Chaos in the Cobweb Model with a New Learning Dynamic

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Abstract

The new learning dynamic of Brown, von Neuman and Nash (1950) is introduced to macroeconomic dynamics via the cobweb model to describe switching between rational and naive forecasting strategies. This dynamic has appealing properities such as positive correlation and inventiveness. There is persistent heterogeneity in the forecasts and chaotic behavior with bifurcations between periodic orbits and strange attractors for the same range of parameter values as in previous studies. Unlike Brock and Hommes (1997), however, there exist intuitively appealing steady states where one strategy dominates, and there are qualitative differences in the resulting dynamics of the two approaches. There are similar bifurcations in a parameter that represents how aggressively agents switch to better performing strategies. When agents are sufficently aggressive, there is minimal variation in the price.

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1 Introduction

While modeling with representative agents will be a dominant paradigm in macroeconomics for some time, this approach involves very strong assumptions. In some models, fixed portions of a population have different strategies or information available to them, but such methods do not permit agents to change their mind or use new sources of data. There are a number of papers that do allow for dynamic switching of strategies, though many of the particular dynamics are open to criticism.

The cobweb model with rational and naive expectations is one area where the modeling of heterogeneous forecasting strategies has been studied in detail[1]. In particular, chaotic behavior is apparent in this model for certain parameter values under different dynamics describing the evolution of agents' choices of forecasting strategies. The present paper introduces the dynamic of Brown, von Neumann and Nash (1950) to the literature on macroeconomic dynamics within the context of the cobweb model. In comparison to some alternative approaches, this dynamic has appealing properties such as *positive correlation* (Sandholm 2006a), a weak monotonicity condition implying poor performing strategies tend to lose adherents, and *inventiveness* (Weibull 1994), which means agents can adopt good performing strategies that are new to the population. A closely related feature of the dynamic is *Nash stationarity* that implies the Nash equilibria of a static game are steady states under the learning dynamic. Under Brown, von Neumann and Nash (1950) updating of the choice of forecasting strategies, the cobweb model shows chaotic behavior for parameter ranges similar to those in previous studies, though there are qualitative differences in the nature of the dynamical system.

Brock and Hommes (1997) study chaos in the cobweb model with a multinomial logit model of heterogeneous forecasts. In their model there are bifurcations in both the ratio of supply and demand elasticities and the *search intensity* parameter, which determines how aggressively agents switch to better performing strategies. Branch and McGough (2005) analyze the cobweb model using a modification of a replicator dynamic similar to that of Sethi and Franke (1995), who first demonstrated the existence of persistent heterogeneity of forecasting strategies in this model with rational and naive expectations. The model of Branch and McGough (2005) shows bifurcations for parameters analogous to those in Brock and Hommes (1997). Both approaches also demonstrate the existence of strange attractors, where the possible paths of the dynamical system form irregular dense sets.

In the cobweb model where the Brown, von Neumann and Nash (1950) dynamic describes the switching between rational and naive forecasts, there is chaotic behavior given that the ratio of supply and demand elasticities is sufficiently large, as in Brock and Hommes (1997) and Branch and McGough (2005). There are multiple bifurcations between periodic orbits and strange attractors in both this ratio and the parameter

[1] There are multiple studies, surveyed in Hommes (2006), of chaotic behavior in asset pricing models, Brock and Hommes (1998) being an early example.

determining how quickly agents switch strategies, again as in the previous studies.

There are qualitative differences in the dynamics of the model with the Brown, von Neumann and Nash (1950) dynamic, however. This model has steady states where one strategy dominates, which are not feasible in the multinomial logit approach of Brock and Hommes (1997). When there is no difference between the costs of the predictors, there is no reason to use the less accurate naive forecast, and there is a stable steady state where all agents use the rational forecast under the Brown, von Neumann and Nash (1950) dynamic. Furthermore, if there is a cost to using the rational forecast and the market is stable, firms have no incentive to incur the extra cost and there is a stable steady state where all agents use the naive forecast. While both models have chaotic behavior associated with unstable period two cycles, the nature of the bifurcations is different[2].

One can naturally extend the selection dynamic of Brown, von Neumann and Nash (1950) to include varying degrees of switching intensity analogous to varying the search intensity parameter in the multinomial logit dynamic. As in Brock and Hommes (1997), there are multiple bifurcations in the switching intensity parameter, but the behavior for large values of these parameters is different for the two dynamics. Under logit dynamics, the steady state becomes a repeller for large values of the search intensity parameter. In contrast, for sufficiently high switching intensity under the extended Brown, von Neumann and Nash (1950) dynamic, the price shows little deviation from a steady state value.

The paper is organized as follows. Section 2 gives a detailed comparison of the prominent selection dynamics in the literature. Sections 3 and 4 describe the cobweb model and the forecasting strategies. Sections 5 and 6 derive the steady states and study their stability properties for various parameter values, while section 7 demonstrates the chaotic behavior of the model. Section 8 extends the analysis of the dynamic to one with varying switching intensity, and section 9 concludes.

2 Selection Dynamics

The Brown, von Neumann and Nash (1950) dynamic and related *excess payoff dynamics* satisfy the properties of positive correlation and inventiveness while neither of the leading alternative approaches to modeling the evolution of strategy choice meet both criteria, which represent mild conditions for an intuitively appealing, dynamic model of the choice of heterogenous strategies. Positive correlation is a monotonicity condition implying the strategies with higher payoffs tend to gain adherents, and inventiveness means that a strategy with no followers can gain some if it's payoff is higher than the population average. The replicator dynamic

[2] There seem to be some similarities in the dynamics of Branch and McGough (2005) and the present approach, though their model has a discontinuity at the steady state, making analytic results difficult to obtain.

and related imitative dynamics do not satisfy inventiveness, while logit dynamics do not meet the positive correlation condition.

Let $q_{k,t}$ be the fraction of followers and $\pi_{k,t}$ be the payoff of strategy k in time t. A general dynamic Φ_k describes the evolution of $q_{k,t}$ within the simplex $\bar{q} = \left\{ (q_{1,t}, \ldots.., q_{H,t}) \mid \sum_{h=1}^{H} q_{h,t} = 1 \right\}$ according to the vector of payoffs $\pi_t = (\pi_{1,t}, \ldots.., \pi_{H,t})$ such that $q_{k,t+1} = \Phi_k (q_t, \pi_t)$ where $q_t \in \bar{q}$. The population average payoff $\bar{\pi}_t = \sum_{h=1}^{H} q_{h,t} \pi_{h,t}$ determines the excess payoff $\widehat{\pi}_{k,t} = \pi_{k,t} - \bar{\pi}_t$ for strategy k in time t. The discrete time formulation of positive correlation relates the excess payoffs and the change in population shares $\Delta q_{k,t} = q_{k,t+1} - q_{k,t}$.

Definition 1 *The dynamic Φ_k satisfies* ***positive correlation*** *iff* $\sum_{h=1}^{H} \Delta q_{h,t} \widehat{\pi}_{h,t} > 0$ *unless* $\pi_{h,t} = 0$, *for all h.*

Positive correlation requires that the change of the fractions of the population using different strategies is correlated with their payoffs, so strategies with higher payoffs gain adherents on average, ensuring out-of-equilibrium dynamic paths reflect strategic incentives. Note that this is a relatively weak condition, since it is possible that individual strategies with positive excess payoff could lose followers while the aggregate condition in Definition 1 is met.

Definition 2 *The dynamic Φ_k satisfies* ***inventiveness*** *if strategies with positive excess payoff in time t have a positive fraction of followers in time $t+1$.*

Inventiveness guarantees that successful strategies can be introduced into the population, and that extinction of a strategy need not be permanent. We can now study the dynamics from the literature in light of these minimal desiderata.

Discrete time excess payoff dynamics take the form

$$q_{k,t+1} = \frac{q_{k,t} + \sigma(\widehat{\pi}_{k,t})}{1 + \sum_{h=1}^{H} \sigma(\widehat{\pi}_{h,t})} \tag{1}$$

where the choice function $\sigma(\cdot)$ is continuous and $\sigma(\widehat{\pi}_{k,t}) = 0$ for $\widehat{\pi}_{k,t} \leq 0$ and $\sigma(\widehat{\pi}_{k,t}) > 0$ for $\widehat{\pi}_{k,t} > 0$. The simplest choice function with these characteristics is

$$\sigma(\widehat{\pi}_{k,t}) = [\widehat{\pi}_{k,t}]_+ , \tag{2}$$

which, along with (1), yields the Brown, von Neumann and Nash dynamic[3] (BNN). The behavioral founda-

[3] The notation $[\pi]_+$ is equivalent to $\max(0, \pi)$. The conditions on the choice function $\sigma(\cdot)$ are the most general specification

tions for the BNN dynamic have an intuitive description. Agents pick a strategy at random and compare its payoff to the population average. If the payoff of the chosen strategy is above average, they switch strategies with probability proportional to the excess payoff.

Proposition 3 *The excess payoff dynamic (1) satisfies positive correlation and inventiveness.*

That the excess payoff dynamic (1) displays inventiveness is evident by inspection. Any positive excess payoff $\widehat{\pi}_{k,t} > 0$ implies that $\sigma\left(\widehat{\pi}_{k,t}\right) > 0$ and $q_{k,t+1} > 0$. The proof of Proposition 3 for positive correlation is in Appendix A.

The most common alternative dynamic in macroeconomics is the multinomial logit model, see Hommes (2006) for a survey. Using the notation above, the dynamic is given by

$$q_{k,t+1} = \frac{\exp\left(\beta\pi_{k,t}\right)}{\sum\limits_{h=1}^{H}\exp\left(\beta\pi_{h,t}\right)} \tag{3}$$

where β is the search intensity parameter, representing how quickly agents adjust to better strategies. Logit dynamics satisfy inventiveness, in fact, every available strategy has a positive fraction of followers regardless of its payoff, unless the search intensity parameter or payoffs are infinite[4]. However, the logit model does not satisfy positive correlation. A simple counterexample is the case of two strategies where one strategy always has a superior payoff to the other. There exists a fraction of agents using the superior strategy sufficiently large so that, according to (3), some agents adopt the inferior strategy in the next period, for a finite β. Hence, there is a positive change in the fraction using the strategy with a negative excess payoff and a negative $\Delta q_{k,t}$ for the strategy with a positive excess payoff, so the condition for positive correlation in Definition 1 is violated.

There are further difficulties with the implementation and interpretation of the logit dynamic. As the search intensity parameter β approaches infinity, the logit dynamic approximates best reply behavior, ameliorating some of the concerns above. However, this observation shows that this single parameter governs both the speed of adjustment and the minimum fraction of agents using a strategy, for bounded payoffs, making the choice of the correct β problematic for a given application. Furthermore, consider the situation when all payoffs are equal. Under the logit dynamic (3), all strategies would then have an equal share of followers, regardless of the population state in the previous period. How agents coordinate on such an outcome is difficult to interpret.

The most studied dynamic in the evolutionary game theory literature is the replicator, so named for its

found in Sandholm (2006b) who provides a detailed development of excess payoff dynamics and positive correlation in continuous time.

[4] This point is made by Branch and Evans (2007) and Parke and Waters (2006), among others.

roots in biology, though it also has social/behavioral foundations, see Weibull (1997), Schlag (1998) and Binmore and Samuelson (1999). The replicator is a special case of imitative dynamics, which can take the form in discrete time

$$q_{k,t+1} = q_{k,t} \frac{\omega\left(\pi_{k,t}\right)}{\sum\limits_{h=1}^{H} q_{h,t}\omega\left(\pi_{h,t}\right)}. \tag{4}$$

The presence of $q_{k,t}$ on the right hand side shows the role of imitation, as popular strategies tend to remain popular in the next period. The weighting function $\omega(\pi)$ must be positive and increasing. If it is linear then the dynamic (4) corresponds to the replicator, while a convex $\omega(\pi)$ yields convex monotonic dynamics, studied in Hofbauer and Weibull (1996) and Samuelson and Zhang (1992).

Imitative dynamics satisfy positive correlation (Fudenberg and Levine 1998) but not inventiveness. In equation (4), if $q_{k,t} = 0$ then $q_{k,t+1} = 0$, and strategy k has no followers in all future periods. Hence, all the edges of the simplex $\bar{q}$ are steady states for imitative dynamics. Strategies with no followers cannot enter the population, regardless of their payoff performance.

There are alternative, perturbed versions of imitative dynamics that recover inventiveness. Such methods include the introduction of minimum fractions of followers of strategies (Parke and Waters 2006), the inclusion of *drift* where small fractions of agents using different strategies are continually introduced[5] (Binmore and Samuelson 1999) and combining an imitative dynamic with an excess payoff dynamic (Sandholm 2005). All of these approaches are viable if *ad hoc*, though their superiority over simpler excess payoff dynamics is questionable. Sandholm (2006a) also advocates for *pairwise comparison dynamics* that satisfy a *scarcity of data* condition meaning agents are not required to know all the payoffs as in the dynamics above, but, for macroeconomic applications, this condition is not crucial.

Inventiveness is related to the additional criterion of Nash stationarity, which implies that steady states of a dynamic correspond to the Nash equilibria of a game. Though this criterion is defined within static game theory, and the interpretation in macroeconomic dynamics depends on the specific model, the behavior of dynamics in a simple environment has important information about their characteristics. All Nash equilibria are steady states of imitative dynamics, but the converse need not be true, since all the edges of the simplex are steady states. In continuous time, steady states of imitative dynamics satisfying the additional requirement of Lyapunov stability correspond to the Nash equilibria, see Weibull (1997). Nachbar (1990) shows an analogous result for discrete time. For logit dynamics with finite search intensity, no point on the edge and therefore no pure strategy equilibrium can be a steady state, and Nash stationarity is not satisfied. Excess payoff dynamics (1) do satisfy Nash stationarity (Sandholm 2006a), in fact, the discrete time version

[5] Droste, Hommes and Tuinstra (2002) use such an approach with the replicator in their study of a Cournot duopoly and demonstrate the existence of chaotic dynamics. They refer to the continual introduction of small fractions using each strategy as *mutational noise*.

of the BNN dynamic first appears in Nash's (1951) proof of the existence of equilibria. In a continuous time framework, Hofbauer (2000) shows that evolutionarily stable Nash equilibria are steady states of the BNN dynamic.

The BNN dynamic offers a parsimonious model of the evolution of strategy choice with appealing properties not found in other approaches. Inventiveness guarantees that strategies that perform well can enter, while positive correlation ensures that poor performing strategies lose support. However, the dynamic is not above criticism. While the fractions following different strategies is continuous in the payoffs, it may not be globally continuously differentiable due to the construction that eliminates negative excess payoffs. Consequently, application of results from the theory of non-linear dynamical systems could be problematic. For the present model, however, we are able to provide a thorough description of the dynamics.

3 The Cobweb Model with Heterogeneous Forecasts

Firms have heterogeneous forecasts of future prices. Before specifying the different forecasts, realized profits and resulting dynamics, we must describe the firm production decision and the aggregated market. Firm expectations of future prices determine the quantity they produce. Equating the aggregate supply of the firms with demand gives the realized price.

