers the target returns r_e and r_b.

Life cycle profiles We compute empirical life-cycle profiles by taking simple averages across SCF age-group household means for holdings of safe assets, net worth, and non-asset income. The youngest age group corresponds to households aged $20 - 29$, and the sixth and oldest age group corresponds to households aged 70 and above.

Budget constraints of households in the model can be written as

$$\begin{aligned} c_i &= (1-\theta)\varepsilon_i + R_i y_i - y_{i+1} \text{ for } i = 1, \ldots, N-1 \\ c_N &= (1-\theta)\varepsilon_N + R_N y_N, \end{aligned}$$

where y_{i+1} is savings for age group i (net worth for age group $i+1$) and where the gross return on savings between age i and $i+1$ is given by

$$R_i = \lambda_i(1 + r_e) + (1 - \lambda_i) r_b.$$

We measure $\{\varepsilon_i\}_2^N$ as ten times average annual earnings of age group i, $\{y_i\}_2^N$ as the average net worth of age group i, and $\{\lambda_i\}_1^N$ as the fraction of risky assets in aggregate net worth for age group i. Note that returns vary by age because λ_i is age-varying and $r_e > r_b$. Because agents in our model enter the economy with zero initial wealth, we re-categorize asset income for the youngest group in the SCF as labor income: thus we set $y_1 = 0$, and ε_1 equal to ten times average annual earnings for age group one plus $R_1 y_1$.

Given the sequences $\{\varepsilon_i\}$, $\{y_i\}$ and $\{\lambda_i\}$, the budget constraints imply a life-cycle consumption profile, $\{c_i\}$. This consumption profile can be used to back out the sequence of time discount factors that supports the age-varying profile for returns. In particular, in a non-stochastic version of the model, the household's inter-temporal first order condition implies

$$\beta_{i+1} = \left(\frac{c_{i+1}}{c_i}\right)^{\sigma} \frac{1}{R_{i+1}}.$$

Note that the consumption profile is derived directly from household budget constraints, and is pinned down by data on labor income, net worth, and returns. Thus, the consumption profile is independent of preference parameters, and in particular of the choice for risk aversion, σ. However, *to support this consumption profile as an equilibrium outcome* requires a discount factor profile $\{\beta_i\}$ that does depend on σ. For example, suppose $\lambda_i = \lambda$ and thus $R_i = R$ for all i. Then, the larger is σ, the more sensitive is the implied profile for β_i to age variation in consumption.

Figure 5 below shows life cycle profiles for consumption, net worth and labor income from the nonstochastic version of our model, and Figure 6 displays the implied calibrated profile $\{\beta_i\}_2^N$. Note that β_i is generally larger than one. This reflects the fact that the data indicate strong growth in income and consumption over the life-cycle between the 20-29 age group, and the 50-59 age group. However, β_i should not be interpreted solely as capturing pure time preference: it also captures the effects of age variation in family size and composition on the marginal utility from consumption.

Technology Because the model is calibrated to replicate observed life-cycle profiles for earnings, net worth and portfolio composition, it will also replicate the ratio of aggregate safe assets to aggregate net worth, and the ratio of aggregate net worth to aggregate (10 year) labor income.[21]

[21] In our model, each age group is assumed to be of equal size. Thus, given that we replicate SCF portfolios for each age group, the appropriate aggregate targets are simple unweighted averages across age groups. Because these age groups are not of identical size in the SCF, these aggregate targets do not correspond exactly to SCF

