# Monetary Policy and Learning from the Central Bank's Forecast*

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**Abstract**

We examine the expectational stability (E-stability) of rational expectations equilibrium (REE) in a standard New Keynesian model in which private agents refer to the central bank's forecast in the process of adaptive learning. In this environment, to satisfy the E-stability condition, the central bank must respond more strongly to the expected inflation rate than the so-called Taylor principle suggests. On the other hand, the central bank's strong reaction to the expected inflation rate raises the possibility of indeterminacy of the REE. In considering these problems, a robust policy is to respond to the current inflation rate to a certain degree.

**Keywords**: Adaptive Learning, E-stability, New Keynesian Model, Monetary Policy, Taylor principle.

**JEL Classification**: E52, D84.

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# 1 Introduction

Since the development of adaptive learning in macroeconomics, many studies have investigated the expectational stability (E-stability) conditions of rational expectations equilibrium (REE) in various macroeconomic models (Evans and Honkapohja (2001)). One of the important applications to monetary economics is provided by Bullard and Mitra (2002). They examine the E-stability condition in a simple class of the New Keynesian model, which consists of an IS equation, a New Keynesian Phillips curve (NKPC), and a Taylor-type monetary policy rule.[1] Their results indicate that the so-called Taylor principle, which requires the central bank to adjust the nominal interest rate by more than one-for-one with the inflation rate, corresponds to the E-stability condition under some versions of Taylor-type monetary policy rules, including a forward-looking rule incorporating the expectations for the future inflation rate and output gap, which are assumed to be homogeneous between the central bank and private agents.[2]

Honkapohja and Mitra (2005) extend the analysis of Bullard and Mitra (2002) to introduce heterogeneous expectations between the central bank and private agents. They show that, even if the central bank and private agents initially have different expectations, the correspondence between the E-stability condition and the Taylor principle holds, as long as the learning algorithms used by these two agents are asymptotically identical. However, they further show that, if the difference of learning algorithms remains even in the long run, the Taylor principle does not generally correspond to the E-stability condition. Therefore,

[1] Evans and Honkapohja (2003a, 2008) review the studies of adaptive learning in New Keynesian models.

[2] The issue of stability under learning when the central bank introduces an interest-rate rule is originally raised by Howitt (1992) in an IS-LM model with a New Classical Phillips curve.

their analysis points out that heterogeneity between the central bank and private agents is a key issue for determining the E-stability condition in a standard New Keynesian model.

However, we can view that the environments of these previous studies are still quite simple because the studies assume that the central bank and private agents homogeneously (or simultaneously) engage in adaptive learning. In other words, the previous studies assume that there is no interaction in the learning process of the central bank and private agents. Of course, as Honkapohja and Mitra (2005) noted, this assumption is introduced as a natural benchmark.[3] However, the validity of this assumption is empirically arguable when we take into account possible interactions between the central bank and private agents.

Especially, in recent years, many central banks have published the forecasts of future economic development in order to enhance the transparency and accountability of monetary policy-making. In this environment, if private agents consider that the central bank's forecast is reliable, it is possible that private agents will use the information from the central bank's forecast in making their forecast. Actually, Fujiwara (2005) provides empirical evidence that, in Japan's survey data, the central bank's forecast significantly influences the forecast of private agents (not vice versa). Therefore, his results present a possibility that the central bank is the leader and private agents are the followers of expectation

[3] Honkapohja and Mitra (2005) stated that, "We will focus on the situation in which both the private sector and the central bank use their own forecasts in their decision-making and the forecasts are not available to the other agents. Consequently, the forecasts have no strategic role. This case can be seen as a natural benchmark."

formations.[4]

In this study, we examine the E-stability of the REE in a standard New Keynesian model in which the central bank is the leader and private agents are the followers of expectation formations. This means that private agents refer to the central bank's forecast in the process of adaptive learning. This kind of leader-follower relationship of adaptive learning has already been introduced by Granato, Guse, and Wong (2007) in the traditional "cobweb" model. However, their analysis investigates heterogeneous expectations among private agents. In contrast, the distinctive feature of our study is that it investigates heterogeneous expectations between the policymaker (namely, the central bank) and private agents.

As we assume that private agents refer to the central bank's forecast, our study introduces heterogeneity concerning the perceived law of motion (PLM) used by the central bank and private agents. However, as for the learning algorithm, we assume that both the central bank and private agents use recursive least squares (RLS) with decreasing gain, which is the most standard algorithm in the literature.[5] In these respects, the environment of our study contrasts sharply with that of Honkapohja and Mitra (2005), which assumes

[4] Fujiwara (2005) suggests, "In the learning context, it would be better to suppose that the central bank is a leader rather than a follower when analyzing monetary policy in Japan, since the results in this paper indicate that professional forecasters tend to learn from the central bank rather than to influence it (Fujiwara 2005, p. 261)."