At time t, a firm using strategy k uses forecast of prices $p^e_{k,t+1}$ to generate the profit maximizing quantity Q to produce. The quantity supplied $S\left(p^e_{k,t+1}\right)$ is given by

$$S\left(p^e_{k,t+1}\right) = \arg\max_Q \left\{p^e_{k,t+1}Q - c\left(Q\right)\right\}$$

where $c\left(Q\right)$ is the cost of production. Maximizing yields

$$S\left(p^e_{k,t+1}\right) = \left(c'\right)^{-1}\left(p^e_{k,t+1}\right) \tag{5}$$

assuming an invertible marginal cost function. To allow for heterogeneous forecasts, let $q_{k,t}$ be the fraction of firms using forecasting strategy k. The equilibrium price p_{t+1} is given by equating market demand $D\left(p_{t+1}\right)$ to market supply as follows.

$$D\left(p_{t+1}\right) = \sum_{h=1}^{H} q_{h,t} S\left(p^e_{h,t+1}\right) \tag{6}$$

Firms update their forecasting strategies, switching to those that have performed well in the past, according to the BNN dynamic (1, 2). Payoffs to a strategy are the associated profits adjusted for the cost of using a particular forecasting strategy. Different forecasting strategies k may have different costs C_k, reflecting the idea that some forecasts are more computationally intensive, as discussed in Ben-Sasson, Kalai and Kalai (2006), or require hiring consultants external to the firm. Therefore, the payoff to strategy k is

$$\pi_{k,t} = p_{t+1} S\left(p^e_{k,t+1}\right) - c\left(S\left(p^e_{k,t+1}\right)\right) - C_k \tag{7}$$

where the realized price p_{t+1} is determined by (6). Changes in the fractions of followers $q_{k,t}$ depend on the strategies' excess payoffs $\widehat{\pi}_{k,t}$, the difference between the payoffs and the population average $\bar{\pi}_t$, the expected payoff of a randomly chosen firm. Branch and McGough (2005) and Sethi and Franke (1995) use a simple mean, but the present approach follows the evolutionary game theory literature weighting profits according to popularity to compute the population average.

4 Rational vs. Naive Forecasts

This section specifies a particular version of the model, also studied by Brock and Hommes (1997) and Branch and McGough (2005), specifying two forecasting strategies, rational and naive. In the present non-stochastic environment the rational forecast $p^e_{R,t+1}$ is equivalent to perfect foresight, meaning $p^e_{R,t+1} = p_{t+1}$. Let q_t be the fraction using the rational forecast. The alternative naive forecast $p^e_{N,t+1}$ simply uses the previous periods price p_t as the forecast so $p^e_{N,t+1} = p_t$. Market demand is linear such that

$$D\left(p_{t+1}\right) = A - Bp_{t+1}.$$

Assuming quadratic cost $c\left(Q\right) = \dfrac{Q^2}{2b}$, an individual firm supplies $S\left(p^e_{k,t+1}\right) = bp^e_{k,t+1}$, according to (5), so equating market demand and market supply as in (6) yields

$$A - Bp_{t+1} = q_t b p_{t+1} + \left(1 - q_t\right) b p_t.$$

As in Brock and Hommes (1997), without loss of generalization assume $A = 0$ so that p_t is now the deviation from the zero steady state. Further, let the parameter $\widehat{b}$ be the ratio of supply and demand elasticities such that $\widehat{b} = b/B$, so the equation of motion is

$$p_{t+1} = -\left[\frac{\widehat{b}\left(1 - q_t\right)}{\widehat{b}q_t + 1}\right] p_t. \tag{8}$$

Since $q_t \in [0, 1]$, if $\widehat{b} < 1$ then $[\cdot] < 1$, the mapping (8) is a contraction, and $p_t = 0$ is the unique steady state of the price deviation. However, for $\widehat{b} > 1$ there is the possibility of a non-zero 2-cycle with oscillating price deviations.

The naive forecast is assumed to be costless, but the rational forecast has cost $C \geq 0$ as it has higher informational and computational requirements. Given the realization of the price p_{t+1}, the payoffs to the rational and naive forecasts, $\pi_{R,t}$ and $\pi_{N,t}$ respectively, are

$$\pi_{R,t} = bp_{t+1}^2 - \frac{b}{2}p_{t+1}^2 - C \tag{9}$$

$$\pi_{N,t} = bp_{t+1}p_t - \frac{b}{2}p_t^2. \tag{10}$$

Note that these time t payoffs can be expressed in terms of time t variables using the price dynamics equation (8). For a two strategy game, the excess payoffs of the strategies take a particularly simple form.

$$\widehat{\pi}_{R,t} = (1-q_t)(\pi_{R,t} - \pi_{N,t}) \tag{11}$$

$$\widehat{\pi}_{N,t} = q_t(\pi_{N,t} - \pi_{R,t}). \tag{12}$$

The difference in payoffs between the two strategies is clearly a key expression so we define the following.

Definition 4 *The payoff difference function $\gamma(p_t, q_t)$ is such that $\gamma(p_t, q_t) = \pi_{R,t} - \pi_{N,t}$.*

This function can be calculated using the expressions for the payoffs (9) and (10) and the excess payoffs (11) and (12).

$$\gamma(p_t, q_t) = p_t^2 \left[\frac{B\widehat{b}\left(\widehat{b}+1\right)^2}{2\left(\widehat{b}q_t+1\right)^2} \right] - C \tag{13}$$

The motion of (p_t, q_t) is determined by the payoff difference function, the price dynamics and the BNN learning dynamic. The analysis of the system is complicated by the non-negativity restriction within the BNN dynamic (1, 2), so to clarify the analysis we describe the motion of q_t in the form below using the excess payoffs (11, 12) and the payoff difference function in Definition 4.

$$q_{t+1} = \left\{ \begin{array}{ll} \dfrac{q_t}{1 - q_t\gamma(p_t, q_t)} & \text{for } \gamma(p_t, q_t) \leq 0 \\ \\ 1 - \dfrac{1-q_t}{1 + (1-q_t)\gamma(p_t, q_t)} & \text{for } \gamma(p_t, q_t) > 0 \end{array} \right\} \tag{14}$$

Whether q_t rises of falls depends on whether $\gamma(\cdot)$ is positive or negative.

The definition for the function F describing the motion of (p_t, q_t) follows.

Definition 5 *The evolution function F where $(p_{t+1}, q_{t+1}) = F(p_t, q_t)$ is determined by the price dynamics (8), and the evolution of q_t (14) along with the payoff difference function $\gamma(p_t, q_t)$ in (13).*

5 Steady States

The steady states depend on the ratio of supply and demand elasticities $\widehat{b}$ and the cost of using the rational forecast C. Clearly, the origin in the (p, q) plane is a steady state of the model for any parameter values, but there is also the possibility of a two period cycle for a sufficiently large ratio of supply and demand elasticities where the price (deviation) oscillates around zero. The case with zero cost to the rational forecast has special characteristics.

Proposition 6 *Let the parameters $b, B > 0$.*

i) The point $(p, q) = (0, 0)$ is a steady state of F from Definition 5 for $C > 0$.

ii) If $C = 0$, the set of points where $p = 0$ for any $q \in [0, 1]$ are steady states.

iii) For $C > 0$ and $\widehat{b} > 1$, there is a 2-cycle of F given by

$$(\pm p^*, q^*) = \left(\pm\sqrt{\frac{C}{2b}}, \frac{\widehat{b}-1}{2\widehat{b}}\right).$$

iv) For $C > 0$ and $\widehat{b} = 1$, the set of points such that $q = 0$ and $|p| \leq \sqrt{\frac{C}{2b}}$ are 2-cycles.

Proof. *i)* The value $p = 0$ is a steady state for the price dynamics equation (8). If $p = 0$ then equation (13) shows that $\gamma(0, q) \leq 0$ for $C \geq 0$, which implies that $q = 0$ is a steady state value for equation (14) for the evolution of q_t. Hence, $(p, q) = (0, 0)$ is a steady state.

ii) If $C = 0$ and $p = 0$ then $\gamma(0, q) = 0$ which implies that any value of q is a steady state of (14).

iii) The computation of (p^*, q^*) when $\widehat{b} \geq 1$ is straightforward. The bracketed term in (8) must be one, determining q^*, and the payoff difference must be zero so $\gamma(p^*, q^*) = 0$, which determines p^*.

iv) At $\widehat{b} = 1$ and $q_t = 0$, the bracketed term $[\cdot]$ in the price dynamics equation (8) is one, so p_t can take any value in a 2-cycle. If $|p_t| \leq \sqrt{\frac{C}{2b}}$, then $\gamma(p_t, 0) \leq 0$ and $q_t = 0$ is a steady state for any such p_t. ■

Figure 1 identifies the steady state at the origin and 2-cycle in (p, q) space for $\widehat{b} > 1$ and $C > 0$. The 2-cycle is the two intersections of the curve where $\gamma(p, q) = 0$ and the horizontal line $q = q^*$ derived from (8) in the proof above.

Note that the origin would be a steady state and $(\pm p^*, q^*)$ would be a 2-cycle under imitative dynamics as well, but the point $(0, 1)$ where q_t achieves its maximum would also be a steady state for any parameter values. Imitative dynamics lack inventiveness so if there are no agents using the naive forecasting strategy, none can enter. The price dynamics (8) show that all agents using the rational forecast leads to $p = 0$, this implies that the payoff difference $\gamma(p, q) = -C$ is negative meaning the naive forecast has a higher payoff the the rational forecast. Hence, a steady state where $q = 1$ and all agent use the rational forecast is hard to justify for $C > 0$, pointing up the desirability of inventiveness.

One difficulty with using the BNN dynamics (1, 2) is the potential lack of a continuous derivative where the excess payoff of a strategy is zero. The following proposition shows that this is an issue for the present model.

Proposition 7 *The evolution function F from Definition 5 is not continuously differentiable at any point $(\bar{p}, \bar{q})$ such that $\gamma(\bar{p}, \bar{q}) = 0$ unless $\bar{p} = 0$ or $q = \frac{1}{2}$.*

For any $\widehat{b} > 1$, F is not continuously differentiable at (p^*, q^*) or $(-p^*, q^*)$ from iii) of Proposition 6.

Proof. See appendix B ■

For analysis of games with a fixed payoff matrix, differentiability is not an issue since the payoffs do not depend on the endogenous variables, but for many applications in macroeconomic dynamics, this will not be true. A primary analytic contribution of this paper is to characterize the stability of the 2-cycle $(\pm p^*, q^*)$. The usual method would involve studying the eigenvalues of the Jacobian of the second iterate of the evolution function F, but there are two technical hurdles to overcome in the present model. First, F is piecewise smooth, and the matrix of partial derivatives is different for a neighborhood of (p^*, q^*) to the right and left of the curve $\gamma(p_t, q_t) = 0$ (Figure 1). Furthermore, the 2-cycle is unstable for reasonable parameter values so it is necessary to show the stability of an inverse. The following function exploits the symmetry of F around $p = 0$ to aid the analysis.

Definition 8 *The evolution function $\widehat{F}$ where $(p_{t+1}, q_{t+1}) = \widehat{F}(p_t, q_t)$ is determined by the price dynamics equation*

$$p_{t+1} = \left[\frac{\widehat{b}(1 - q_t)}{\widehat{b}q_t + 1}\right] p_t, \tag{15}$$

and the evolution of q_t (14) along with the payoff difference function $\gamma(p_t, q_t)$ in (13).

The function $\widehat{F}$ is identical to F, except that negative price deviations are reflected to positive values, assuming a positive starting value $p_0 > 0$. Since p_t enters as a squared term in the equations shared by F and $\widehat{F}$, the evolution of (p_t, q_t) is otherwise identical. Furthermore, the 2-cycle $(\pm p^*, q^*)$ under F from *iii)* of Propostion 6 becomes a steady state (p^*, q^*) under $\widehat{F}$, and the dynamics of the second iterates of both functions are identical. The dynamics of $\widehat{F}$ are represented in Figure 1, restricting attention to positive or negative p, depending on the starting value p_0. The equations in the above definition determine the motion shown in the figure. For example, for the case of (p_t, q_t) such that $q_t > q^*$ and $\gamma(p_t, q_t) > 0$, the bracket term in the price dynamics equation for $\widehat{F}$ (15) is less than one so $p_{t+1} < p_t$, and the second equation for the evolution of q_t (14) shows that $q_{t+1} > q_t$. Similar arguments can be made for the other four cases determining the counter-clockwise rotation around (p^*, q^*).

6 Stability

As with other studies of the cobweb model, the ratio of supply and demand elasticities $\widehat{b}$ is a key parameter determining the stability of the steady state at the origin and the 2-cycle $(\pm p^*, q^*)$ identified in Proposition 6. Assuming $C > 0$, for $\widehat{b} < 1$, the origin is the unique steady state, and it is stable, but if $\widehat{b} > 1$, the origin and the 2-cycle are both unstable. First, we formally define the appropriate stability concepts, see Lakshmikantham and Trigiante (2002) among others.

Definition 9 *For evolution function F, a steady state $x^* \in \boldsymbol{R}^n$ where $F(x^*) = x^*$ is* ***stable*** *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, if x is such that $\|x - x^*\| < \delta$, then $\|F^m(x) - x^*\| < \varepsilon$ for any positive integer m, where $\|\cdot\|$ is the Euclidian norm. A steady state x^* is* ***unstable*** *if it is not stable.*

A steady state $x^* \in \boldsymbol{R}^n$ is ***asymptotically stable (unstable)*** *if $F^k(x_t) \to x^*$ as $k \to \infty$ $(k \to -\infty)$.*

These definitions apply to a cycle of period m with periodic point y^* where $F^m(y^*) = y^*$ if the conditions are satisfied for the evolution function F^m.

Consider the role of the cost to the rational forecast C. If $C = 0$, there is no reason to use the naive forecast since the rational forecast is more accurate and there is no difference in costs. For a positive cost, however, the situation may change in favor of the naive forecast. The analysis of the stability of the steady states formalizes these notions.

Let $C = 0$, so any point where $p_t = 0$ is a steady state according to case *ii)* of Proposition 6, since $\gamma(p_t, q_t) = 0$, but only the point $(p_t, q_t) = (0, 1)$ is stable. For any $p_t > 0$, the payoff difference $\gamma(p_t, q_t)$ is positive so q_t will rise. Hence, any point such that $p_t = 0$ is not stable unless q_t is at its maximum so the only stable point is where all agents adopt the rational forecast.

If $C > 0$, however, the situation is quite different since $p_t = 0$ means that $\gamma(p_t, q_t)$ is negative, leading agents to adopt the naive forecast, so the origin is a steady state, as noted in
Proposition 6. The stability of these steady states is summarized below.

Proposition 10 *i) If $C = 0$, the evolution function F has a unique stable steady state at $(p_t, q_t) = (0, 1)$.*

ii) If $C > 0$, the steady state at $(p_t, q_t) = (0, 0)$ is stable iff $\widehat{b} \le 1$.

Proof. $i)$ For $C = 0$, see discussion above[6].

ii) For $C > 0$, the Jacobian around $(0,0)$ can be computed using the price dynamics equation (8) and the $\gamma(p_t, q_t) < 0$ case in (14) to find $J_{(0,0)} = \begin{pmatrix} -\widehat{b} & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the steady state is non-hyperbolic[7] with eigenvalues $-\widehat{b}$ and 1, so $(0,0)$ is a stable steady state for $\widehat{b} \leq 1$ and unstable for $\widehat{b} > 1$. ■

The stable steady state at $(p_t, q_t) = (0,1)$ when $C = 0$ is sensible, though it is at odds with the multinomial logit treatments of the model. If there is no cost to the rational forecast, there is no reason to use the less accurate naive forecast, but in Brock and Hommes (1997), there is a stable steady state where the population splits evenly between the two forecasting strategies when $C = 0$. Under the multinomial logit dynamic, the naive forecast is not driven out even though it is inferior in this case, an example of the dynamic's failure to satisfy positive correlation.