Figure 5: **Life Cycle Profiles for Consumption, Net Worth and Labor Income**

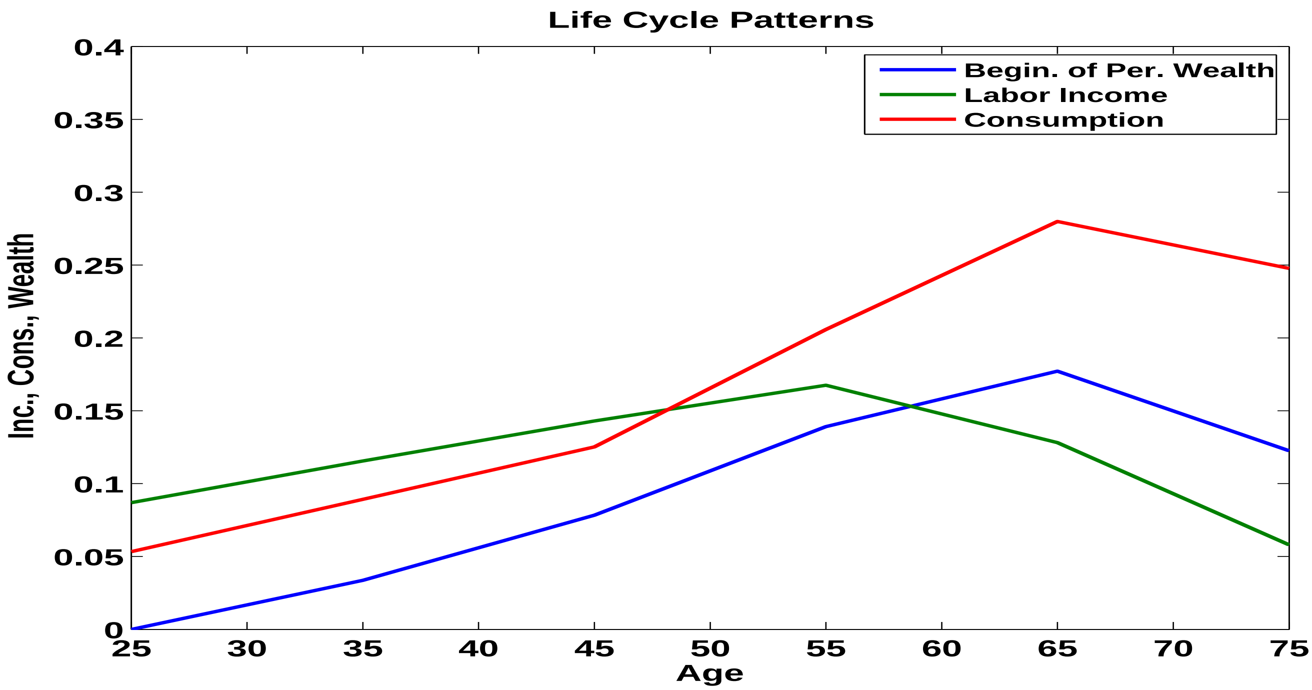
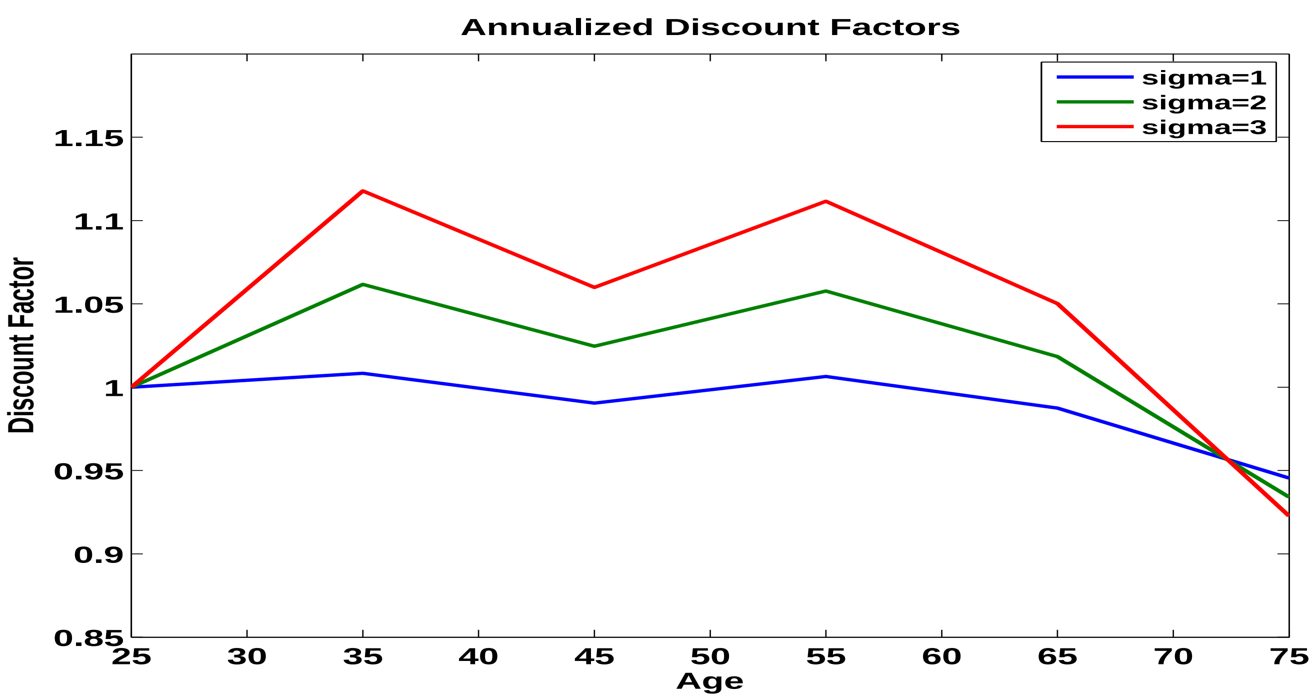


Figure 6: **Implied Discount Factors for various Elasticities of Substitution**



Suppressing state-dependent notation these ratios in the SCF are, respectively

$$\frac{q\widehat{B}}{p+q\widehat{B}} = 0.082 \qquad \text{and} \qquad \frac{p+q\widehat{B}}{1-\theta} = \frac{1}{10} \times 7.84.$$

The expressions for returns in equations (40-41) define q and p as functions of θ and $\widehat{B}$. Substituting in these functions, the two ratios above can be used to solve for θ and $\widehat{B}$. The solutions are $\theta = 0.30$ and $\hat{B} = 0.048$.

Aggregate risk We assume that the aggregate shock z takes one of two values, $z \in Z = \{z_l, z_h\}$. We assume that the ratio $\frac{z_l}{z_h} = 1.1$. We assume that z is *iid* over time, where the probability that $z = z_h$ is equal to 0.85. Given that a period is 10 years, this implies that the expected duration of periods of high productivity is $10/0.15 = 66.7$ years, while the expected duration of periods of low productivity is $10/0.85 = 11.8$ years. Thus in our calibration, a recession involves a very large and quite persistent decline in output, but is a very rare event. Finally, we normalize so that average output is equal to one: $0.15 \times z_l + 0.85 \times 1.1 \times z_l = 1$.