[5] An alternative algorithm is RLS with constant gain, which is typically used to describe a situation in which agents take account of the possibility of structural changes (as is explained by Evans and Honkapohja (2001)). Honkapohja and Mitra (2005) introduce heterogeneous constant gains between the central bank and private agents. They show that, if the difference of constant gains remains in the long run, then it matters for the E-stability condition.

that PLMs are homogeneous and that the learning algorithms are heterogeneous.

In this study, we restrict our attention to a Taylor-type simple monetary policy rule. In this sense, our study is distinct from the studies which examine the E-stability under optimal monetary policy, such as Evans and Honkapohja (2003b, 2006). In a recent study, Preston (2008) examines a situation in which the central bank and private agents have different expectations since they have different knowledge on the economic structure. Although the environment of his study is somewhat similar to ours, his study introduces a targeting rule, rather than simple Taylor-type rule.[6] In addition, his study does not examine the issue of interactions between the central bank's forecast and private agents' expectations.[7]

The rest of this paper is organized as follows. In Section 2, we introduce our framework. In addition, by using this framework, we confirm that, if the central bank and private agents are homogeneously learning, the E-stability condition corresponds to the Taylor principle, as reported by Bullard and Mitra (2002). In Section 3, we examine the E-stability condition when private agents are learning from the central bank's forecast. In Section 4, we investigate the relationship between the E-stability and the determinacy (uniqueness) of the REE. In Section 5, we provide some further analysis. In Section 6, we conclude our analysis.

[6] As a result, in his framework, the central bank responds to private agents' expectation, rather than to the central bank's internal forecast.

[7] Furthermore, his model introduces private agents' long-horizon forecast, rather than one-period-ahead forecast, following Preston (2005).

# 2 Framework

## 2.1 Model

Our study is based on the standard New Keynesian model, which is used by Bullard and Mitra (2002) and Honkapohja and Mitra (2005). It consists of three equations, which are the IS equation, the NKPC, and a forward-looking monetary policy rule.

The IS equation and the NKPC are given as follows:

$$x_t = E_t^P x_{t+1} - \sigma(r_t - r_t^n - E_t^P \pi_{t+1}), \tag{1}$$

$$\pi_t = \beta E_t^P \pi_{t+1} + \kappa x_t, \tag{2}$$

where $x_t$ is the output gap, $\pi_t$ is the inflation rate, $r_t$ is the nominal interest rate, and $r_t^n$ is the natural rate of real interest. Each variable is defined as the deviation from its steady state. In particular, $r_t$ is the deviation of the nominal interest rate from its steady-state level, which is consistent with zero inflation and steady-state output growth. $E_t^P$ denotes private agents' subjective (possibly nonrational) expectation. $\sigma$, $\beta$, and $\kappa$ are the structural parameters which satisfy $\sigma > 0$, $1 \geq \beta > 0$, and $\kappa > 0$.

The process of natural rate of real interest is given by

$$r_t^n = \rho r_{t-1}^n + \varepsilon_t, \tag{3}$$

where $\rho$ satisfies $1 > \rho > 0$ and $\varepsilon_t$ follows i.i.d. with zero mean.

The central bank introduces a forward-looking monetary policy rule:

$$r_t = \phi_\pi E_t^{CB} \pi_{t+1} + \phi_x E_t^{CB} x_{t+1}, \tag{4}$$

where $\phi_\pi$ is the responsiveness to the expected inflation rate and $\phi_x$ is the responsiveness to the expected output gap. $E_t^{CB}$ denotes the central bank's subjective expectation.

In this study, we investigate mainly the situation in which the central bank and private agents have heterogeneous expectations at least during the transition to equilibrium. Therefore, the central bank's expectations ($E_t^{CB}$) and private agents' expectations ($E_t^P$) are potentially different.

However, in order to provide analytical results, we temporarily introduce the following assumptions as for the monetary policy rule (4). First, we assume that the central bank can observe private agents' expectation for future output gap. Second, we assume that the central bank completely offsets the variations of private agents' expectation for future output gap. These assumptions imply that $E_t^{CB}x_{t+1} = E_t^P x_{t+1}$ and $\phi_x = \sigma^{-1}$ in (4). This kind of restriction partially reflects the theoretical results of Evans and Honkapohja (2003b), which show that appropriate response to private agents' expectations yields desirable outcome under private agents' learning. However, it is apparent that the feasibility of this policy response crucially depends on the central bank's precise knowledge on $E_t^P x_{t+1}$ and $\sigma$. Therefore, in Section 6, we numerically investigate the robustness of our main results without imposing the assumptions on the central bank's precise knowledge on these aspects.