With a positive cost to the rational forecast, the stability of the origin depends on the ratio of supply and demand elasticities $\widehat{b}$. Since the origin is non-hyperbolic when $\widehat{b} \leq 1$ and $C > 0$, case *ii)* of Proposition 6, it is stable but not asymptotically stable in the sense of Definition 9.

Again, the stability of the origin is an intuitive outcome that differs from that of the multinomial logit approach in Brock and Hommes (1997). In a stable market $\left(\widehat{b} < 1\right)$, firms have no incentive to incur the cost of using the rational forecast, so that strategy is driven from the population under the BNN dynamic, but there is no steady state where $q = 0$ under multinomial logit for finite search intensity[8].

If $\widehat{b} > 1$, unless $p_t = 0$, the dynamic F takes the system away from the origin. As $\widehat{b}$ rises above one, the 2-cycle $(\pm p^*, q^*)$ of F (steady state of $\widehat{F}$) appears, and this case is the primary focus of the discussion of the dynamics in the next section. The stability analysis is complicated by the lack of differentiability of F and $\widehat{F}$ along the curve defined by $\gamma(p_t, q_t) = 0$, but the Jacobians can be computed for the sets to the left and right of the curve.

Proposition 11 *Let $\widehat{JL}$ and $\widehat{JR}$ be the Jacobians of $\widehat{F}$ around its steady state (p^*, q^*) for $\gamma(p_t, q_t) < 0$ and $\gamma(p_t, q_t) > 0$, respectively. For any $\widehat{b} > 1$ and $0 < C < 4$, the eigenvalues of $\widehat{JL}$ and $\widehat{JR}$ are complex with modulus greater than one.*

Proof. See Appendix C. ■

The condition on C in the above propositions is not particularly restrictive considering the parameterizations in the literature and could be relaxed for larger values of $\widehat{b}$. The proof gives more refined conditions involving both $\widehat{b}$ and C.

[6] This case can be analyzed algebraically. The steady state $(0,1)$ is non-hyperbolic and the Jacobian has eigenvalues 1 and -1. The discussion preceding the proposition gives more insight about the cases where $q_t < 1$.

[7] A non-hyperbolic point has a Jacobian with at least one eigenvalue with unit magnitude, see Kuznetzov (2004).

[8] There is such a steady state for infinite search intensity, though it is not locally stable.

Surprisingly, the condition on the eigenvalues does not ensure that the steady state is unstable. Intuitively, the Jacobians might interact in a perverse way, allowing a dynamic path that converges to the steady state. Lakshmikantham and Trigiante (2002, section 4.5) provide a counterexample involving a difference equation that periodically switches between two linear functions. In spite of this complication, a continuous, piecewise quadratic Lyapunov function can be constructed showing that the steady state (p^*, q^*) is asymptotically unstable under the parameter conditions in the above proposition.

Proposition 12 *For any $\widehat{b} > 1$ and $0 < C < 4$, the point (p^*, q^*) is an asymptotically unstable steady state of $\widehat{F}$, and so the pair $(\pm p^*, q^*)$ is an asymptotically unstable 2-cycle of F.*

Proof. See Appendix D. ■

Locally, the evolution function F leads dynamic paths away from $(\pm p^*, q^*)$, but the phase plot in Figure 1 shows that the price deviations remain bounded. For example, as the price deviation become positive and large, $\gamma(p_t, q_t)$ becomes positive and q_t increases, but that in turn leads to a p_t lower in magnitude.

In the non-generic case *iv*) of Proposition 6 when $\widehat{b} = 1$, there is a set of stable points while the steady state on the curve $\gamma(p_t, q_t) = 0$ is neither stable nor unstable.

Proposition 13 *For $\widehat{b} = 1$ and $C > 0$, the set of points such that $q = 0$ and $|p| < \sqrt{\frac{C}{2b}}$ are stable 2-cycles of F. The 2-cylce $(\pm p^*, q^*) = \left(\pm\sqrt{\frac{C}{2b}}, 0\right)$ is neither stable nor unstable.*

Proof. See Appendix E. ■

Dynamic paths starting from a neighborhood of $(p^*, q^*) = \left(\sqrt{\frac{C}{2b}}, 0\right)$ could have increasing $|p_t|$ and q_t for a number of periods while $\gamma(p_t, q_t) > 0$, but q_t eventually approaches zero when $\widehat{b} = 1$.

7 Chaos

Chaos exists for this model given parameter restrictions similar to previous studies, though there are important differences in the resulting dynamics. For a positive cost to the rational forecast $C > 0$ and a ratio of the demand and supply elasticities greater than one $\widehat{b} > 1$, the system shows chaotic behavior with orbits around the asymptotically unstable 2-cycle. There are multiple bifurcations in $\widehat{b}$ and the orbits can be cycles of various periods or aperiodic strange attractors for different values of $\widehat{b}$.

The analytic results given in the propositions and proofs so far are suggestive of chaotic dynamics for varying $\widehat{b}$. At $\widehat{b} = 1$ the Jacobian $JL_{\widehat{F}}$ of $\widehat{F}$ for $\gamma(p_t, q_t) < 0$ has two real eigenvalues equal to one, but as $\widehat{b}$ rises above one, the eigenvalues become complex conjugates with modulus greater than one. Hence, the same is true of $JL^2_{\widehat{F}}$, which is relevant for the dynamics of F^2, yielding the following observations.

- The 2-cycle is non-hyperbolic at $\widehat{b}=1$, but hyperbolic for $\widehat{b}>1$, suggesting the presence of a bifurcation.

- As $\widehat{b}$ exceeds one, the eigenvalues of $JL_{\widehat{F}}$ are complex and rise above one in modulus, as with a Hopf bifurcation (Devaney 1986).

- The eigenvalues of $JL_{\widehat{F}}$ pass through a double root of one, corresponding to a fold bifurcation in the plane (Kuznetsov 2004).

The above statements are not rigorous, since the eigenvalues the Jacobian $JR_{\widehat{F}}$ of $\widehat{F}$ for $\gamma(p_t, q_t) < 0$ are complex conjugates with modulus greater than one for any $\widehat{b} > 0$ and $C < 4$ (see Appendix C). However, piecewise smooth dynamical systems[9] often behave similarly to related smooth systems, and the simulation results do not contradict any of these observations.

There is chaotic behavior in the case of the 2-cycle in the model of Brock and Hommes (1997) though the dynamics are qualitatively different from the model in the present work and depend on the search intensity in their model with multinomial logit. For low search intensity, the 2-cycle is stable but at higher values it is an unstable saddle with eigenvalues both greater and less than one in modulus, which never arises under the BNN dynamic. There is a secondary bifurcation for high search intensity with two eigenvalues at negative one. So for high search intensity, the 2-cycle is asymptotically unstable as with $(\pm p^*, q^*)$ from Proposition 6, though the eigenvalues of $JL^2_{\widehat{F}}$ for the bifurcation at $\widehat{b}=1$ are both positive one, demonstrating a qualitative difference in the dynamics of the two models.

Simulations of the system given by the evolution function F in Definition 5 confirm the existence of a bifurcation at $\widehat{b}=1$ and cycling behavior for $\widehat{b}>1$. There are further bifurcations in $\widehat{b}$ between cycles of different periods and aperiodic behavior as well. In the aperiodic cases, the orbits form irregular dense sets that are typical of strange attractors.

Figures 2a and 2b show the observed values[10] of q_t and p_t, respectively, for $\widehat{b} \in (0.9, 3.25)$ and demonstrate the bifurcation at $\widehat{b}=1$ and chaotic behavior for $\widehat{b}>1$. The other parameters B and C are all set to unity[11] so the condition on C in Propositions 11 and 12 is satisfied. For $\widehat{b}<1$, the steady state at $q_t = 0$ is stable, but for $\widehat{b}>1$ it is asymptotically unstable, and q_t achieves a wide range of values. The bifurcation at $\widehat{b}=1$ is even more dramatic for p_t shown in Figure 2b. For $\widehat{b}<1$, the price deviation remains at zero, but as $\widehat{b}$ rises above unity, the price deviation varies across a range from -0.5 to 0.5. Further bifurcations appear for higher values of $\widehat{b}$. In Figures 2a and 2b, for some values of $\widehat{b}$, the possible values of q_t and p_t are sparse

[9] There is a literature on *border-collision* bifurcations (see di Bernardo, Budd, Champneys and Kowalczyk 2008) that studies dynamics at the interestion of the set where the evolution function is not smooth and the set of fixed points for some varying parameter, but in the present model these sets coincide. To my knowledge, this case has not been analyzed in detail.

[10] In all of the simulation results presented here, the system is allowed to run for 2000 periods before values are recorded. None of the results presented here are sensitive to changes in the initial conditions unless noted. The phase plots and and bifurcation diagrams, Figures 2-5, 8 and 10, come from simulations of length 100,000.

[11] The parameters B and C remain one throughout the simultations, which implies $\widehat{b}=b$.

and the orbits are periodic. There are cases of typical period doubling bifurcations as $\widehat{b}$ increases. For some values, the orbits appear to be aperiodic and the values of (p_t, q_t) that the system achieves form a dense set in the plane. For orbits for a number of values of $\widehat{b}$, there were no cycles of periods less than 10^6, so for any potential application they would be considered aperiodic.

Figures 3-5 show phase plots in the (p, q) plane for particular values of $\widehat{b}$. The phase plot in Figure 3 for $\widehat{b} = 1.56$ shows period 30 cycles, while Figure 4 for the slightly higher $\widehat{b} = 1.573$ shows cycles of period 60, demonstrating a period doubling bifurcation[12].

The plot in Figure 5 makes a case for the existence of strange attractors. This phase plot for $\widehat{b} = 1.7$ forms an irregular shape surrounding the repelling steady state. Close inspection reveals multiple curves folding back on the themselves typical of strange attractors, for example see Gluckenheimer and Holmes' (1983) discussion of Duffing's equation.

The plots of the largest Lyapunov exponents[13] for varying $\widehat{b}$ in Figure 6 confirm that these dense sets are strange attractors. The largest Lyapunov exponent measures the growth rate of small deviations from an orbit, and values above zero are evidence of strange attractors. Comparing the bifurcation graph in Figure 2 and the largest Lyapunov exponent graph in Figure 6 shows that values of $\widehat{b}$ leading to periodic orbits correspond to negative exponents, while values of $\widehat{b}$ that lead to dense sets correspond to a largest Lyapunov exponent greater than zero, solidifying the case for the presence of strange attractors.

The plots in Figure 7 show typical paths for the price deviations and the fraction of agents using the rational forecast. They show sample time series for p_t and q_t in the case of $\widehat{b} = 1.56$, corresponding to the periodic phase plot in Figure 3. Time series graphs for other values of $\widehat{b}$ are quite similar, even for the cases with strange attractors, and visual inspection reveals very little qualitative difference. The strange attractors appear to have cycles of similar periods, and only close examination of the data shows slightly larger fluctuations and deviations from periodic behavior. A paper by de Vilder (1996) argues that only strange attractors can explain the variation seen in economic data. The analysis here suggests that the existence of strange attractor may not explain fully such variation either. Of course, this observation does not rule out chaotic behavior as an explanatory factor for economic variation, but that chaos may be only a partial explanation and other sources of variability such as exogenous shocks are also required. Preliminary simulations show that the model with exogenous shocks with parameters near bifurcations can produce interesting time series data. Another possible explanation of more complicated behavior is the switching between coexisting attractors as studied in Brock and Hommes (1997) and is possible in the present model. Naturally, these ideas need to be tested in more sophisticated economic models.

[12] There are different coexisting attractors of the same period for different starting values.

[13] Graphs of largest Lyupanov exponents were made using *E&F Chaos* software package. See Diks, Hommes, Panchenko and van der Weide (2007) for details.

8 Bifurcations with varying switching intensity

The BNN learning rule can be naturally extended to study dynamics under varying speeds of adjustment, similar to other dynamics. Logit rules can have different search intensity parameters as in Brock and Hommes (1997). Many of their results require the search intensity to be above a certain level for chaotic behavior to occur. Imitative dynamics such as the replicator can be similarly extended as in Hofbauer and Weibull (1997), who develop a particular example of convex monotonic dynamics with an exponential weighting function $\omega(\widehat{\pi}_{k,t}) = \exp(\delta\widehat{\pi}_{k,t})$ in (4), where δ parameterizes switching intensity. Parke and Waters (2007) use a version of this approach to study heterogeneous expectations with applications to asset pricing. The analogous extension of the model in this work shows multiple bifurcations in the switching intensity parameter, like the model of Brock and Hommes (1997), but the behavior for high values of the adjustment parameter varies considerably between the two approaches.

The $\alpha-$BNN dynamics with switching intensity parameter $\alpha > 0$ are defined setting the choice function to be $\sigma(\widehat{\pi}_{k,t}) = \left([\widehat{\pi}_{k,t}]_+\right)^\alpha$ in the dynamic (1) to get

$$q_{j,t+1} = \frac{q_{j,t} + \left([\pi_{j,t} - \overline{\pi}]_+\right)^\alpha}{1 + \sum_{h=1}^{H}\left([\pi_{h,t} - \overline{\pi}]_+\right)^\alpha}, \tag{16}$$

which was introduced by Weibull[14] (1994) and is discussed by Hofbauer (2000) and Sandholm (2006c). The α-BNN dynamic retains the appealing features of excess payoff dynamics including positive correlation and inventiveness (Proposition 3). For static games, the asymptotic behavior of (16) is identical to the original BNN dynamic (1 and 2) in that it retains Nash stationarity. However, for higher α, agents are switching more aggressively to strategies with superior payoffs[15].

Simulation results for the above cobweb model with $\alpha-$BNN dynamics, given by equations (8), (11), (12) and (16), show multiple bifurcations in α between periodic and strange attractors for a wide range of $\alpha > 0$. However, for sufficiently large α, the variation in the price deviation virtually disappears. Figures 8a and 8b show the values of p_t and q_t achieved in the simulations for different levels of α with $\widehat{b} = 1.5$. The graph of the price deviation in Figure 8a has a sufficiently small scale to show the alternation between solid sections corresponding to strange attractors, similar to the behavior represented in the phaseplot in Figure 5, and sparse sections corresponding to periodic behavior similar to the phase plot in Figure 3. The graph of the largest Lyapunov exponents in Figure 9 again show positive values for settings of α corresponding to dense

[14] Part of this paper is in the Nobel seminar 1994.

[15] An alternative approach would introduce a scaling parameter multiplicatively with the excess payoff terms. The analytic results for the model with $\alpha-$BNN dynamics involve a number of subtleties and are beyond the scope of the present work. Note, however, that the derivative with respect to the payoffs is continuous for $\alpha \geq 1$.

sets (strange attractors) in Figure 8a and negative values for settings corresponding to periodic dynamics. Figure 10 shows a magnified phase plot for parameters settings $\alpha = 1.15$ and $\widehat{b} = 1.5$ demonstrating a clear example of a strange attractor.