6 Results

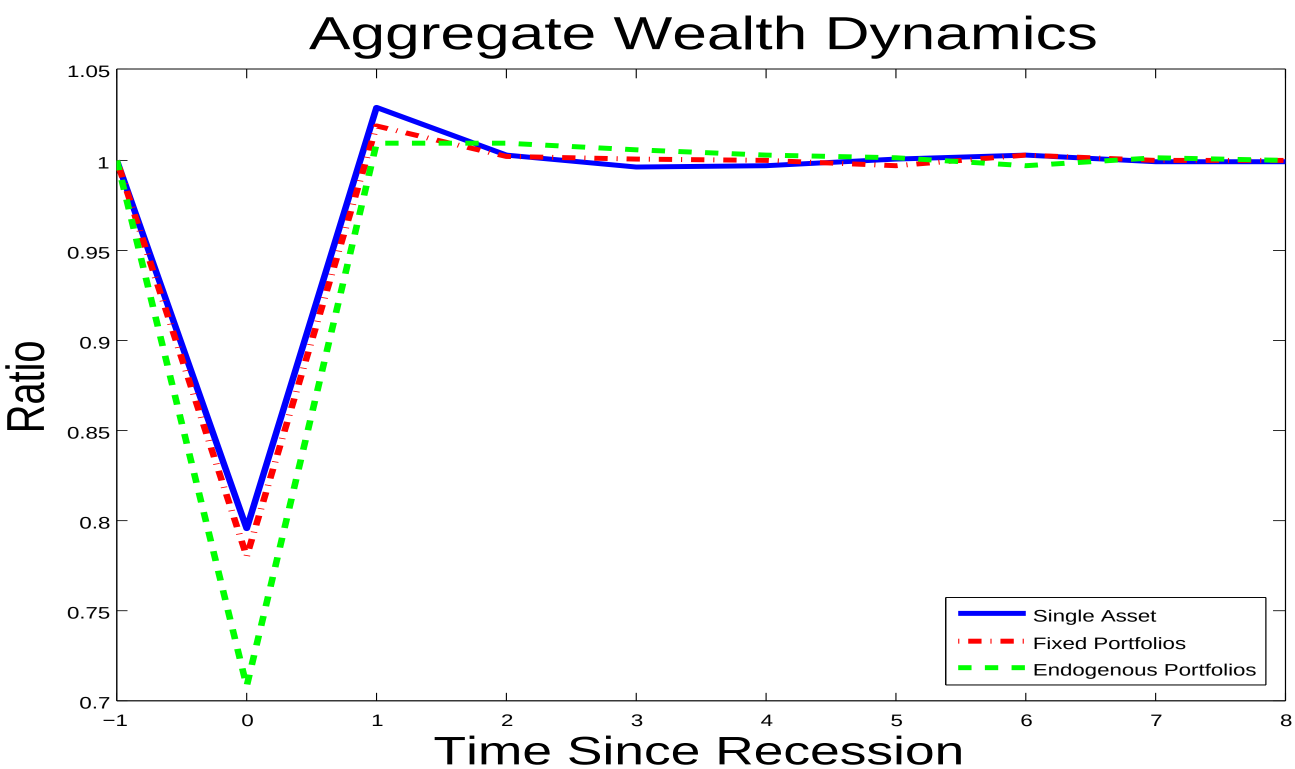
We now document the asset price and welfare implications of a large recession. The nature of the experiment is as follows. We simulate each alternative model economy assuming the sequence for aggregate productivity involves a long period of normal times, a recession in period zero, and a return to normal times in subsequent periods. Recall that the shock involves a 10% decline in productivity, and that agents assign a 15% probability to a recession occurring in any period. Our baseline value for the inter-temporal elasticity of substitution is one third: $\sigma = 3$. We begin by describing the results for the one-asset economy in which only stocks are traded, and then move to the two other economies, the two-asset economy with exogenous portfolios, and the model in which portfolios are endogenous.

One asset economy Figure 7 displays the implied paths for asset prices. In the one asset economy, the stock price falls by 20% in the period of a recession. The finding that stock prices decline more than output is consistent with the intuition developed in Section 4 that for an elasticity of substitution smaller than one, asset prices are more volatile than the underlying shocks. The

population averages, but the differences turn out to be small.

elasticity of prices to output is 2.04 (see Table 5). [22] Quantitatively, the decline in the model stock price is smaller than the corresponding change from asset price peak to trough observed in quarterly US data, which was approximately three times as large as the output decline over the same period. However, recall that the period length in our model is 10 years. In this light, we note that asset prices in the United States have partially recovered since their trough and thus we view a model price decline somewhere between 20% and 30% as a reasonable approximation to the medium run decline in net worth associated with the great recession.

Figure 7: **Equilibrium Asset Prices for all Calibrated Economies**



In the recovery period after the shock, the stock price over-shoots, rising above its long run value. This reflects endogenous wealth dynamics in the model. When the shock hits, older households, and especially those in the 60–69 year-old age group, sell additional equity to fund consumption. Thus, in the period after the recession, a larger share of aggregate wealth is held by younger cohorts, who are net savers, while less is held by older cohorts, who are net borrowers. This translates into higher net demand for equity, and thus a high stock price.

[22] Recall that the corresponding value for the two-period OLG economy described in Section 4 was very similar, at 1.99.

Table 5 shows that the asset price decline is declining in the inter-temporal elasticity of substitution. With log consumption, prices move one-for-one with output, as in the simpler economies described in Section 4. As in the two-period economy described in Section 4, the price elasticity is a concave function of the coefficient of relative risk aversion: the magnitude of the additional price response as σ is increased from 3 to 5 is much smaller than that when σ goes from 1 to 3. Recall that in an analogous representative agent economy with iid shocks, the elasticity is just σ.

Table 5: **Relative price decline** $\left(\frac{\%\Delta(p_0/p_{-1})}{\%\Delta(z_0/z_{-1})}\right)$ **for Each Economy**

Economy	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
Single Asset	1.00	2.04	2.62
Fixed Portfoliios			
–Stock	1.04	2.17	2.81
–Bond	1.02	2.52	3.40
–Wealth	1.04	2.20	2.87
Endogenous Portfolios			
–Stock	1.00	2.91	5.00
–Bond	1.00	2.94	4.98
–Wealth	1.00	2.91	5.00

Table 6 displays the welfare consequences of a model recession by age group, with $i = 1$ denoting households that become economically active in the recession period.[23]

In the one-asset economy, the welfare consequences of a recession are monotone in age, with older generations suffering more. For $\sigma = 3$, the loss for the oldest households is equivalent to a 15% decline in consumption. In the model this age group finances roughly half of consumption from income (evenly split between dividends and non-asset income) and finances the other half by selling assets. In a recession, income (output) declines by 10% and asset prices decline by 20%, translating into a 15% decline in consumption for this age group.

The welfare loss is much smaller for younger age groups for two reasons. First, welfare losses are expressed in units of lifetime consumption, so relatively small losses for younger households partly reflect the fact that one period of recession accounts for a small fraction of remaining lifetime for the young. Second, the old finance part of consumption by selling stocks, and their welfare losses are amplified by the fact that stock prices decline by more than output. The flip side of large

[23]Welfare gains are measured as the percentage increase in consumption (in all periods of life) under a no-recession scenario needed to make households indifferent between a one period recession and the no recession alternative. Negative numbers reflect welfare losses from a recession.

Table 6: **Welfare Gain from Recession**

Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
Single Asset Economy			
1	−1.96%	−1.01%	−0.56%
2	−2.47%	−1.35%	−0.83%
3	−2.93%	−1.66%	−0.98%
4	−4.32%	−3.35%	−2.65%
5	−6.49%	−7.59%	−7.98%
6	−10.04%	−15.16%	−18.07%
Fixed Portfolio Economy			
1	−2.13%	−0.85%	−0.12%
2	−3.15%	−2.44%	−2.10%
3	−3.22%	−2.09%	−1.44%
4	−4.30%	−3.31%	−2.60%
5	−6.19%	−7.21%	−7.63%
6	−9.43%	−14.60%	−17.66%
Endogenous Portfolio Economy			
1	−1.96%	0.50%	4.17%
2	−2.49%	−3.39%	−3.67%
3	−2.94%	−2.48%	−1.01%
4	−4.33%	−4.66%	−4.55%
5	−6.49%	−7.49%	−9.07%
6	−10.00%	−11.12%	−14.26%

capital losses for older households is that young households get to buy stocks at fire-sale prices. As the economy recovers in subsequent periods, stock prices bounce back, and younger generations enjoy substantial capital gains. The lower is the intertemporal elasticity of substitution (and thus the larger is the recession-induced price decline) the more unevenly are welfare costs distributed across generations, with larger losses for the old, and smaller losses for younger households.