When we introduce the above restrictions in monetary policy rule (4), the model, which consists of (1), (2) and (4), can be reduced to the univariate model of inflation dynamics:

$$\pi_t = A + BE_t^P \pi_{t+1} + CE_t^{CB} \pi_{t+1} + Dr_t^n, \tag{5}$$

where $A = 0$, $B = \kappa\sigma + \beta$, $C = -\kappa\sigma\phi_\pi$, and $D = \kappa\sigma$.

## 2.2 E-stability under Homogeneous Learning

Before moving on to our main analysis, we present the E-stability condition of the REE in the benchmark situation where the central bank and private agents use homogeneous procedure of adaptive learning. As usual in the literature, we assume that the PLM used by central bank and private agents has the following form:

$$\pi_t = \widetilde{a} + \widetilde{b} r_t^n, \tag{6}$$

where $\widetilde{a}$ and $\widetilde{b}$ are coefficients, which are updated in every period. Since the functional form of (6) corresponds to the minimal state variables (MSV) solution of the system (5), we call the learning process of (6) "MSV learning."[8]

Based on PLM, the one-period-ahead expectation is calculated as follows:

$$E_t^{CB}\pi_{t+1} = E_t^P \pi_{t+1} = \widetilde{a} + \rho \widetilde{b} r_t^n. \tag{7}$$

By substituting (7) into (5), we derive the actual law of motion (ALM) as follows:

$$\pi_t = A + (B+C)\widetilde{a} + (\rho(B+C)\widetilde{b} + D) r_t^n. \tag{8}$$

From (6) and (8), the mapping functions (T-maps) from PLM to ALM are as follows:

$$T_a(\widetilde{a}) = A + (B+C)\widetilde{a}, \tag{9}$$

$$T_b(\widetilde{b}) = \rho(B+C)\widetilde{b} + D. \tag{10}$$

The REE with the MSV form (MSV solution) is obtained as the fixed point of T-maps. The parameters of the MSV solution ($\overline{a}$ and $\overline{b}$) are computed as follows:

$$\overline{a} = (1-(B+C))^{-1}A, \ \overline{b} = (1-\rho(B+C))^{-1}D.$$

Note that the combination of $\overline{a}$ and $\overline{b}$ is unique. It means that, if we restrict attention to the MSV form, the solution is unique, regardless of the values of structural parameters. For the moment (except for Section 4), we focus on the MSV solution.

In this study, we assume that both the central bank and private agents use RLS with decreasing gain. Then, the E-stability of the REE is defined as the local asymptotic stability

[8] See McCallum (1983) for the details of MSV solution.

of the ordinary differential equations (ODEs) associated with the T-maps ((9) and (10)):

$$h_a(\widetilde{a}) \equiv \frac{da}{d\tau} = T_a(\widetilde{a}) - \widetilde{a} = A + (B + C - 1)\widetilde{a}, \tag{11}$$

$$h_b(\widetilde{b}) \equiv \frac{db}{d\tau} = T_a(\widetilde{b}) - \widetilde{b} = (\rho(B + C) - 1)\widetilde{b} + D, \tag{12}$$

where $\tau$ is "notional" or "artificial" time.

From these ODEs, the E-stability condition is derived as two inequalities:

$$Dh_a(\widetilde{a}) = B + C - 1 < 0, \tag{13}$$

$$Dh_b(\widetilde{b}) = \rho(B + C) - 1 < 0. \tag{14}$$

Since $1 > \rho > 0$, (14) holds if (13) holds. Therefore, the necessary and sufficient condition for the E-stability of the REE is (13). (13) is rewritten as follows:

$$\phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{15}$$

(15) requires that the responsiveness to the expected inflation rate is strictly larger than some threshold value $(1 + \frac{\beta-1}{\kappa\sigma})$. This result can be viewed as a special case of the results of Bullard and Mitra (2002), which shows that, in the absence of restrictions in monetary policy rule (4), the necessary and sufficient condition for the E-stability of the REE is given by the so-called Taylor principle, which is expressed as $\phi_\pi + \frac{1-\beta}{\kappa}\phi_x > 1$. In our framework, the Taylor principle corresponds to (15), since we have imposed the restriction of $\phi_x = \sigma^{-1}$. Therefore, the result indicates that the Taylor principle is the necessary and sufficient condition for the E-stability of the REE even if we introduce the restrictions in monetary policy rule (4).