As α increases the range of values for q_t initially decreases, and for values of $\alpha \geq 5.17$, p_t remains very close to zero in the bifurcation diagram in Figure 8a. Figure 8b has a larger scale to show the variation in q_t at values of alpha where p_t is fixed. As α increases further, the variation in q_t decreases and it converges to one half. For very large α, the cost of the rational forecast C becomes relatively insignificant to the payoffs and the choice of forecast makes little difference. The value that q_t approaches as α becomes large depends on the initial conditions and the cost parameter C.

The dramatic decrease in the variation in the price deviation for sufficiently large α is worthy of further investigation. This behavior is in direct contrast to the cobweb model with multinomial logit dynamics in Brock and Hommes (1997) where instability and chaos only appear for a minimum value of the search intensity parameter β and bifurcations between periodic and chaotic behavior continue as β increases[16]. One interpretation is that chaotic behavior is not an automatic feature of the cobweb model with these two predictors but depends on the learning dynamic. Alternatively, the behavior of the $\alpha-$BNN dynamic for very large α appears to be perverse, so the choice of α for applications must be made with care.

9 Conclusion

The learning dynamic of Brown, von Neumann and Nash (1950) has appealing properties that give it great promise for applications in macroeconomic dynamics. Besides tractability, the dynamic is continuous in the payoffs, and has the properties of positive correlation, inventiveness and Nash stationarity, avoiding the problematic behavior near the edges of the simplex for the commonly used multinomial logit or replicator dynamics. Under BNN dynamics, poor performing strategies will have monotonically decreasing fractions of followers, and good performing strategies with no followers can gain some. The lack of a globally continuous derivative will often present a challenge for the analysis of models of macroeconomic dynamics, but the present work shows that it needn't be insurmountable.

The cobweb model presented here shows chaotic dynamics as in Brock and Hommes (1997) and Branch and McGough (2005). In all these papers there are bifurcations between periodic orbits and strange attractors in the ratio of supply and demand elasticities and a switching intensity parameter. In contrast, the model with the BNN dynamic has steady states where one strategy dominates that have a natural interpretation in terms of the cost of the rational forecast. Furthermore, there are qualitative difference

[16] Branch and McGough (2005) also show increased ranges for price and population fractions for larger values of the speed of adjustment parameter in their model.

in the resulting dynamics of the different approaches, so the choice of model of heterogeneous forecasts is non-trivial.

Models showing chaos are serious candidates for explaining economic fluctuations. Modeling switching between heterogenous strategies is a field that offers rich possibilities for future work. The learning dynamic of Brown, von Neumann and Nash (1950) promises to make contributions in both areas.

Appendix A

This appendix presents the proof of Proposition 3.

Proof. Using the specification of the excess payoff dynamic (1) to compute $\Delta q_{k,t}$ the condition for positive correlation in Definition 1 becomes

$$\sum_{k=1}^{K} \pi_{k,t} \left(\frac{\sigma\left(\widehat{\pi}_{k,t}\right) - q_{k,t} \sum\limits_{h=1}^{H} \sigma\left(\widehat{\pi}_{h,t}\right)}{1 + \sum\limits_{h=1}^{H} \sigma\left(\widehat{\pi}_{h,t}\right)} \right) > 0,$$

which is equivalent to

$$\left(1 + \sum_{h=1}^{H} \sigma\left(\widehat{\pi}_{h,t}\right)\right)^{-1} \sum_{k=1}^{K} \left[\left(\pi_{k,t}\right) \sigma\left(\widehat{\pi}_{k,t}\right) - \bar{\pi}_t \sum_{h=1}^{H} \sigma\left(\widehat{\pi}_{h,t}\right) \right] > 0,$$

where $\bar{\pi}_t$ is the population average payoff. The second term in $[\cdot]$ does not depend in k so using a change of index and the specification of the excess payoff $\widehat{\pi}_{k,t} = \pi_{k,t} - \bar{\pi}_t$ yields the following.

$$\left(1 + \sum_{h=1}^{H} \sigma\left(\widehat{\pi}_{h,t}\right)\right)^{-1} \sum_{k=1}^{K} \widehat{\pi}_{k,t} \sigma\left(\widehat{\pi}_{k,t}\right) > 0$$

Since the choice function $\sigma\left(\widehat{\pi}_{k,t}\right)$ is positive for $\widehat{\pi}_{k,t} > 0$ and zero otherwise, the above condition is true unless $\widehat{\pi}_{k,t} = 0$ for all k. ∎

Appendix B

This appendix presents the proof of Proposition 7..

Proof. The dynamic F from Definition 5 is continuously differentiable at $(\bar{p}, \bar{q})$ such that $\gamma\left(\bar{p}, \bar{q}\right) = 0$ if the partial derivatives are the same for the $\gamma\left(p_t, q_t\right) \leq 0$ and the $\gamma\left(p_t, q_t\right) > 0$ cases in the specification for the evolution of q_t (14). The terms $\frac{dp_{t+1}}{dp_t}$ and $\frac{dp_{t+1}}{dq_t}$ will be the same since only the price dynamics equation (8) is involved. Differentiation of the $\gamma\left(p_t, q_t\right) < 0$ case of (14) with respect to p_t yields the following.

$$\frac{dq_{t+1}}{dp_t} = \left(1 - q_t \gamma\left(p_t, q_t\right)\right)^{-2} \left(q_t^2 \frac{d}{dp_t} \gamma\left(p_t, q_t\right) \right)$$

At $(\bar{p}, \bar{q})$ such that $\gamma(\bar{p}, \bar{q}) = 0$, so we have

$$\frac{dq_{t+1}}{dp_t} = \bar{q}^2 \frac{d}{dp_t} \gamma(\bar{p}, \bar{q}) \qquad \text{for } \gamma(p_t, q_t) \leq 0.$$

By similar reasoning, we have

$$\frac{dq_{t+1}}{dp_t} = (1 - \bar{q})^2 \frac{d}{dp_t} \gamma(\bar{p}, \bar{q}) \qquad \text{for } \gamma(p_t, q_t) > 0.$$

Differentiating the two cases of (14) with respect to q_t yields these two relations.

$$\begin{aligned} \frac{dq_{t+1}}{dq_t} &= 1 + \bar{q}^2 \frac{d}{dq_t} \gamma(\bar{p}, \bar{q}) \qquad \text{for } \gamma(p_t, q_t) \leq 0 \\ \frac{dq_{t+1}}{dq_t} &= 1 + (1 - \bar{q})^2 \frac{d}{dq_t} \gamma(\bar{p}, \bar{q}) \qquad \text{for } \gamma(p_t, q_t) > 0 \end{aligned}$$

For $\frac{dq_{t+1}}{dp_t}$ to be the same for positive and negative $\gamma(p_t, q_t)$ requires either $\bar{p} = 0$, which is necessary for $\frac{d}{dp_t} \gamma(\bar{p}, \bar{q}) = 0$, or $\bar{q} = \frac{1}{2}$. There is an analogous statement for $\frac{dq_{t+1}}{dq_t}$.

From *iii*) in Proposition 6, note that $p^* > 0$. Furthermore, q^* is strictly increasing in $\widehat{b}$ and as $\widehat{b} \to \infty$, $q^* \to \frac{1}{2}$ so for a finite $\widehat{b}$, $q^* < \frac{1}{2}$. Therefore, F cannot be continuously differentiable at (p^*, q^*) or $(-p^*, q^*)$.

■

Appendix C

This appendix proves Proposition 11.

The Jacobians $\widehat{JL}$ and $\widehat{JR}$ of $\widehat{F}$ for $\gamma(p_t, q_t) < 0$ and $\gamma(p_t, q_t) > 0$, respectively, can be calculated directly using Definition 8 or from the definition of F in Definition 5, multiplying the result by the reflection matrix $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, as in the proof of Theorem 3.3 in Brock and Hommes (1997).

$$\widehat{JL} = \begin{pmatrix} 1 & -\sqrt{\frac{2C}{b}} \left(\frac{2\widehat{b}}{\widehat{b}+1} \right) \\ \sqrt{\frac{bC}{2}} \left(\frac{\widehat{b}-1}{\widehat{b}}. \right)^2 & 1 - \frac{C\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)} \end{pmatrix}$$

Similarly, the Jacobian of $\widehat{F}$ at (p^*, q^*) for $\gamma(p_t, q_t) > 0$ is as follows.

$$\widehat{JR} = \begin{pmatrix} 1 & -\sqrt{\frac{2C}{b}}\left(\frac{2\widehat{b}}{\widehat{b}+1}\right) \\ \sqrt{\frac{bC}{2}}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2 & 1-\frac{C\left(\widehat{b}+1\right)}{\widehat{b}} \end{pmatrix}$$

The proof of Proposition 11 follows.

Proof. Let λ be an eigenvalue of $\widehat{JL}$ and $\widehat{\lambda} = 1-\lambda$. The characteristic polynomial $P(\lambda)$ of $\widehat{JL}$ is

$$P\left(1-\widehat{\lambda}\right) = \widehat{\lambda}^2 - \widehat{\lambda}C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)} + 2C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)}.$$

The discriminant for $P\left(1-\widehat{\lambda}\right) = 0$ is

$$\left(C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)}\right)^2 - 8C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)},$$

which is negative if $C < \frac{8\widehat{b}\left(\widehat{b}+1\right)}{\left(\widehat{b}-1\right)^2}$, so the condition $C < 4$ guarantees complex eigenvalues for any $\widehat{b} > 1$.

For complex roots of $P(\lambda) = 0$, the eigenvalues $\lambda, \bar{\lambda}$ are conjugates that have modulus greater than one if $\lambda\bar{\lambda} > 1$. Computation yields

$$\lambda\bar{\lambda} = \left(1 - C\frac{\left(\widehat{b}-1\right)^2}{2\widehat{b}\left(\widehat{b}+1\right)}\right)^2 + \frac{1}{4}\left(8C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)} - \left(C\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)}\right)^2\right),$$

and so $\lambda\bar{\lambda} > 1$ iff $C\left(\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)}\right) > 0$, which holds if $\widehat{b} > 1$ and $C > 0$.

Similiarly, let λ be an eigenvalue of $\widehat{JR}$ and $\widehat{\lambda} = 1-\lambda$. The characteristic polynomial $P(\lambda)$ of $\widehat{JR}$ is

$$P\left(1-\widehat{\lambda}\right) = \widehat{\lambda}^2 - \widehat{\lambda}C\left(\frac{\widehat{b}+1}{\widehat{b}}\right) + 2C\left(\frac{\widehat{b}+1}{\widehat{b}}\right).$$

The discriminant here is

$$C^2\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2 - 8C\left(\frac{\widehat{b}+1}{\widehat{b}}\right),$$

which is negative if $C < \dfrac{8\widehat{b}}{\widehat{b}+1}$ which holds for $C < 4$ and $\widehat{b} > 1$.

The product of the eigenvalues is

$$\lambda\bar{\lambda} = \left(1 + C\left(\frac{\widehat{b}+1}{2\widehat{b}}\right)\right)^2 + \frac{1}{4}\left(8C\left(\frac{\widehat{b}+1}{\widehat{b}}\right) - \left(C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)\right)^2\right),$$

and so $\lambda\bar{\lambda} > 1$ if $C\left(\frac{\widehat{b}+1}{\widehat{b}}\right) > 0$, which is true if $\widehat{b}, C > 0$. ■

Appendix D

Assume the parameter restrictions in Proposition 11 hold throughout this appendix. The first step in the proof of Proposition 12 is to make the following coordinate transformation.

Definition 14 *Define the space* $(\bar{p}_t, \bar{q}_t)$ *such that* $(\bar{p}_t, \bar{q}_t) = h(p_t, q_t) = (p_t - g(q_t), q_t - q^*)$ *where* $g(q_t)$ *is such that* $\gamma(g(q_t), q_t) = 0$. *The evolution function* $\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t) = (\bar{p}_{t+1}, \bar{q}_{t+1})$ *is determined by* $\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t) = h\left(\widehat{F}\left(h^{-1}(\bar{p}_t, \bar{q}_t)\right)\right)$, *where* $\widehat{F}$ *is given by Definition 8.*

Since p_t and q_t are taken to be non-negative, $h(p_t, q_t)$ is a homeomorphism. The origin is the steady state of $\overline{\widehat{F}}$ associated with the steady state (p^*, q^*) of $\widehat{F}$, and the local dynamics for both are equivalent[17]. Furthermore, the conditions $\gamma(p_t, q_t) \lesseqgtr 0$ are equivalent to $\bar{p}_t \lesseqgtr 0$. Studying the stability of the origin requires computing the following Jacobians $\overline{\widehat{JL}}$ and $\overline{\widehat{JL}}$ of $\overline{\widehat{F}}$ around the origin for $\bar{p}_t < 0$ and $\bar{p}_t > 0$, respectively. The requirement $\gamma(g(q_t), q_t) = 0$ implies that the function $g(q_t)$ has the following form $g(q_t) = \sqrt{\dfrac{2C}{b}\left(\dfrac{\widehat{b}q_t + 1}{\widehat{b}+1}\right)}$. The new evolution function $\overline{\widehat{F}}$ can be written explicitly using the equations from Definition 8, substituting q_t with $\bar{q}_t + q^*$, q_{t+1} with $\bar{q}_{t+1} + q^*$, p_t with $\bar{p}_t + g(\bar{q}_t + q^*)$, p_{t+1} with

[17] See the discussion of *topological conjugacy* in di Bernardo, Budd, Champneys and Kowalczyk (2007) for details. This also guarantees that the eigenvalues of the Jacobians are the same for $\widehat{F}$ and $\overline{\widehat{F}}$, which is verified below.

$\bar{p}_{t+1} + g\left(\bar{q}_{t+1} + q^*\right)$. The following Jacobians can now be computed[18].

$$
\begin{aligned}
\overline{\widehat{JL}} &= \begin{pmatrix} 1 - \dfrac{C\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)} & -\sqrt{\dfrac{2C}{b}}\left(\dfrac{2\widehat{b}}{\widehat{b}+1}\right) \\ \\ \sqrt{\dfrac{bC}{2}}\left(\dfrac{\widehat{b}-1}{\widehat{b}}.\right)^2 & 1 \end{pmatrix} \\
\overline{\widehat{JR}} &= \begin{pmatrix} 1 - \dfrac{C\left(\widehat{b}+1\right)}{\widehat{b}} & -\sqrt{\dfrac{2C}{b}}\left(\dfrac{2\widehat{b}}{\widehat{b}+1}\right) \\ \\ \sqrt{\dfrac{bC}{2}}\left(\dfrac{\widehat{b}+1}{\widehat{b}}\right)^2 & 1 \end{pmatrix}
\end{aligned} \tag{17}
$$

The only change in the Jacobians after the coordinate transformation is the switch between the $(1,1)$ and $(2,2)$ elements of the matrices, so the eigenvalue condition from Proposition 11 still holds for $\overline{\widehat{JL}}$ and $\overline{\widehat{JR}}$.