In order to isolate the immediate effect of the recession from the effect of age, we also report the immediate fall in consumption for each cohort in the period the recession hits. We immediately see how the relative importance of labor vs. asset income is over the life cycle. The oldest generation suffers the greatest fall in consumption since most of their income is derived from wealth and the value of assets falls by at least as much as TFP. The fall in consumption for all other generations is due to both the fall in their income as well as the endogenous change in their asset purchases.

Exogenous portfolios economy We now turn to our model with two assets, in which we impose the substantial heterogeneity in portfolio composition across age groups observed in the Survey of Consumer Finances (see Section 3.1.2). Recall that in this version of the model, older households hold a significant share of wealth in safe assets, while younger households are leveraged, and thus more exposed to asset price declines. Again, Figure 7 shows the time path for aggregate net worth for this economy, while Table 5 breaks down price declines for stocks and bonds in the period of the recession. Note that each bond pays off one unit of consumption in the period of the recession, as in every other period: this is the definition of a safe asset. However, the equilibrium price of new bonds (and stocks) must both adjust so that markets clear, given agents' (optimal) saving choices, and the (suboptimal) portfolio shares they are forced to adopt.

The two-asset model generates a slightly larger decline in asset values in a recession than the stock-only model described above. To understand why, recall that because younger households are leveraged in this economy, they suffer a larger decline in wealth relative to the one asset economy. Thus in the period of the recession, younger households require an even larger decline in asset prices (relative to the one asset model) in order to be willing to absorb the extra assets older households are selling.

Table 6 indicates larger welfare losses for households aged 30–49 in the two asset economy, relative to the one asset model, while households aged 60 and older fare better in the two asset model. These differences are readily interpreted: because 30–49 year-old households are now leveraged they take a bigger hit in the period of the crisis, while households aged 60 and older have more safe assets, and suffer less.

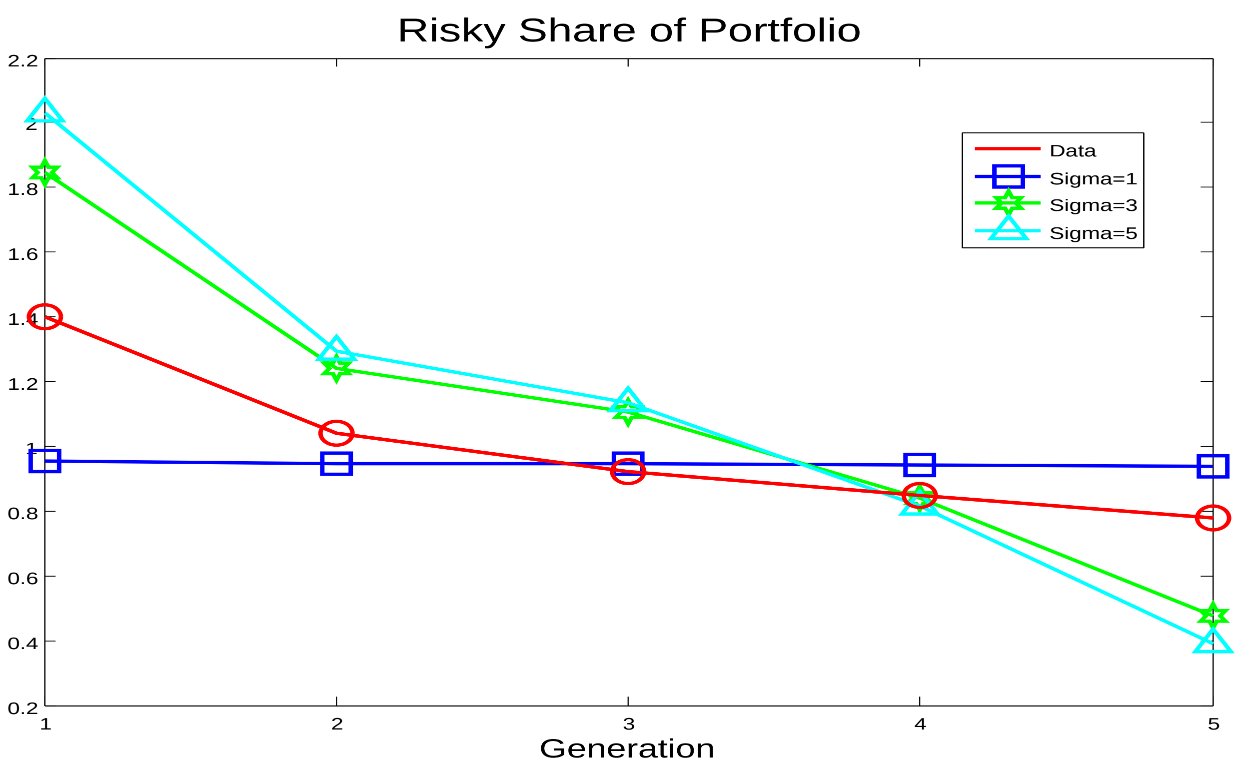
Table 7: **Consumption Changes at Time of Recession**

Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
Single Asset Economy			
1	−10.0%	−6.76%	−5.12%
2	−10.0%	−6.76%	−5.13%
3	−10.0%	−6.81%	−5.13%
4	−10.0%	−7.56%	−6.08%
5	−10.0%	−10.21%	−10.14%
6	−10.0%	−15.16%	−18.07%
Fixed Portfolio Economy			
1	−10.22%	−6.71%	−4.85%
2	−10.67%	−8.03%	−6.63%
3	−10.43%	−7.57%	−5.94%
4	−10.16%	−7.84%	−6.35%
5	−9.87%	−10.07%	−10.00%
6	−9.43%	−14.60%	−17.66%
Endogenous Portfolio Economy			
1	−10.00%	−3.18%	8.40%
2	−10.00%	−10.49%	−8.67%
3	−10.00%	−8.34%	−4.35%
4	−10.00%	−9.93%	−9.45%
5	−10.00%	−10.89%	−12.85%
6	−10.00%	−11.11%	−14.26%

Endogenous portfolios economy We now move to our final economy, the economy in which households choose portfolios freely at each age. Recall that there are only two values for the aggregate shock, and thus trade in two assets effectively completes markets. Because a constant amount of asset income flows to bond holders in a risk-free fashion, dividend income and stock returns are more volatile than in the corresponding one-asset economy.