# 3 E-stability under Learning from the Central Bank's Forecast

## 3.1 Basic Assumptions

In this section, we examine the E-stability condition when private agents are learning from the central bank's forecast. In order to set up the environment, we introduce five basic assumptions.

First, we assume that private agents cannot observe the natural rate of real interest, although the central bank can observe it. This assumption implies that it is very hard, or very costly, for an individual private agent to observe directly the state of the aggregate economy, although the central bank can observe it by using many internal research resources.

Second, we assume that the central bank is the leader and private agents are the followers in the process of expectation formations. This means that the central bank calculates and announces the forecast before private agents form their forecast. Since private agents can observe the value of the central bank's forecast, private agents can use this information in their process of adaptive learning.

Third, we assume that the central bank does not disclose the model used to make the forecast. Because of this assumption, private agents cannot know how the central bank's forecast is calculated. So private agents are unable to infer the value of natural rate of real interest from the central bank's forecast. This assumption is based on the fact that, in reality, no central bank to discloses the exact methodology used to make an economic forecast.

Fourth, we assume that the central bank announces the forecast simply to enhance the transparency and accountability of monetary policy-making. In other words, the announcement of the forecast is not necessarily the optimal strategy for the central bank. We assume that the central bank is legally required to provide views on future economic development as a background to monetary policy-making so as to maintain the central bank independence.

Fifth, we assume that private agents do not automatically follow the central bank's forecast. Rather, private agents determine how to use the information in the central bank's forecast, depending on their evaluation on the historical performance of the central bank's forecast. We consider this assumption reasonable because private agents do not have any reason to blindly follow the central bank's forecast if the historical performance of the central bank's forecast is very poor.

## 3.2 Learning from the Central Bank's Forecast

Based on the basic assumptions, we introduce the following setup for the learning mechanisms of the central bank and private agents. We assume that, as in the previous section, the central bank is MSV learning. Then, the central bank's PLM is as follows:

$$\pi_t = \widetilde{a} + \widetilde{b} r_t^n. \tag{16}$$

At the beginning of period $t$, the central bank updates the parameters of $\widetilde{a}$ and $\widetilde{b}$ by using the data of period $t-1$ ($y_{t-1}$ and $r_{t-1}^n$). Then, the central bank observes the realization of the natural rate of real interest at period $t$ ($r_t^n$). By using the newest estimates of $\widetilde{a}$ and $\widetilde{b}$, the central bank calculates the forward-looking expectations as follows:

$$E_t^{CB} \pi_{t+1} = \widetilde{a} + \rho \widetilde{b} r_t^n. \tag{17}$$

After calculating (17), the central bank announces this forecast to private agents.

Private agents observe the central bank's forecast $E_t^{CB} \pi_{t+1}$. Then, private agents determine how to utilize the central bank's forecast in forming their expectations by evaluating the historical performance of the central bank's forecast. Specifically, we assume that private agents estimate the following PLM:

$$\pi_t = \widetilde{c} + \widetilde{d} E^{CB}_{t-1} \pi_t. \tag{18}$$

By estimating (18) with RLS, private agents assess the historical performance of the central bank's forecast.[9] If the forecast has historically performed well, the constant term $\widetilde{c}$ approximates zero, and the slope $\widetilde{d}$ should be close to unity. In contrast, if the central bank's forecast has performed poorly, $\widetilde{c}$ approximates the sample average of $\pi_t$, and $\widetilde{d}$ should be close to zero.

Private agents update the parameters of $\widetilde{c}$ and $\widetilde{d}$ by using the data of period $t-1$ ($\pi_{t-1}$ and $E^{CB}_{t-2}\pi_{t-1}$). Since private agents are the followers, they can use the central bank's forecast $E^{CB}_t \pi_{t+1}$ in forming their expectations at period $t$ ($E^P_t \pi_{t+1}$). To calculate $E^P_t \pi_{t+1}$, private agents use their evaluation of the performance of the central bank's forecast as follows:

$$E^P_t \pi_{t+1} = \widetilde{c} + \widetilde{d} E^{CB}_t \pi_{t+1}. \tag{19}$$

(19) indicates that the forecast of private agents is influenced by the central bank's forecast. As we can see, the impact of the central bank's forecast on private agents' forecast is determined by the estimated parameter $\widetilde{d}$. Therefore, (19) illustrates a situation in which private agents are learning from the central bank's forecast, depending on its historical performance.

By inserting both agents' expectations ((17) and (19)) into the system of (5), we derive ALM for $\pi_t$ as follows:

$$\pi_t = A + B(\widetilde{c} + \widetilde{d}\widetilde{a}) + C\widetilde{a} + (\rho(B\widetilde{d} + C)\widetilde{b} + D) r^n_t. \tag{20}$$

[9] As is seen in the next subsection, the use of RLS in estimating (18) is consistent with the REE.