Lemma 15 *For any $\widehat{b} > 1$ and $0 < C < 4$, the eigenvalues of $\overline{\widehat{JL}}$ and $\overline{\widehat{JR}}$ are complex with modulus greater than one.*

Proof. The characteristic equations for $\overline{\widehat{JL}}$ and $\overline{\widehat{JR}}$ are identical to those for $\widehat{JL}$ and $\widehat{JR}$ in Appendix C so proof is identical to that for Proposition 11. ■

Before postulating a Lyapunov function for $\overline{\overline{F}}$, first note that for a general difference equation $y_{t+1} = Ay_t$, where $y_t \in \mathcal{R}^n$ and A is an nxn matrix with eigenvalues of modulus less then one, the origin is asymptotically stable and there is a Lyapunov function $V(y) = y^T By$ for some symmetric, positive definite B, such that $B - A^T BA$ is positive definite. The key attributes for a Lyapunov function are $V(0) = 0$, $V(y) > 0$ for $y \neq 0$, and $V(y_{t+1}) < V(y_t)$, which is guaranteed by the condition on the matrices. See Lashmikantham and Trigiante (2002, section 4.9) for details.

Since the eigenvalues of $\overline{\widehat{JL}}$ have modulus greater than one, the eigenvalues of $\overline{\widehat{JL}}^{-1}$ have modulus less than one, and there exists a symmetric, positive definite $BL = \begin{pmatrix} bl_{11} & bl_{12} \\ bl_{12} & bl_{22} \end{pmatrix}$ such that $BL - \left(\overline{\widehat{JL}}^{-1}\right)^T BL\left(\overline{\widehat{JL}}^{-1}\right)$ is positive definite. Multiplying on the left by $\overline{\widehat{JL}}^T$ and on the right by $\overline{\widehat{JL}}$ shows that an equivalent condition is for $\overline{\widehat{JL}}^T BL\overline{\widehat{JL}} - BL$ to be positive definite.

Now, for studying the behavior of $\overline{\overline{F}}$ around the origin, let the candidate Lyapunov function be defined

[18] Some care is required since $\bar{p}_{t+1}$ and $\bar{q}_{t+1}$ appear simultaneously in the price evolution equation.

as follows.

$$V\left(\bar{p}_t,\bar{q}_t\right)=\left\{\begin{array}{ll}\left(\bar{p}_t,\bar{q}_t\right)BL\left(\bar{p}_t,\bar{q}_t\right)^T & \text{for } \bar{p}_t\leq 0\\ \left(\bar{p}_t,\bar{q}_t\right)BR\left(\bar{p}_t,\bar{q}_t\right)^T & \text{for } \bar{p}_t>0\end{array}\right\} \tag{18}$$

For $V\left(\bar{p}_t,\bar{q}_t\right)$ to be continuous along $\bar{p}_t=0$, it must be the case that $\left(0,\bar{q}_t\right)BL\left(0,\bar{q}_t\right)^T=\left(0,\bar{q}_t\right)BR\left(0,\bar{q}_t\right)^T$, which requires[19] that $bl_{22}=br_{22}$, where $BR=\begin{pmatrix} br_{11} & br_{12}\\ br_{12} & br_{22}\end{pmatrix}$. The next task is to show the existence of a BR with the attributes of BL described above for an arbitrary $br_{22}>0$. Note that for a symmetric 2x2 matrix B, positive definiteness is equivalent to $b_{11}>0$ and the determinant $|B|>0$. For both of those conditions to hold it must also be true that $b_{22}>0$.

Lemma 16 *Given the parameter restrictions in Proposition 11, for an arbitrary $br_{22}>0$, there exists a symmetric, positive definite BR such that $D=\overline{\widehat{JR}}^T BR\overline{\widehat{JR}}-BR$ is positive definite.*

Proof. Using the specifications for D in the lemma and $\overline{\widehat{JR}}$ from above (17), computation yields

$$\begin{aligned}|D| &= -\frac{8C}{b}\left(\frac{\widehat{b}}{\widehat{b}+1}\right)^2 br_{11}^2-\frac{bC}{2}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^4 br_{22}^2-\left(C^2\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2+8C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)\right)br_{12}^2\\ &\quad +4C\sqrt{\frac{2C}{b}}b_{11}^R b_{12}^R+4C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)br_{11}br_{22}+2C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^3 br_{12}br_{22}\end{aligned}$$

Let br_{11}^* be the value of br_{11} that maximizes $|D|$, given br_{12} and br_{22}. Solving the first order condition $\frac{d\,|D|}{dbr_{11}}=0$ gives

$$br_{11}^*=\frac{b}{4}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2\left[\sqrt{\frac{2C}{b}}br_{12}+\left(\frac{\widehat{b}+1}{\widehat{b}}\right)br_{22}\right].$$

So $br_{11}^*>0$ for any $br_{12},br_{22}>0$.

Furthermore, $|B|=br_{11}br_{22}-br_{12}^2$ so given any $br_{11},br_{22}>0$, there exists $\varepsilon^*>0$ such that for $br_{12}>0$, if $br_{12}<\varepsilon^*$, then $|B|>0$.

The upper left element of D, d_{11} can be computed using $\overline{JR}$ in (17).

$$d_{11}=\left(C^2\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2-2C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)\right)br_{11}+2\sqrt{\frac{bC}{2}}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2\left(1-C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)\right)br_{12}+\frac{bC}{2}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^4 br_{22}$$

Evaluating d_{11} at $br_{11}=br_{11}^*$ yields

$$d_{11}|_{br_{11}=br_{11}^*}=\frac{1}{2}\sqrt{\frac{bC}{2}}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2\left(C^2\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^2-6C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)+4\right)br_{12}+\frac{bC^2}{4}\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^5 br_{22}.$$

[19] The coordinate transformation makes possible this simple requirement for continuity.

Since the coefficient on br_{22} is positive, there exists $\varepsilon^{**} > 0$ such that for $br_{12} > 0$, if $br_{12} < \varepsilon^{**}$, then $d_{11}|_{br_{11}=br_{11}^*} > 0$.

Evaluating $|D|$ at $br_{11} = br_{11}^*$ yields

$$|D|\,|_{br_{11}=br_{11}^*} = -8C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)(br_{12})^2 + 2C\left(\frac{\widehat{b}+1}{\widehat{b}}\right)^3\left(1+\sqrt{\frac{bC}{2}}\right)br_{12}br_{22}.$$

The coefficient on $br_{12}br_{22}$ is positive so, there exists $\varepsilon^{***} > 0$ such that for $br_{12} > 0$, if $br_{12} < \varepsilon^{***}$, then $|D|\,|_{br_{11}=br_{11}^*} > 0$.

Therefore, for any $br_{22} > 0$, taking $br_{11} = br_{11}^*$, there exists a $br_{12} > 0$ such that $br_{12} < \min\{\varepsilon^*, \varepsilon^{**}, \varepsilon^{***}\}$ that guarantees $|B|, d_{11}, |D| > 0$ so both BR and D are positive definite, as required. ■

Therefore, since BR, specified in the lemma above, can be constructed for any $br_{22} = bl_{22} > 0$, there exists a continuous $V(\bar{p}_t, \bar{q}_t)$ where both $\overline{\widehat{JL}}^T BL\overline{\widehat{JL}} - BL$ and $\overline{\widehat{JR}}^T BR\overline{\widehat{JR}} - BR$ are positive definite. Intuitively, the above lemma shows that $V(\bar{p}_t, \bar{q}_t)$ is increasing over time when the linearization of $\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t)$ determines the dynamics. The following lemma formally extends this idea to $\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t)$ within a neighborhood of the origin.

Lemma 17 *There exist BL and BR such that the function $V(p_t, q_t)$ given by (18) has the property that $V\left(\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t)\right) - V(\bar{p}_t, \bar{q}_t) > 0$ for $(\bar{p}_t, \bar{q}_t)$ sufficiently close to the origin.*

Proof. Since $\overline{\widehat{JR}}^T BR\overline{\widehat{JR}} - BR$ is positive definite, from lemma 16, there exists an $\varepsilon > 0$ such that

$$\frac{(\bar{p}_t, \bar{q}_t)\left(\overline{\widehat{JR}}^T BR\overline{\widehat{JR}} - BR\right)(\bar{p}_t, \bar{q}_t)^T}{\|(\bar{p}_t, \bar{q}_t)\|} > \varepsilon,$$

for all $(\bar{p}_t, \bar{q}_t) \neq (0,0)$. Refering to the specification for $V(\bar{p}_t, \bar{q}_t)$ in (18), this condition implies

$$\frac{V\left(\overline{\widehat{JR}}(\bar{p}_t, \bar{q}_t)^T\right) - V(\bar{p}_t, \bar{q}_t)}{\|(\bar{p}_t, \bar{q}_t)\|} > \varepsilon, \tag{19}$$

for $(\bar{p}_t, \bar{q}_t)$ such that $\bar{p}_t > 0$.

Using the definition of the derivative to examine deviations from the origin, again for $\bar{p}_t > 0$, given $\varepsilon' > 0$, there exists a $\delta > 0$ such that for $(\bar{p}_t, \bar{q}_t)$ where $\|(\bar{p}_t, \bar{q}_t)\| < \delta$,

$$\frac{\left\|\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t) - \overline{\widehat{JR}}(\bar{p}_t, \bar{q}_t)^T\right\|}{\|(\bar{p}_t, \bar{q}_t)\|} < \varepsilon'. \tag{20}$$

Further, $V(\bar{p}_t, \bar{q}_t)$ is continuous and bounded on any compact set, so it is Lipschitz continuous, and there exists a $K > 0$ such that

$$\left| V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(\widehat{\overline{JR}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)\right| < K \left\| \widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right) - \widehat{\overline{JR}}\left(\bar{p}_t, \bar{q}_t\right)^T \right\|. \tag{21}$$

Reasoning by contradiction, assume that $V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) \le V\left(\bar{p}_t, \bar{q}_t\right)$. This condition implies

$$\frac{V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(\widehat{\overline{JR}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)}{\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|} + \frac{V\left(\widehat{\overline{JR}}\left(\bar{p}_t, \bar{q}_t\right)^T\right) - V\left(\bar{p}_t, \bar{q}_t\right)}{\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|} \le 0.$$

Applying the condition (19) derived from the positive definiteness of $\widehat{\overline{JR}}^T BR\widehat{\overline{JR}} - BR$ to the inequality above implies that

$$\frac{V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(\widehat{\overline{JR}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)}{\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|} < -\varepsilon$$

and so

$$\left| V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(JR_{\widehat{\overline{F}}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)\right| > \varepsilon \left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|. \tag{22}$$

However, the Lipschitz continuity condition (21) with the derivative condition (20) implies

$$\left| V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(JR_{\widehat{\overline{F}}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)\right| < K\varepsilon' \left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|$$

for $\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|$ sufficiently small. Choose $\varepsilon' > 0$ so that $\varepsilon' < \varepsilon/K$, so there exists a $\delta > 0$ such that for $\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\| < \delta$,

$$\left| V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(JR_{\widehat{\overline{F}}}\left(\bar{p}_t, \bar{q}_t\right)^T\right)\right| < \varepsilon \left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|,$$

which contradicts (22). Therefore, for $\left\|\left(\bar{p}_t, \bar{q}_t\right)\right\|$ sufficiently small, the assumption above is false, and it must be true that $V\left(\widehat{\overline{F}}\left(\bar{p}_t, \bar{q}_t\right)\right) - V\left(\bar{p}_t, \bar{q}_t\right) > 0$.

So far, the proof applies only to point to the right of the axis $\bar{p}_t = 0$. However, exactly the same argument can be made for $\left(\bar{p}_t, \bar{q}_t\right)$ such that $\bar{p}_t < 0$ using $JL_{\widehat{\overline{F}}}$. Furthermore, since V and $\widehat{\overline{F}}$ are continuous, the condition in the lemma is true for points where $\bar{p}_t = 0$ as well. ■

To show asymptotic instability, it is necessary to examine the inverse function $\widehat{\overline{F}}^{-1}$. The following results show that the inverse is well-defined and continuous.

Clarke (1976) defines the *generalized gradient* for a piecewise differentiable function as the convex hull of the Jacobian matrices that exist. For the purposes of this paper, the following is sufficient.

Definition 18 *The* ***generalized gradient*** $\mathcal{J}$ *of* $\widehat{\overline{F}}$ *is the set of matrices* $\tau\overline{JL} + (1-\tau)\widehat{\overline{JR}}$ *where* $\tau \in [0, 1]$.

His central result follows.

Proposition 19 *(F. H. Clarke 1976) If a function is Lipschitz continuous on a neighborhood of $x_0 \in \mathcal{R}^n$ and its generalized gradient around x_0 has maximal rank, then the function has a Lipschitz continuous inverse in a neighborhood around x_0.*

The function $\overline{\widehat{F}}$ is continuous and bounded on any compact set so it must be Lipschitz continuous. Proposition 19 thereby applies when $\mathcal{J}$ is non-singular for all $\tau \in [0,1]$.

Lemma 20 *For $\widehat{b}, C > 0$, $\mathcal{J}$ is non-singular and has full rank for all $\tau \in [0,1]$.*

Proof. Computation of the determinant of $\mathcal{J}$ given by Definition 18 yields using (17) the following.

$$|\mathcal{J}| = 1 + C\left[\tau\frac{\left(\widehat{b}-1\right)^2}{\widehat{b}\left(\widehat{b}+1\right)} + (1-\tau)\left(\frac{\widehat{b}+1}{\widehat{b}}\right)\right]$$

Hence, $|\mathcal{J}| > 0$ and is non-singular for any $\widehat{b}, C > 0$. ■

Now, we can proceed to the proof of Proposition 12

Proof. Lemma 17 guarantees that there exists a $V(\cdot)$ such that $V\left(\overline{\widehat{F}}(\bar{p}_t, \bar{q}_t)\right) - V(\bar{p}_t, \bar{q}_t) > 0$ in a neighborhood of the origin, and Proposition 19 and Lemma 20 show that $\overline{\widehat{F}}$ has a continuous inverse. Therefore, it is also true that $V\left(\overline{\widehat{F}}^{-1}(\bar{p}_t, \bar{q}_t)\right) - V(\bar{p}_t, \bar{q}_t) < 0$. Clearly, $V(0,0) = 0$ and $V(\bar{p}_t, \bar{q}_t) \neq 0$ for $(\bar{p}_t, \bar{q}_t) \neq 0$. Therefore, there exists a specification of $V(\bar{p}_t, \bar{q}_t)$ given in (18) with BL and BR chosen according to Lemma 16, that is a continuous Lyapunov function. So the origin is an asymptotically stable steady state of $\overline{\widehat{F}}^{-1}$, by Theorem 4.9.1 of Lashmikantham and Trigiante (2002), and therefore an asymptotically unstable steady state of $\overline{\widehat{F}}$.

Further, this means that the point (p^*, q^*) is an asymptotically unstable steady state of $\widehat{F}$ and the pair $(\pm p^*, q^*)$ is an asymptotically unstable 2-cycle of F. ■

Appendix E

The proof of Proposition 12 follows.

Proof. The Jacobian of $\widehat{F}$ around any (p,q) where $q = 0$ and $0 < p < \sqrt{\frac{C}{2b}}$ is a lower triangular matrix with ones on the diagonal given by $\widehat{JL}$ where $\widehat{b} = 1$, so these points are stable. Hence, the associated 2-cycle under F is also stable.