In Figure 8 we plot how equilibrium portfolios in this model vary with age.[24] We do this for three values for risk aversion, $\sigma = 1$, 3 and 5. In the log case ($\sigma = 1$) portfolios are age-invariant, reflecting the fact that in this case, as in the other economies we have considered, asset values decline one-for-one with output. In this equilibrium, saving choices do not change in response to the recession shock and consumption of each generation declines by the same amount as output.

Figure 8: **Porftolio Choices of Households in Data and Models**



For $\sigma > 1$, aggregate asset values become more volatile than output, for the standard reason: when agents are less willing to substitute inter-temporally, prices must adjust more to induce agents to tolerate fluctuations in consumption. Because asset prices fluctuate by more than output,

[24] These are the portfolio shares after a long sequence of good shocks $z = z_h$.

younger households who have little wealth relative to earnings – require a more leveraged portfolio (a higher equity to debt ratio) to face the same exposure to aggregate risk as older (and wealthier) households. Thus, the equilibrium share of equity in household portfolios is decreasing with age, consistent with the downward-sloping profile for the share of risky assets in net worth observed in the SCF. While this qualitative pattern is an important success for the model, for $\sigma = 3$ or 5 age variation in portfolio composition is larger in the model than in the data. Thus, for these values for σ, observed US portfolios are not extreme enough to share risk efficiently across generations: older Americans are over-exposed to aggregate risk in the dta, relative to what is optimal in the model.

Figure 7 indicates that prices decline by more in the endogenous portfolios model than in the other two model economies. This reflects the fact that with endogenous portfolios younger households are heavily leveraged. Thus, compared to the other economies, younger households take a bigger hit in the period of the shock. This translates into a larger decline in equilibrium consumption for younger households, and necessitates a larger fall in asset prices to preserve asset market clearing. Table 6 indicates that price changes in the complete markets economy are extremely similar to those that would emerge in a representative agent economy, with a price to output decline elasticity roughly equal to σ (see Section 4.1).

Turning to welfare, when portfolios are endogenous older households suffer less (relative to the other economies) in a recession, reflecting the fact that they hold more safe assets. Conversely, younger households generally fare worse, reflecting greater leverage. The differences in welfare losses by age between the models with exogenous and endogenous portfolios offer one metric for how far observed US portfolios are from those that maximize inter-generational risk-sharing through the trade of financial assets.

Note that there is no way for agents alive in the period before the recession to share risk with the new cohort that enters only in the recession period. Since prices fall by so much in the endogenous portfolios economy, the youngest households are actually better off entering the economy during a recession compared to normal times: for $\sigma = 3$ their welfare gain is equivalent to a half percent increase in life-time consumption. Thus the endogenous portfolios economy delivers an interesting twist: the fact that existing generations diversify aggregate risk efficiently magnifies asset price declines. These larger asset price declines in turn work to decrease effective risk sharing between generations alive in the previous period and new labor market entrants.

7 Conclusion

In this paper we have analyzed the distributional consequences of a large recession across different age cohorts. For a quantitative version of our stochastic overlapping generations economy restricted to match life cycle income and asset profiles from the SCF we find that older households suffer large welfare losses from a severe recession. Young households, in contrast, lose less and might even benefit from the economic downturn. The key statistic determining these welfare consequences is the price decline of assets, relative to the fall in wages and output. If households have low intertemporal elasticity of substitution then older households are pressed to sell their assets in the downturn in order to smooth consumption, putting additional pressure on asset prices, inducing larger welfare losses for older households and small welfare gains for households that become economically active in the recession.

Our model also has strong predictions how the recession affects the wealth distribution across households of different ages. Once the 2010 SCF is available we can evaluate whether the model predictions along this dimension are born out by the actual wealth data prior and after the great recession the U.S. economy is currently experiencing. We must defer this to future work.

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Appendix

A Asset Prices in the Economy with Exogenous Portfolios

In this section we briefly relate asset prices in the two asset economy with exogenous portfolios to the price of the stock in the one asset economy. To do so, first note that in the two-asset economy, if all age groups were to have the *same* exogenous portfolio allocations, $\lambda_i = \lambda$, then the two market clearing conditions for stocks and bonds imply a parametric relationship between $\widehat{p}(z, A)$ and $\widehat{q}(z, A)$:

$$\frac{\widehat{p}(z,A)}{\lambda} = \frac{\widehat{B}\ \ \widehat{q}(z,A)}{(1-\lambda)}$$

Moreover, the realized gross real return to the portfolio is identical across age cohorts, and is given by

$$R(z,A;z',A') = \lambda\frac{\left[\widehat{p}(z',A') + \widehat{d}(z'')\right]}{\widehat{p}(z,A)} + (1-\lambda)\frac{1}{\widehat{q}(z,A)} = \frac{\widehat{p}(z'') + \lambda\theta\, z'}{\widehat{p}(z,A)}$$

where the equality follows from substituting in the definition for $\widehat{d}(z', A')$ and the equation relating $\widehat{p}(z, A)$ and $\widehat{q}(z, A)$.