## 3.3 Equilibrium

Next, we derive the T-maps from PLM to ALM. From (16) and (20), the T-maps about parameters of $\widetilde{a}$ and $\widetilde{b}$ are given as follows:

$$T_a(\widetilde{a}) = A + B(\widetilde{c} + \widetilde{d}\widetilde{a}) + C\widetilde{a}, \tag{21}$$

$$T_b(\widetilde{b}) = \rho(B\widetilde{d} + C)\widetilde{b} + D. \tag{22}$$

Since private agents' PLM (18) is not the MSV form, we must derive the T-maps from the relevant orthogonality conditions.[10] From (17) and (18), private agents' "projected" ALM is defined as follows:

$$\pi_t = T_c + T_d(\widetilde{a} + \rho\widetilde{b}r^n_{t-1}). \tag{23}$$

The corresponding orthogonality conditions are given by

$$E\left[1 \cdot \left(\pi_t - T_c - T_d(\widetilde{a} + \rho\widetilde{b}r^n_{t-1})\right)\right] = 0, \tag{24}$$

$$E\left[(\widetilde{a} + \rho\widetilde{b}r^n_{t-1})\left(\pi_t - T_c - T_d(\widetilde{a} + \rho\widetilde{b}r^n_{t-1})\right)\right] = 0. \tag{25}$$

In order to calculate $T_c$ and $T_d$, we substitute (20) into (24) and (25). Then, by solving (24) and (25), we obtain the following expressions of $T_c$ and $T_d$:

$$T_c(\widetilde{c}) = A + B\widetilde{c} + (1-\rho)(B\widetilde{d} + C)\widetilde{a} - \widetilde{b}^{-1}D\widetilde{a}, \tag{26}$$

$$T_d(\widetilde{d}) = \rho B\widetilde{d} + \rho C + \widetilde{b}^{-1}D. \tag{27}$$

The equilibrium is derived as the fixed points of the T-maps ((21), (22), (26), and (27)). The coefficients at the equilibrium are given as follows:

$$\overline{a} = (1-(B+C))^{-1}A, \overline{b} = (1-\rho(B+C))^{-1}D, \overline{c} = 0, \overline{d} = 1.$$

[10] See Branch (2004) for the derivation of T-maps using orthogonality conditions.

Note that, at the equilibrium, (19) becomes as follows:

$$E_t^P \pi_{t+1} = E_t^{CB} \pi_{t+1} = (1 - (B + C))^{-1} A + \rho(1 - \rho(B + C))^{-1} D r_t^n. \tag{28}$$

Therefore, at the equilibrium, expectations are homogeneous between the central bank and private agents. Furthermore, these expectations are the same as the expectation at the MSV solution in Section 2. Therefore, the expectations of (28) are the rational expectations and this equilibrium is the REE. This means that the economic dynamics at equilibrium are exactly the same in the two cases: (i) the case in which the central bank and private agents are homogeneously learning and (ii) the case in which private agents are learning from the central bank's forecast. However, as the analysis in the next subsection shows, the E-stability conditions of the REE can differ between these two cases.

## 3.4 E-stability

The E-stability of the equilibrium is the local asymptotic stability of ODEs associated with the T-maps of (21), (22), (26), and (27). Although these T-maps are interdependent, $T_b(\widetilde{b})$ and $T_d(\widetilde{d})$ only depend on $\widetilde{b}$ and $\widetilde{d}$. Therefore, we can examine the stability of $\widetilde{b}$ and $\widetilde{d}$, independently of the stability of $\widetilde{a}$ and $\widetilde{c}$.

To examine the stability of $\widetilde{b}$ and $\widetilde{d}$, we define the ODEs associated with the T-maps of $\widetilde{b}$ and $\widetilde{d}$ ((22) and (27)) as follows:

$$h \begin{pmatrix} \widetilde{b} \\ \widetilde{d} \end{pmatrix} \equiv \begin{pmatrix} T_b(\widetilde{b}) - \widetilde{b} \\ T_d(\widetilde{d}) - \widetilde{d} \end{pmatrix} = \begin{pmatrix} \rho(B\widetilde{d} + C)\widetilde{b} + D - \widetilde{b} \\ \rho B\widetilde{d} + \rho C + \widetilde{b}^{-1} D - \widetilde{d} \end{pmatrix}. \tag{29}$$