The steady state of $\widehat{F}$, $(p^*, q^*) = \left(\sqrt{\frac{C}{2b}}, 0\right)$ lies on the curve $\gamma(p_t, q_t) = 0$ so both $\widehat{JL}$ and $\widehat{JR}$ (Appendix C) where $\widehat{b} = 1$ are relevant. The eigenvalues of $\widehat{JL}$ are 1 and -1 but the eigenvalues for $\widehat{JR}$ are both greater the one in magnitude, see Appendix C, so this point is neither stable nor unstable. ■

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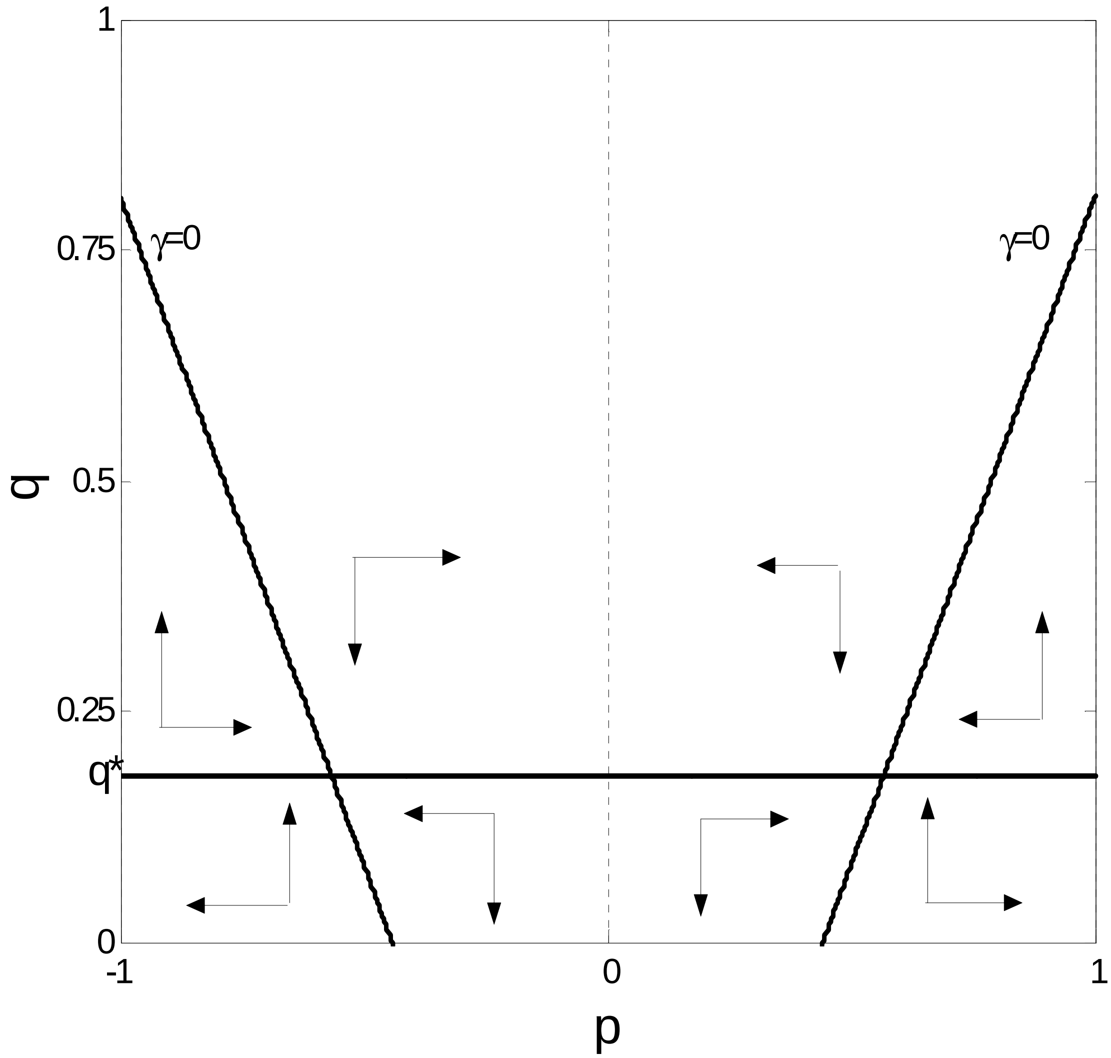
Figure 1



Phasespace for $\widehat{b} = 1.56$. The intersections for $\gamma(p_t, q_t) = 0$ and $q_t = q^*$ show the unstable 2-cycle.

Figure 2a

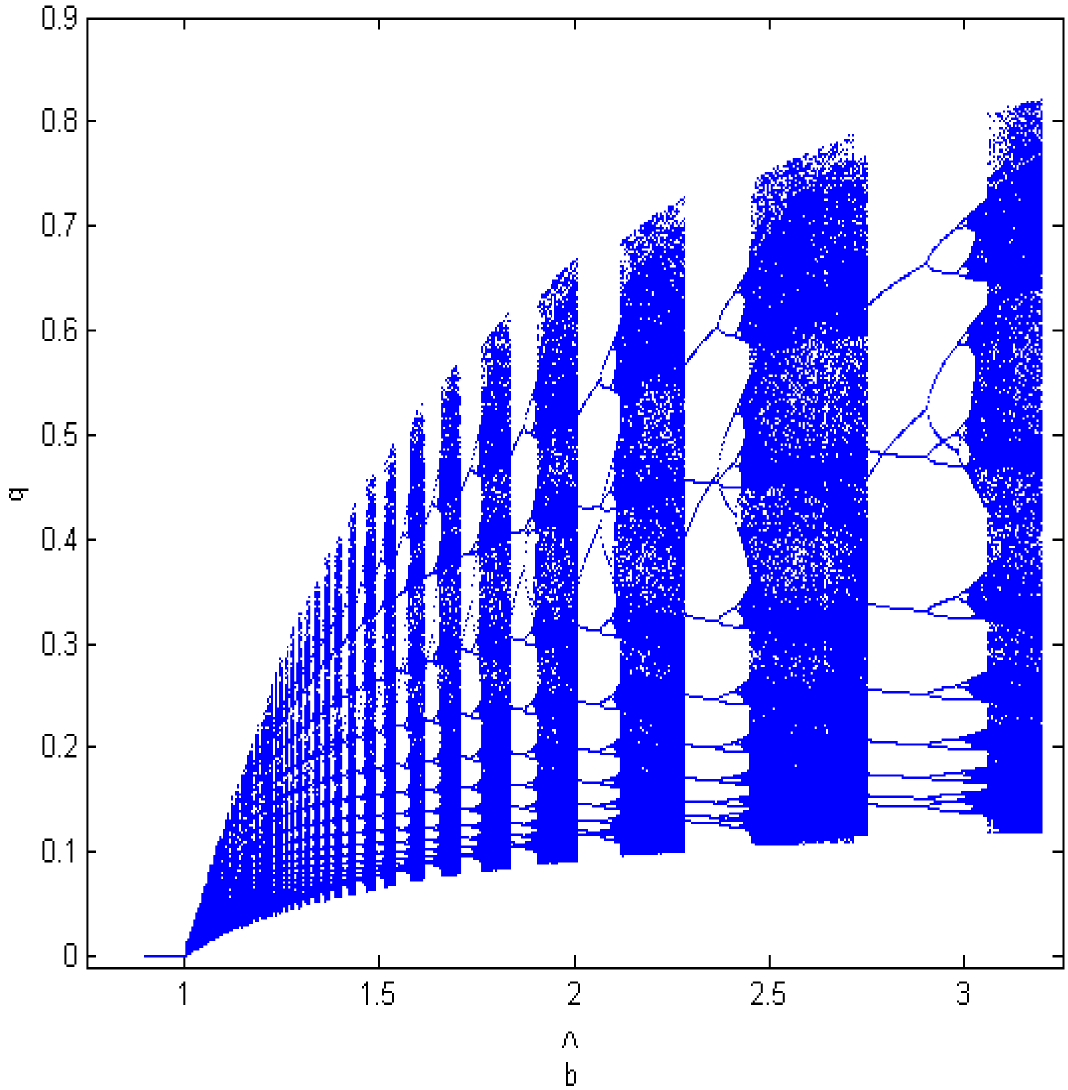


Figure 2b

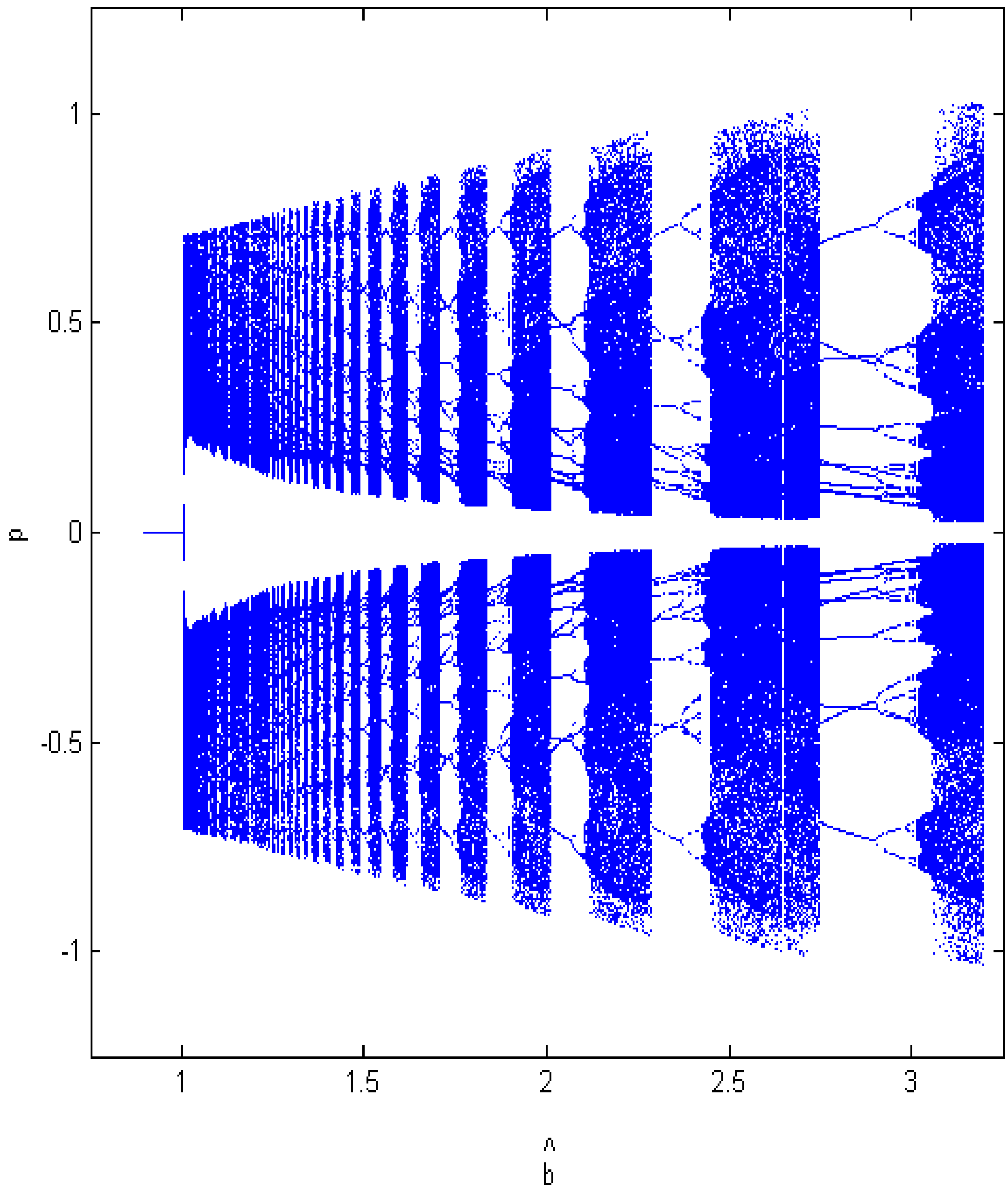
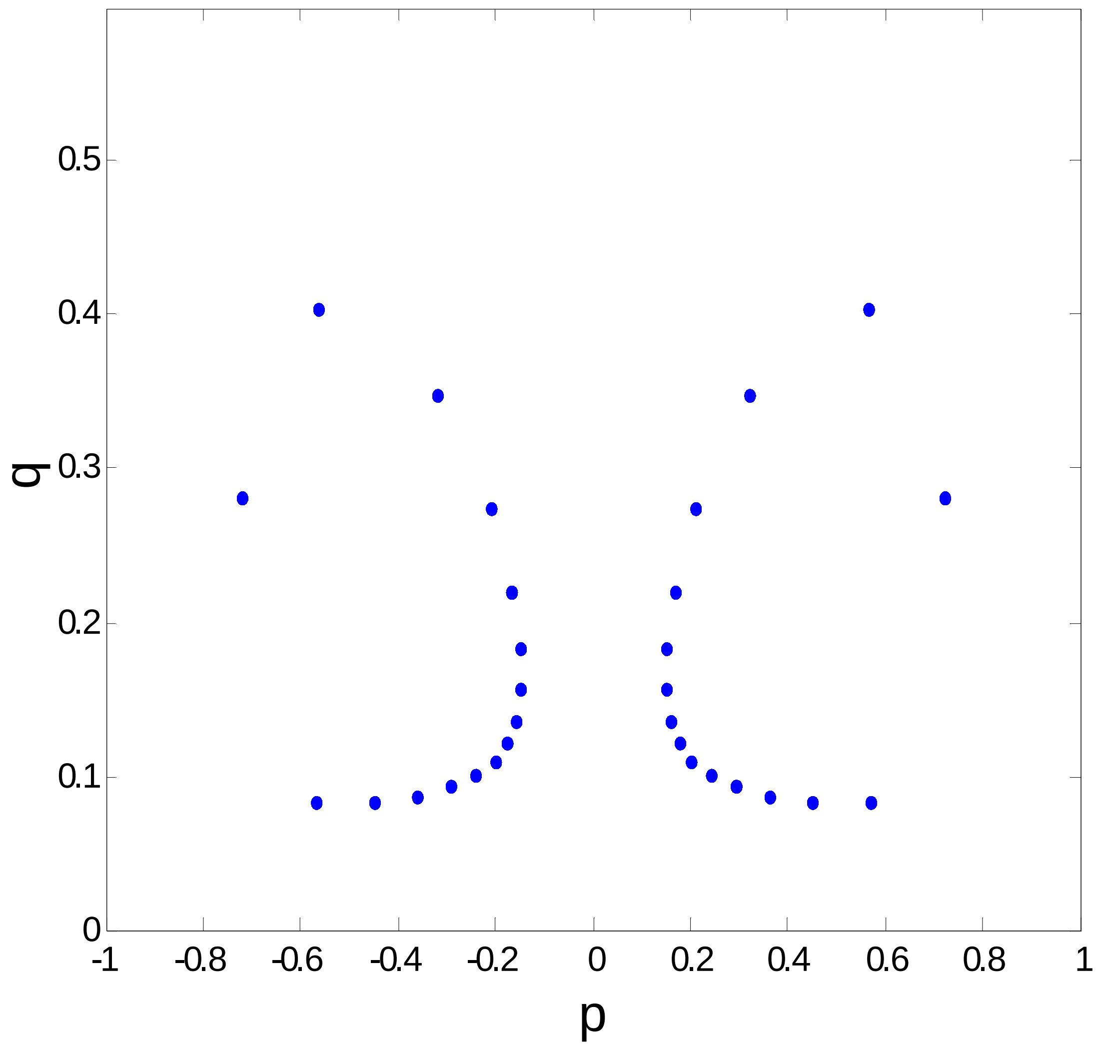
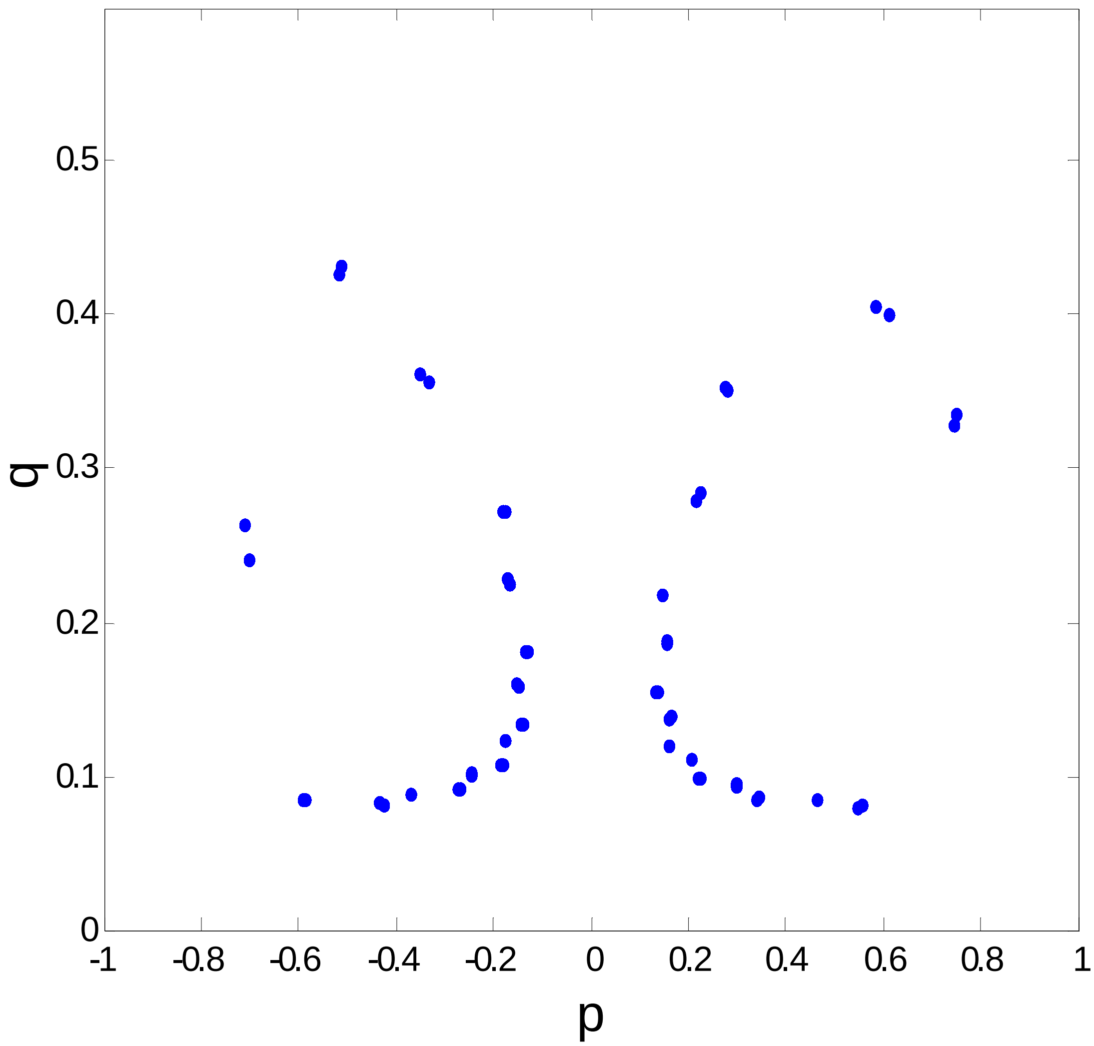