For the case $\lambda = 1$, this reduces to the familiar return on equity from a one-asset economy:

$$R(z,A;z'') = \frac{\widehat{p}(z',A''}{\widehat{p}(z,A)}$$

For $\lambda < 1$, the equilibrium stock price $\widehat{p}$ in the two asset economy is related to the stock price p in the one-asset economy by

$$\widehat{p}(z,A) = \lambda p(z,A).$$

Given this price, the return to saving in the two asset economy (when households are forced to hold a portfolio characterized by λ) is the same, state-by-state, as in the one-asset economy. The resulting equilibrium bond price in the two asset economy is given by

$$\widehat{q}(z,A) = (1-\lambda)p(z,A)/\widehat{B}.$$

and the conditional multiplicative equity premium is given by

$$E\left(\left.\frac{\frac{\widehat{p}(z',A')+\widehat{d}(z'')}{\widehat{p}(z,A)}}{1/\widehat{q}(z,A)}\right| z,A\right) = \frac{1-\lambda}{\lambda}\left[\frac{p(z,A)}{\widehat{B}}E\left(\left.\frac{p(z',A')+\theta z'}{p(z,A)}\right| z,A\right) - 1\right].$$

Thus for given stock returns and prices in the one-asset economy, by choice of λ and $\widehat{B}$ one can generate an arbitrarily large equity premium in the two-asset economy *with exogenous portfolios*. Specifically, for a given $\lambda \in (0,1)$ increasing the exogenous supply of bonds $\widehat{B}$ lowers the bond price, increases bond returns and thus lowers the equity premium. Similarly, fixing $\widehat{B}$ and increasing the portfolio share λ in stocks raises the demand and thus the price of shares, lowering stock returns and thus reducing the equity premium

Now consider the more general case, in which the exogenous portfolio shares vary with age. Denote aggregate savings and the fraction of aggregate savings in stocks by

$$Y(z,A) = \sum_{i=1}^{I} \; y_i(z,A,A_i)$$

and

$$s(z,A) = \frac{\sum_{i=1}^{I} \; \lambda_i \; y_i(z,A,A_i)}{\sum_{i=1}^{I} \; y_i(z,A,A_i)}$$

The market clearing conditions for stocks and bonds imply

$$s(z,A)\widehat{B} \;\; \widehat{q}(z,A) = \widehat{p}(z,A)(1-s(z,A))$$

Now the multiplicative equity premium is given by

$$\begin{aligned} E\left(\left. \frac{\frac{\widehat{p}(z',A')+\widehat{d}(z'')}{\widehat{p}(z,A)}}{1/\widehat{q}(z,A)} \right| z,A \right) &= E\left(\left. \frac{\frac{\widehat{p}(z',A')+\theta\, z' - \left[(1-\widehat{q}(z'A'))\widehat{B}\right]}{\widehat{p}(z,A)}}{1/\widehat{q}(z,A)} \right| z,A \right) = \\ &= \frac{(1-s(z,A))}{s(z,A)\widehat{B}} E\left(\widehat{p}(z',A') + \theta\, z' - \left[(1-\widehat{q}(z'A'))\,\widehat{B} \right] \right| z,A \Big) \\ &= \frac{(1-s(z,A))}{s(z,A)\widehat{B}} E\left(\left. \frac{\widehat{p}(z',A')}{s(z',A')} + \theta\, z' - \widehat{B} \right| z,A \right) \end{aligned}$$

As before, the equity premium is decreasing in $\widehat{B}$ (the relative supply of bonds) and decreasing in $s(z,A)$ (the relative demand for equity). The relative demand for equity in turn depends on the average fraction of savings devoted to stocks, and the covariance across age groups between equity shares and savings:

$$s(z,A) = \frac{cov\left(\lambda_i, y_i(z,A,A_i)\right)}{E\left[y_i(z,A,A_i)\right]} + E\left[\lambda_i\right].$$

In the special case $\lambda_i = \lambda$, $s(z,A) = \lambda$ and the expressions simplify to the ones above.

B Asset Prices in the Representative Agent Model

Suppose the representative agent invests fraction λ of savings in stocks, and $1-\lambda$ in bonds. Let $c(z,a)$ and $y(z,a)$ denote optimal consumption and savings, and let $p(z)$ and $q(z)$ be the equilibrium prices for stocks and bonds. Note that there is no need to keep track of aggregate wealth as a state: by assumption, the supply of capital is constant and equal to one. Thus prices can only depend on the shock z.

The dynamic programming problem for a household is

$$v(z,a) = \max_{c \geq 0, y} \left\{ u(c) + \beta \sum_{z' \in Z} \Gamma_{z,z'} v\left(z', a'(z',y)\right) \right\}$$

subject to

$$c + y = (1-\theta)z + \left(p(z) + d(z) + \widehat{B}\right) a$$

and the law of motion

$$a'(z',y)\left[p(z') + d(z') + \widehat{B}\right] = \left(\frac{\lambda\ [p(z') + d(z')]}{p(z)} + \frac{(1-\lambda)}{q(z)}\right) y$$

The solution to this problem yields decision rules $c(z,a)$, $y(z,a)$.

Given the preferences and technology described above, the market clearing conditions are simply

$$\begin{aligned} \lambda y(z,1) &= p(z) \\ (1-\lambda) y(z,1) &= q(z)\widehat{B} \\ c(z,1) &= z \end{aligned}$$

The individual and aggregate consistency condition is

$$a'(z', y(z,1)) = 1$$

Now suppose the process for z is a 2-state Markov chain. There are just two equity prices to solve for $p(z) \in \{p_l, p_h\}$. Recall, from the OLG version of the model that given an exogenous portfolio split defined by λ, the return to equity is $\left(\lambda\theta z' + p(z')\right)/p(z)$. Thus the equilibrium equity prices are defined by the solutions to the two inter-temporal first order conditions

$$\begin{aligned} p_l u'(c(z_l,a)) &= \beta \sum_{z' \in Z} \Gamma_{z_l,z'} \left[u'', a')\right) \left[\lambda\theta z' + p(z')\right]\right] \\ p_h u'(c(z_h,a)) &= \beta \sum_{z' \in Z} \Gamma_{z_h,z'} \left[u'(c(z'')) \left[\lambda\theta z' + p(z')\right]\right] \end{aligned}$$

From the market clearing conditions, $u'(c(z), a) = z^{-\sigma}$.