Given the convergence of $\widetilde{b}$ and $\widetilde{d}$, we can examine the stability of $\widetilde{a}$ and $\widetilde{c}$ by using the following ODEs:

$$h \begin{pmatrix} \widetilde{a} \\ \widetilde{c} \end{pmatrix} \equiv \begin{pmatrix} T_a(\widetilde{a}) - \widetilde{a} \\ T_c(\widetilde{c}) - \widetilde{c} \end{pmatrix} = \begin{pmatrix} A + B(\widetilde{c} + \widetilde{d}\widetilde{a}) + C\widetilde{a} - \widetilde{a} \\ A + B\widetilde{c} + (1 - \rho)(B\widetilde{d} + C)\widetilde{a} - \widetilde{b}^{-1} D\widetilde{a} - \widetilde{c} \end{pmatrix}. \tag{30}$$

We derive the E-stability condition as the necessary and sufficient condition for the ODEs of (29) and (30) to be locally asymptotically stable around the REE. The result is given by the following proposition.

**Proposition 1** *Suppose that the central bank is MSV learning and all private agents are learning from the central bank's forecast. Then, the REE of (5) is E-stable if and only if (31) and (32) hold.*

$$\phi_\pi > 2 + \frac{2(\beta-1)}{\kappa\sigma}, \tag{31}$$

$$\phi_\pi > 1 + \frac{\beta-1}{\kappa\sigma}. \tag{32}$$

*Furthermore, suppose that $\kappa\sigma > 1-\beta$ holds. Then, the REE of (5) is E-stable if and only if (31) holds.*

***Proof.*** *See Appendix A.* ■

Since $\beta$ is close to unity, $\kappa\sigma > 1-\beta$ holds for a wide range of parameter sets ($\kappa$ and $\sigma$). Then, the Taylor principle, which is expressed as (32), is not a sufficient condition for the E-stability. Alternatively, the necessary and sufficient condition for the E-stability is given by (31). This means that, to satisfy the E-stability condition, the central bank must adjust the nominal interest rate by more than double the rise of central bank's expected inflation rate.

Thus, the E-stability condition in this situation is quite different from the condition in the benchmark case analyzed in Section 2. Although the equilibrium dynamics of these two cases are identical, the E-stability condition is severer in the environment of this section. This means that, if private agents are learning from the central bank's forecast, the central bank must respond to the expected inflation rate more strongly than the Taylor principle suggests.

The basic intuition about the result arises from the fact that, in the situation of this section, private agents' forecast errors, which are defined as deviations of private agents' expectations from rational expectations, are magnified, compared to the central bank's forecast errors. The reason is twofold. Firstly, private agents have estimation errors concerning the parameters $\widetilde{c}$ and $\widetilde{d}$ in their PLM (18). These estimation errors are the first source of private agents' forecast errors. Secondly, as in (19), the central bank's forecast errors influence the forecasts of private agents. This contagious effect of the central bank's forecast error is the second source of private agents' forecast errors. Since the parameter $\widetilde{d}$ is almost unity around the equilibrium, the central bank's forecast errors bring about almost the same number of forecast errors as those of private agents. Therefore, if we sum up the two sources of private agents' forecast errors, the total forecast errors of private agents exceed the central bank's forecast errors.

In addition, since the central bank introduces its own forecast in the monetary policy rule (4), the central bank responds to its own forecast errors. Because private agents have larger forecast errors than the central bank, this policy response might be insufficient to offset the forecast errors of private agents. Namely, if the responsiveness to the central bank's inflation forecast in monetary policy rule ($\phi_\pi$) only slightly exceeds unity, it is possible that the real interest rate, which is calculated as the nominal interest rate subtracted from the private agents' inflation forecast, falls (not rises) against the increase of private agents' inflation forecast. In order to avoid this undesirable situation, the central bank must respond very strongly to its own forecast to raise the real interest rate against the rise of private agents' inflation expectation.

## 3.5 E-stability When Part of Private Agents Are Learning from the Central Bank's Forecast

So far, we have assumed that all private agents are learning from the central bank's forecast. However, this could be regarded as an extreme case.[11] In this subsection, therefore, we consider a more realistic environment in which some private agents are learning from the central bank's forecast.

Suppose that a proportion $\mu$ of private agents are learning from the central bank's forecast ($1 \geq \mu \geq 0$). The remaining $1-\mu$ of private agents are MSV learning. Denote $E_t^{P1}\pi_{t+1}$ as the forecast of the former private agents and $E_t^{P2}\pi_{t+1}$ as the forecast of the latter private agents. Note that the forecast made by the latter is just the same as the central bank's forecast. Therefore, the aggregate forecast of private agents ($E_t^P\pi_{t+1}$) can be expressed as follows:[12]

$$\begin{aligned} E_t^P\pi_{t+1} &= \mu E_t^{P1}\pi_{t+1} + (1-\mu)E_t^{P2}\pi_{t+1} \\ &= \mu E_t^{P1}\pi_{t+1} + (1-\mu)E_t^{CB}\pi_{t+1}. \end{aligned} \tag{33}$$

[11] In this respect, Kohn (2005) judges that private agents do not rely perfectly on the central bank's expectation. He remarks that "in the United States, we have some indirect evidence that crowding out of private views has not increased even as the Federal Reserve has become more talkative. Market interest rates have continued to respond substantially to surprises in economic data."