Figure 3



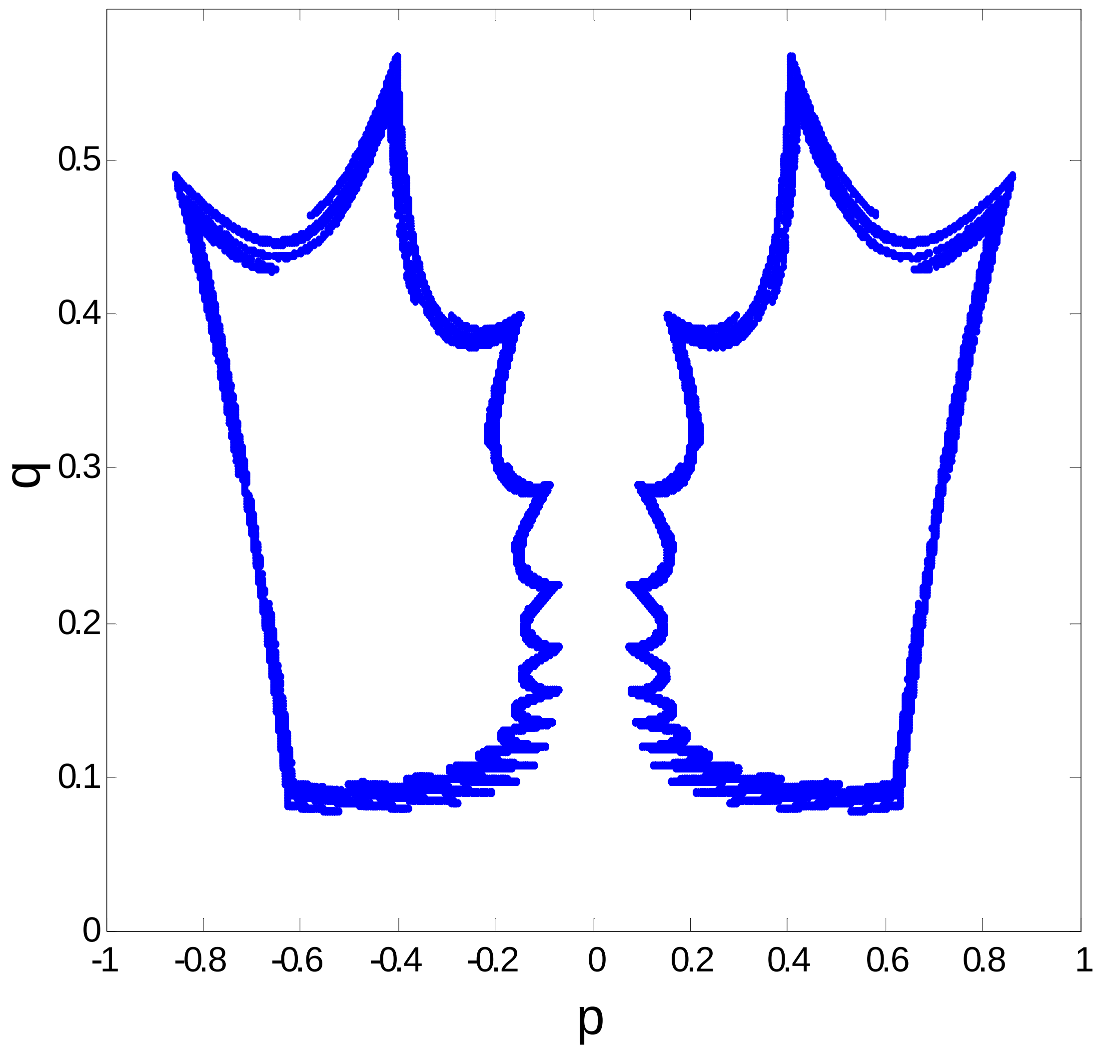
Phaseplot for $\widehat{b} = 1.56$.

Figure 4



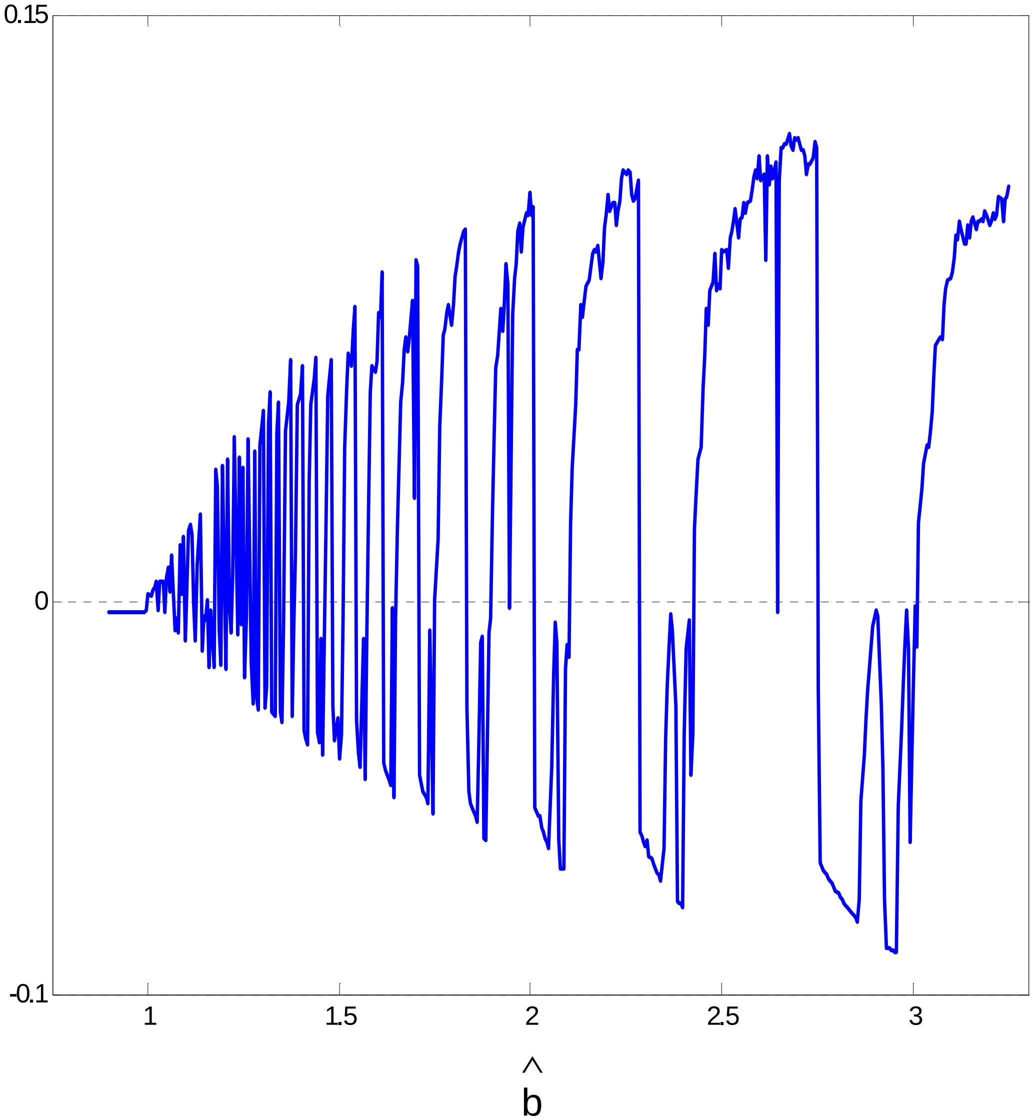
Phaseplot for $\widehat{b} = 1.573$.

Figure 5



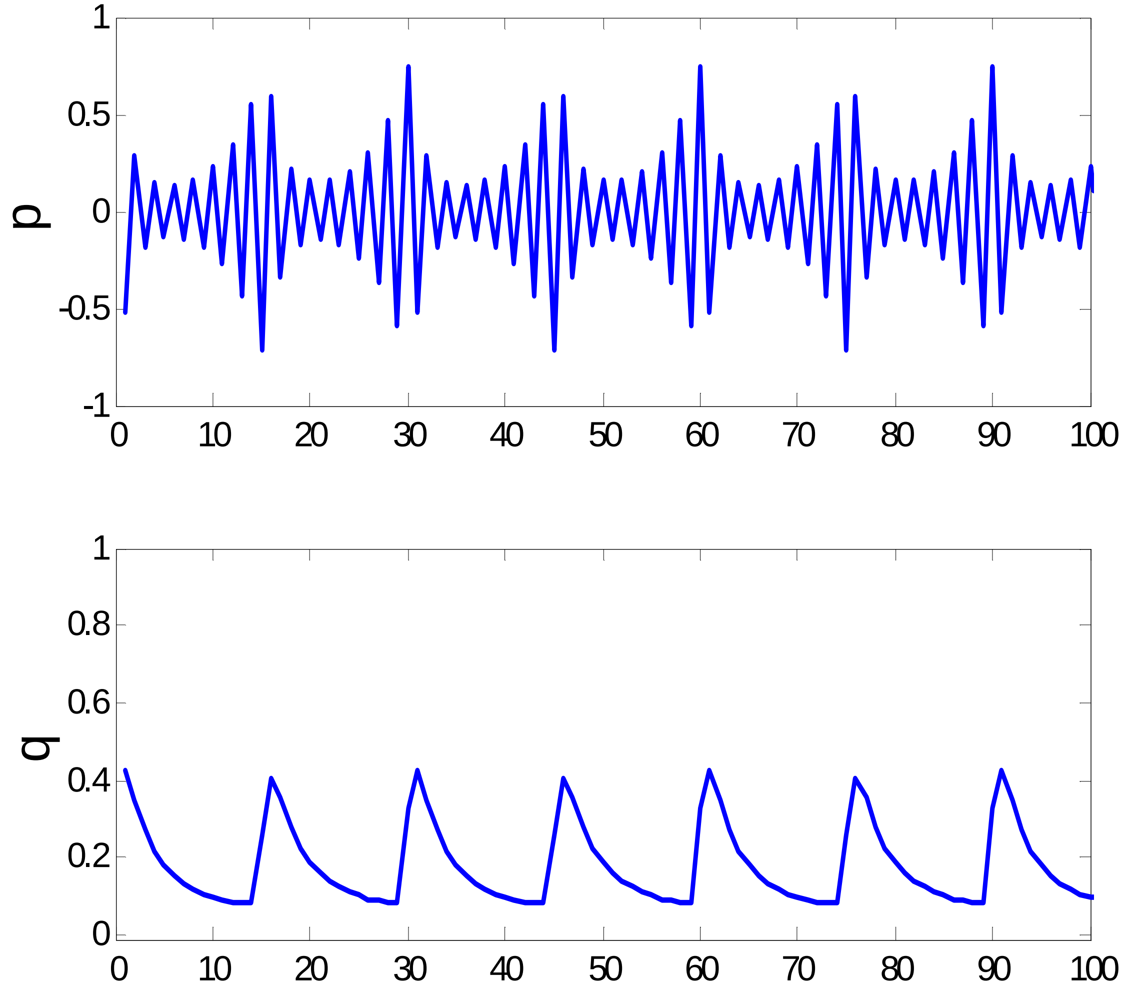
Phaseplot for $\widehat{b} = 1.7$.