Let Γ_h denote the probability of remaining in the high productivity state, and let Γ_l denote the probability of remaining in the low productivity state. Then the equations for pricing stocks in the two states are

$$\begin{aligned} p_h z_h^{-\sigma} &= \beta\Gamma_h z_h^{-\sigma}\left(\lambda\theta z_h + p_h\right) + \beta(1-\Gamma_h) z_l^{-\sigma}\left(\lambda\theta z_l + p_l\right) \\ p_l z_l^{-\sigma} &= \beta\Gamma_l z_l^{-\sigma}\left(\lambda\theta z_l + p_l\right) + \beta(1-\Gamma_l) z_h^{-\sigma}\left(\lambda\theta z_h + p_h\right) \end{aligned}$$

>From the first

$$p_h = \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\sigma}}{z_h^{-\sigma}}\left(\lambda\theta z_l + p_l\right)}{(1-\beta\Gamma_h)}$$

Substituting this into the second

$$p_l = \frac{\beta\Gamma_l z_l^{-\sigma}\lambda\theta z_l + \beta(1-\Gamma_l) z_h^{-\sigma}\left(\lambda\theta z_h + \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\sigma}}{z_h^{-\sigma}}\lambda\theta z_l}{(1-\beta\Gamma_h)}\right)}{z_l^{-\sigma}\left(\frac{(1-\beta)(1+\beta(1-\Gamma_h-\Gamma_l))}{(1-\beta\Gamma_h)}\right)}$$

Since the expression for p_h is symmetric, we can take the ratio to express the ratio of prices across states as a function of fundamentals

$$\frac{p_h}{p_l} = \frac{z_h}{z_l}\left(\frac{(1-\Gamma_h)\, z_l^{1-\sigma} z_h^{\sigma-1} + (\beta + \Gamma_h - \beta\Gamma_h - \beta\Gamma_l)}{(1-\Gamma_l)\, z_h^{1-\sigma} z_l^{\sigma-1} + (\beta + \Gamma_l - \beta\Gamma_h - \beta\Gamma_l)}\right)$$

Note that λ has dropped out here: the ratio of stock prices does not depend on either λ or θ, though the level of prices does. Defining $\widetilde{p} = \frac{p_h}{p_l}$, $\widetilde{z} = \frac{z_h}{z_l}$ we have

$$\widetilde{p} = \widetilde{z}\left(\frac{(1-\Gamma_h)\,\widetilde{z}^{\sigma-1} + (\beta + \Gamma_h - \beta\Gamma_h - \beta\Gamma_l)}{(1-\Gamma_l)\,\widetilde{z}^{1-\sigma} + (\beta + \Gamma_l - \beta\Gamma_h - \beta\Gamma_l)}\right).$$

Note that if aggregate shocks are *iid*, then $1 - \Gamma_l = \Gamma_h$ and

$$\widetilde{p} = \widetilde{z}^{\sigma}$$

The same result is obtained for $\sigma = 1$, even without the *iid* assumption.

C Asset Prices in the Two Period Overlapping Generations Economy

Recall that the budget constraints when young and old are

$$\begin{aligned} c_1(z) &= (1-\theta)z - p(z) \\ c_2(z) &= \theta z + p(z) \end{aligned}$$

while share prices are pinned down the young generation's inter-temporal first order condition:

$$p(z)\left[(1-\theta)z - p(z)\right]^{-\sigma} = \beta \sum_{z' \in \{z_L, z_H\}} \Gamma_{z,z'} \left[\theta z' + p(z')\right]^{-\sigma} \left[\theta z' + p(z')\right]$$

Let $\widetilde{p} = \frac{p(z_H)}{p(z_L)}$, $\widetilde{z} = \frac{z_H}{z_L}$, and $\widetilde{x} = \frac{z_L}{p_L}$. In terms of these variables, the inter-temporal FOCs conditional on the current state being z_L and z_H are, respectively:

$$\begin{aligned} \left((1-\theta)\widetilde{x} - 1\right)^{-\sigma} &= \beta\Gamma_{z_L,z_L}\left(\theta\widetilde{x}+1\right)^{1-\sigma} + \beta\Gamma_{z_L,z_H}\left(\theta\widetilde{z}\widetilde{x}+\widetilde{p}\right)^{1-\sigma} \\ \widetilde{p}\left((1-\theta)\widetilde{z}\widetilde{x} - \widetilde{p}\right)^{-\sigma} &= \beta\Gamma_{z_H,z_L}\left(\theta\widetilde{x}+1\right)^{1-\sigma} + \beta\Gamma_{z_H,z_H}\left(\theta\widetilde{z}\widetilde{x}+\widetilde{p}\right)^{1-\sigma} \end{aligned}$$

Our goal is to solve for $\widetilde{p}$ as a function of $\widetilde{z}$. However, except for the special case $\sigma = 1$, this system of equations cannot be solved in closed form. So instead we will linearize these equations, and look for an approximate solution for relative prices as a linear function of relative productivity. We proceed as follows:

1. Take first-order Taylor-series approximations to these two first order conditions around the non-stochastic steady state values for $\widetilde{p}$, $\widetilde{z}$, and $\widetilde{x}$, which we denote P, Z and X (where $Z = P = 1$). This gives a system of two equations in three first-order terms $(\widetilde{x} - X)$, $(\widetilde{z} - Z)$ and $(\widetilde{p} - P)$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} (\widetilde{x} - X) \\ (\widetilde{z} - Z) \\ (\widetilde{p} - P) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{aligned}
A_{11} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta)- \\
&\quad \left((1-\sigma)\beta\Gamma_{z_L,z_L}(\theta X+1)^{-\sigma}\theta+(1-\sigma)\beta\Gamma_{z_L,z_H}(\theta X+1)^{-\sigma}\theta\right) \\
A_{12} &= -(1-\sigma)\beta\Gamma_{z_L,z_H}(\theta X+1)^{-\sigma}\theta X \\
A_{13} &= -(1-\sigma)\beta\Gamma_{z_L,z_H}(\theta X+1)^{-\sigma} \\
A_{21} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta)- \\
&\quad \left((1-\sigma)\beta\Gamma_{z_H,z_L}(\theta X+1)^{-\sigma}\theta+(1-\sigma)\beta\Gamma_{z_H,z_H}(\theta X+1)^{-\sigma}\theta\right) \\
A_{22} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta)X-(1-\sigma)\beta\Gamma_{z_H,z_H}(\theta X+1)^{-\sigma}\theta X \\
A_{23} &= \sigma\left((1-\theta)X-1\right)^{-\sigma-1}+\left((1-\theta)X-P\right)^{-\sigma}-(1-\sigma)\beta\Gamma_{z_H,z_H}(\theta X+1)^{-\sigma}
\end{aligned}$$

2. Use the the first equation in this system to solve for $(\widetilde{x}-X)$ as a linear function of $(\widetilde{z}-Z)$ and $(\widetilde{p}-P)$:

$$(\widetilde{x}-X)=-\frac{A_{12}}{A_{11}}(\widetilde{z}-Z)-\frac{A_{13}}{A_{11}}(\widetilde{p}-P)$$