[12] Guse (2005) incorporates a convex combination of heterogeneous forecasts into a simple macroeconomic model with multiple equilibria. Branch and McGough (2006) present the underlying assumptions for the validity of a convex combination of heterogeneous forecasts. These include (i) the identical expectations at steady state, (ii) some linearity properties of expectations, and (iii) the law of iterated expectations at both an individual and aggregate level. We assume that all of these assumptions are satisfied.

By substituting (33) into (5), we obtain the following ALM:

$$\pi_t = A + \widehat{B}E_t^{P1}\pi_{t+1} + \widehat{C}E_t^{CB}\pi_{t+1} + Dr_t^n, \tag{34}$$

where $\widehat{B} = \mu B$ and $\widehat{C} = (1-\mu)B + C$.

(34) has the same form as (5). Therefore, in order to examine the E-stability of the REE, we can follow the same steps of the subsections 3.3 and 3.4, by replacing the matrices of $B$ and $C$ with $\widehat{B}$ and $\widehat{C}$. Then, the result for the E-stability of the REE is given by the following proposition.[13]

**Proposition 2** *Suppose that the central bank and a proportion $1-\mu$ of private agents are MSV learning. In addition, suppose that a proportion $\mu$ of private agents are learning from the central bank's forecast. Then, the REE of (5) is E-stable if and only if (35) and (36) hold.*

$$\phi_\pi > 1 + \mu + \frac{(1+\mu)\beta - 2}{\kappa\sigma}, \tag{35}$$

$$\phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{36}$$

*Furthermore, suppose that $\mu \geq (\kappa\sigma+\beta)^{-1}$ holds. Then, the REE of (5) is E-stable if and only if (35) holds. In contrast, if $\mu < (\kappa\sigma+\beta)^{-1}$, the REE of (5) is E-stable if and only if (36) holds.*

***Proof.*** *See Appendix B.* ■

Thus, if $\mu$ is relatively low, then the Taylor principle is the necessary and sufficient condition for the E-stability.[14] However, if $\mu$ is relatively high, to ensure the convergence to the REE, the central bank must respond more strongly to the expected inflation rate than the Taylor principle suggests.

[13] We can easily find that the equilibrium of (34) is just the same as the REE of (5).

[14] Note that, for a wide range of parameter sets, the value of $(\kappa\sigma+\beta)^{-1}$ is between 0 to 1, since $\beta$ is almost unity.

# 4 Determinacy and E-stability

In the previous sections, we have examined the E-stability condition of the REE. However, in the standard analysis, the condition for the determinacy (uniqueness) of the REE is also regarded as the minimum criterion which should be satisfied in monetary policy rules. In this regard, Bernanke and Woodford (1997) point out that the issue of determinacy is especially relevant when the central bank introduces a forward-looking monetary policy rule, such as (4). The reason why the determinacy condition has not been examined in the previous sections is that we have restricted our attention to the MSV solution, which is unique in our model. However, if we broaden our scope to introduce the solution forms other than the MSV form (i.e., sunspot equilibria), we must examine the condition for determinacy of the REE.[15] In particular, we must investigate the relationship between the determinacy condition and the E-stability condition. In this section, we examine this issue.

## 4.1 Determinacy of the REE

The determinacy condition is presented by Blanchard and Kahn (1980). Since the system is reduced as the univariate model of (5), the derivation of determinacy condition is easy. In the REE, the system of (5) is rewritten as follows:

$$\pi_t = A + (B + C)E_t\pi_{t+1} + Dr_t^n. \tag{37}$$

Blanchard and Kahn (1980) show the determinacy condition of (37) as $|B + C| < 1$.

[15] Honkapohja and Mitra (2004) examine the existence of learnable sunspot equilibria in a simple New Keynesian model with a forward-looking Taylor rule. In contrast to our study, they introduce a benchmark assumption that the central bank and private agents are independently learning. They show that learnable sunspot equilibria can exist even if the policy rule satisfies the Taylor principle.

This leads to the following proposition.