Figure 6



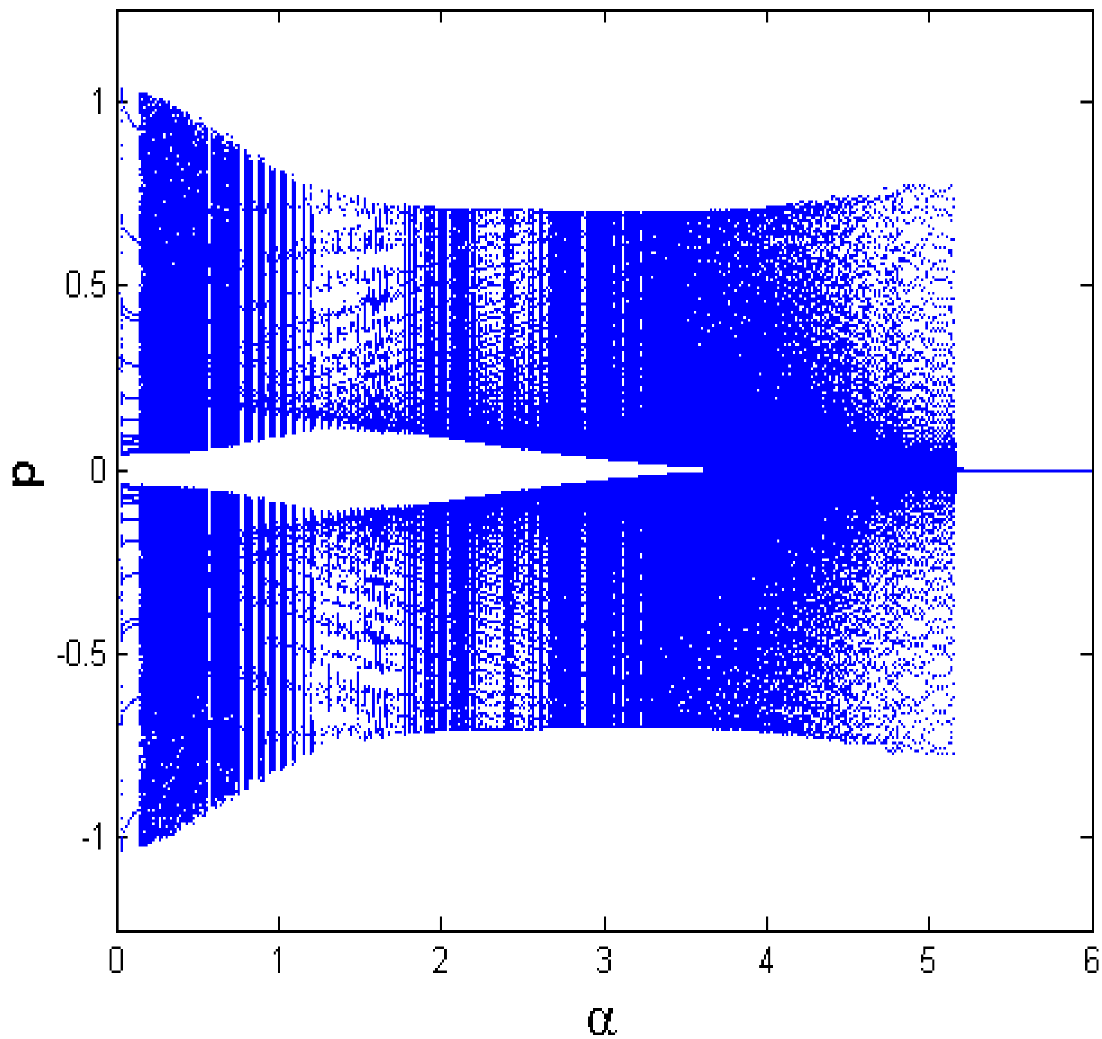
Largest Lyapunov exponent plot.

Figure 7



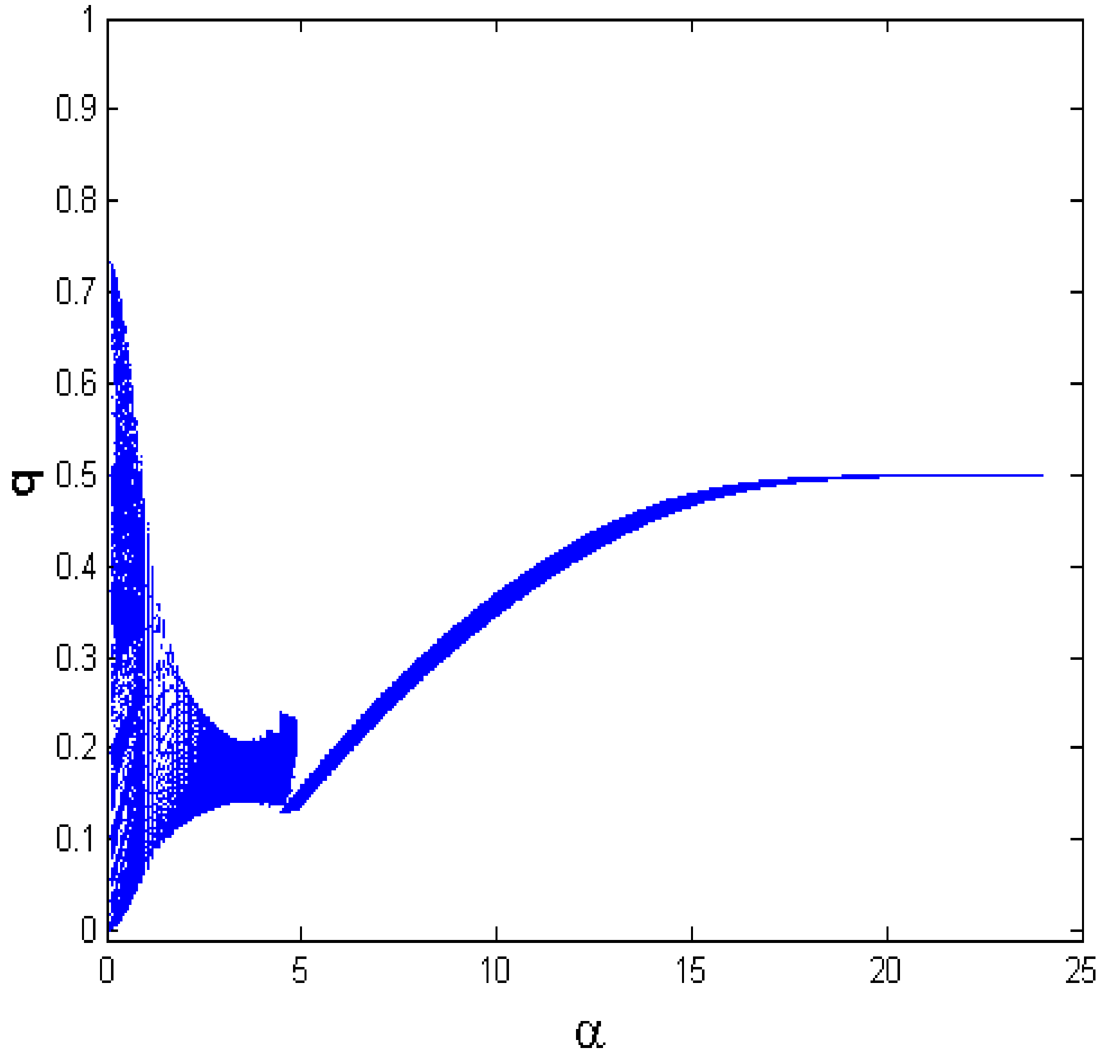
Sample time series for $\widehat{b} = 1.56$.

Figure 8a



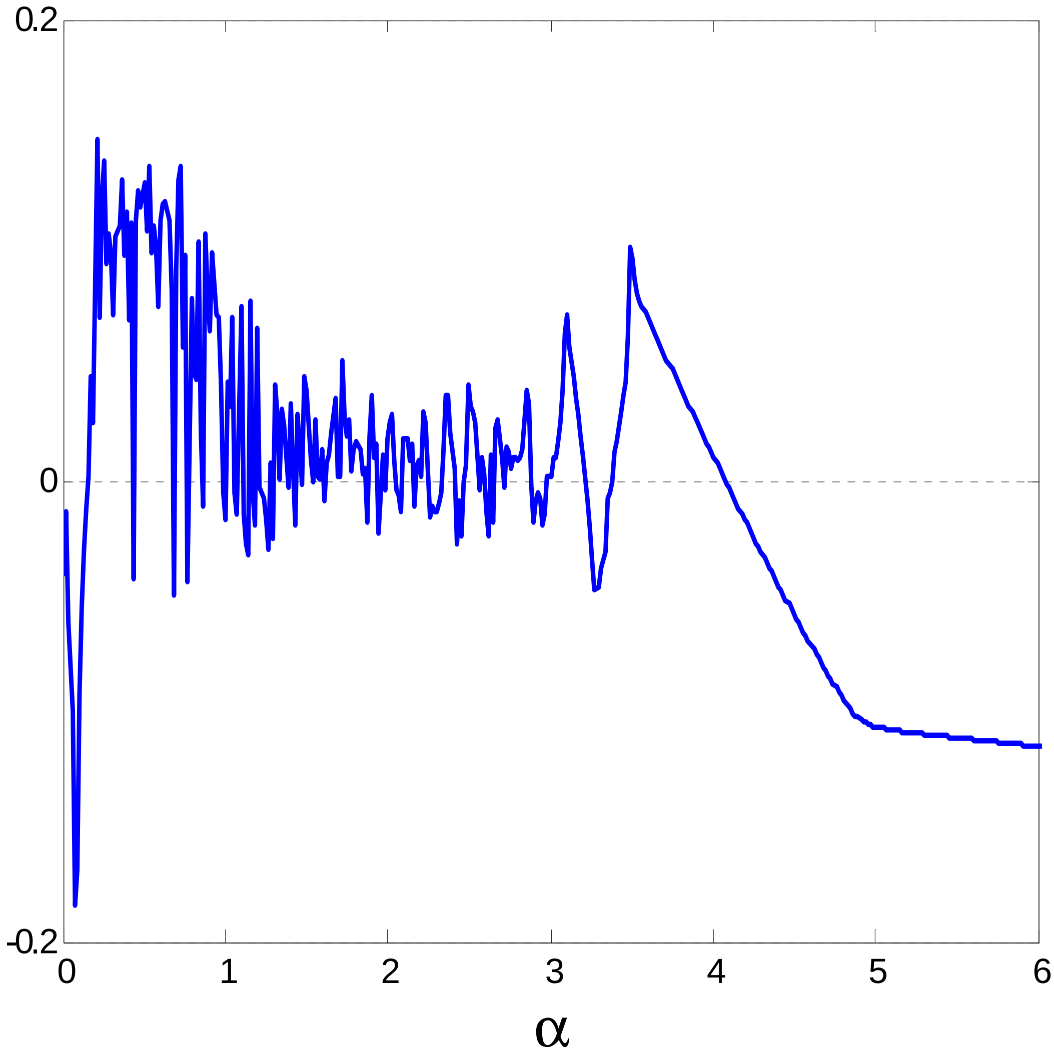
Bifurcation diagram for $\widehat{b} = 1.5$.

Figure 8b



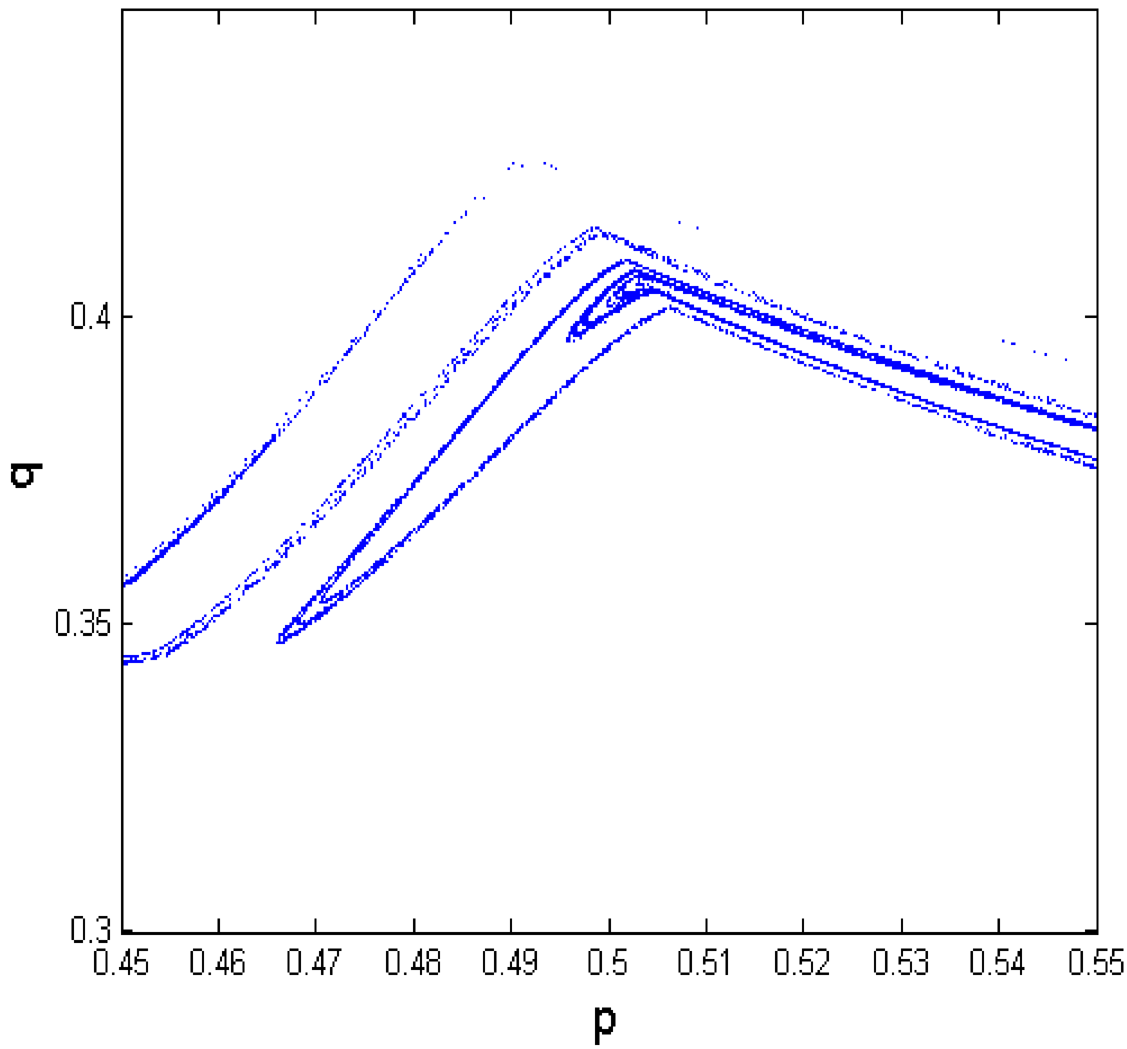
Bifurcation diagram for $\widehat{b} = 1.5$.

Figure 9



Largest Lyapunov exponent plot for $\widehat{b} = 1.5$.

Figure 10



Magnified view of the phaseplot for $\alpha = 1.15$ and $\widehat{b} = 1.5$.