Then substitute this solution into the second equation, and solve for $(\widetilde{p}-P)$ as a function of $(\widetilde{z}-Z)$:

$$\begin{aligned}
(\widetilde{p}-P) &= -\frac{A_{21}}{A_{23}}(\widetilde{x}-X)-\frac{A_{22}}{A_{23}}(\widetilde{z}-Z) \\
&= -\frac{A_{21}}{A_{23}}\left(-\frac{A_{12}}{A_{11}}(\widetilde{z}-Z)-\frac{A_{13}}{A_{11}}(\widetilde{p}-P)\right)-\frac{A_{22}}{A_{23}}(\widetilde{z}-Z)
\end{aligned}$$

Thus

$$\xi^{2p}\approx\frac{\widetilde{p}-P}{\widetilde{z}-Z}=\frac{A_{21}A_{12}-A_{22}A_{11}}{A_{23}A_{11}-A_{21}A_{13}}$$

3. Now assume productivity shocks are i.i.d., so that $\Gamma_{z_L,z_H}=\Gamma_{z_H,z_H}=\Gamma_{z_H}$ and $\Gamma_{z_L,z_L}=\Gamma_{z_H,z_L}=1-\Gamma_{z_H}$. Under this i.i.d. assumption, $A_{11}=A_{21}$ and thus

$$\xi^{2p}\approx\frac{A_{21}A_{12}-A_{22}A_{21}}{A_{23}A_{11}-A_{21}A_{13}}=\frac{A_{12}-A_{22}}{A_{23}-A_{13}}=X\frac{\sigma(1-\theta)}{((X-X\theta-1)+\sigma)}$$

Recall that X is the inverse of the steady state stock price, so we can equivalently write this elasticity in terms of the steady state gross interest rate. where $R=\theta X+1$:

$$\xi^{2p}\approx\frac{\sigma(1-\theta)}{1-\theta\frac{(R-\sigma)}{(R-1)}}$$

This is the expression given in the text.

4. In the special case in which θ is such that $R = \frac{1}{\beta}$, the expression simplifies further. For this interest rate to satisfy the young agent's inter-temporal first order condition in the non-stochastic steady state, it must be the case that $c_1 = c_2$ which implies, by equating the young and old's budget constraints that

$$\theta = \frac{1 - 2P}{2}$$

Then the relationships $R = \frac{\theta}{P} + 1$, and $\beta R = 1$ imply

$$\theta = \frac{1}{2}(1 - \beta)$$

Substituting this into the expression for ξ^{2p} gives

$$\xi^{2p} \approx \frac{\sigma(\beta + 1)}{\sigma\beta + 1}$$

which is the expression for this special case given in the text.

D Computational Appendix

Even for moderate number of generations the state space is large: $N-2$ continuous state variables (plus z). Since we want to deal with big shocks local methods (e.g. perturbation) should be used with caution. Consequently, wherever possible, this is in the economies with either only one asset or exogenous portfolios, we have used global methods based on sparse grids, as in Krueger and Kubler (2004, 2006).

To solve the complete markets economy we solve for undetermined coefficients of the linearized aggregate prices and policies. For each generation $i = 1, .., I - 1$ and each current shock z_j, we have $k = 1, .., N_z$ Euler Equations of the form:

$$0 = u'(c_i(z_j, S, S_i))p_{j,k}(S) - \beta_i \Gamma_{ij} \left(\tilde{p}_k(G_{j,k}(S)) + \theta z_k\right) u'(c_{i+1}(z_j, G_{j,k}(S), g^i_{j,k}(S_i, S)))$$

That gives a total of $(I - 1) \times N_z^2$ conditions. In addition there are N_z^2 market clearing conditions, for a total of $I \times N_z^2$ equations.

For each shock z_j we compute the steady state price in an economy where TFP is constant at that level, call it $\bar{p}_j$. The steady state distribution of wealth shares is constant across these steady states, we call it $\bar{S}$. We will linearize around these points and impose price and policy functions of the form:

$$\begin{aligned}
\Delta p_{j,k}(\Delta S) &= p_{j,k} - \bar{p}_j = \alpha^{p,0}_{j,k} + \alpha^{p,S}_{j,k}\Delta S \\
\Delta g^i_{j,k}(\Delta s_i, \Delta S) &= \alpha^{i,0}_{j,k} + \alpha^{i,s}_{j,k}\Delta s_i + \alpha^{i,S}_{j,k}\Delta S \\
\Delta G^i_{j,k}(\Delta S) &= \beta^{i,0}_{j,k} + \beta^{i,S}_{j,k}\Delta S
\end{aligned}$$

The coefficients for G are determined from g by imposing that each individual in a generation is representative of that generation, so in effect this gives N_z^2 coefficients for prices plus $(I-1)\times N_z^2$ coefficients for policies for a total of $I \times N_z^2$, where coefficients are the constant terms and on each generation's share of aggregate wealth. We linearize the Euler Equations and market clearing conditions with respect to current state contingent prices, the current aggregate share distribution, current personal shares, future prices, and personal share demands. Imposing representativeness, this gives equations of the form (suppressing arguments where obvious):

$$0 = \bar{\phi}^i_{j,k} + A^i_{j,k}\Delta p_{j,\cdot} + B^i_{j,k}\Delta S + C^i_{j,k}\Delta s_i + D^i_{j,k}\Delta G_{j,k} + E^i_{j,k}\Delta p'_{k,\cdot}(\Delta G_{j,k}) + F^i_{j,k}\Delta g^i_{j,k} + H^i_{j,k}\Delta G^{i+1}_{k,\cdot}(\Delta G_{j,k})$$

Where the row vectors A through H are given by the gradient of the Euler Equation and the notation $x_{j,\cdot}$ is the column vector of coefficients produced by fixing current state j and stacking the values for $k = 1, ..., N_z$. The market clearing conditions are given by:

$$0 = \sum_i \Delta G^i_{j,k}(\Delta S)$$

In order for the linearized Euler Equations and market clearing conditions to hold for any ΔS we impose conditions on each term from the aggregates. This gives exactly the same number of coefficients as equations. The systems are stacked and an iterative procedure is used to find the solution.