**Proposition 3** *The economy of (5) has a unique REE if and only if the following condition holds:*

$$1 + \frac{\beta + 1}{\kappa\sigma} > \phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{38}$$

Thus, the determinacy condition sets the upper bound of $\phi_\pi$. This result means that the central bank should not respond to the expected inflation rate very strongly, because such a strong response causes the emergence of sunspot equilibria. This is the issue raised by Bernanke and Woodford (1997).

## 4.2 Relationship between Determinacy and E-stability

Next, we examine the relationship between the determinacy condition and the E-stability condition. Specifically, we investigate a situation in which all private agents are learning from the central bank's forecast.[16] In this case, the E-stability condition is given by Proposition 1. By combining these with Proposition 3, we obtain the following proposition.

**Proposition 4** *Suppose that the central bank is MSV learning and all private agents are learning from the central bank's forecast. Then, the following statements hold.*

*(i) If* $1 > \beta + \kappa\sigma$*, the necessary and sufficient condition for the REE of (5) to be E-stable and determinate is given by*

$$1 + \frac{\beta + 1}{\kappa\sigma} > \phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{39}$$

[16] The extension to the situation in which some private agents are learning from the central bank's forecast is straightforward.

*(ii) If* $3 \geq \beta + \kappa\sigma \geq 1$*, the necessary and sufficient condition for the REE of (5) to be E-stable and determinate is given by*

$$1 + \frac{\beta + 1}{\kappa\sigma} > \phi_\pi > 2 + \frac{2(\beta - 1)}{\kappa\sigma}. \tag{40}$$

*(iii) If* $\beta + \kappa\sigma > 3$*, the REE of (5) cannot be both E-stable and determinate for any value of* $\phi_\pi$*.*

The condition (39) is the same as the determinacy condition (38). This means that, in the case (i), the determinacy condition is a sufficient condition for the E-stability of the REE. However, this is a relatively special case, because $\beta + \kappa\sigma$ is usually more than unity (since $\beta$ is close to unity).

Therefore, for a wide range of the parameter sets, the determinacy is not a sufficient condition for the E-stability of the REE. This is an important finding in the literature, because McCallum (2007) points out that, if a forward-looking model includes one-period-ahead expectation and the current-period information is available in the process of adaptive learning, the determinacy becomes a sufficient condition for the E-stability of the REE, in a broad class of linear models. In contrast to the argument of McCallum (2007), Proposition 4 indicates that the determinacy is not necessarily a sufficient condition for the E-stability, even though both the central bank and private agents calculate the expectations ($E_t^{CB}\pi_{t+1}$ and $E_t^P\pi_{t+1}$) by using the information at period $t$. This result suggests that, in the presence of the leader-follower relationship in adaptive learning, the determinacy does not automatically guarantee the E-stability of the REE[17].

[17] Some studies also provide examples in which the determinacy is not a sufficient condition for the E-stability of REE. Preston (2006) shows that, if a New Keynesian model incorporates private agents' long-horizon forecast, then the determinacy does not guarantee the E-stability of REE under a forward-looking Taylor rule. Bullard and Eusepi (2008) show

Since $\beta+\kappa\sigma$ is usually greater than unity, the cases of (ii) and (iii) deserve our attention. In the case (ii), the region of E-stable and determinate REE is narrow. This means that the central bank's choice of the value $\phi_\pi$ is highly restrictive. The environment of the case (iii) is even severer, because the central bank cannot simultaneously satisfy the conditions of determinacy and E-stability. In the case (iii), we obtain the following proposition.

**Proposition 5** *Suppose that the central bank is MSV learning and all private agents are learning from the central bank's forecast. Then, under the condition of $\beta + \kappa\sigma \geq 3$, the following statements hold.*

*(i) The REE of (5) is E-stable and indeterminate if*

$$\phi_\pi > 2 + \frac{2(\beta - 1)}{\kappa\sigma}. \tag{41}$$

*(ii) The REE of (5) is E-unstable and determinate if*

$$1 + \frac{\beta + 1}{\kappa\sigma} > \phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{42}$$

*(iii) The REE of (5) is E-unstable and indeterminate if*

$$2 + \frac{2(\beta - 1)}{\kappa\sigma} \geq \phi_\pi \geq 1 + \frac{\beta + 1}{\kappa\sigma} \quad \text{or} \quad 1 + \frac{\beta - 1}{\kappa\sigma} > \phi_\pi. \tag{43}$$

Thus, if $\beta + \kappa\sigma > 3$, the central bank must choose either the determinacy or the E-stability. If the monetary policy rule satisfies (41), then the E-stable sunspot equilibria emerge. This is the situation investigated by Honkapohja and Mitra (2004). In this case, the central bank's strong reaction to the expected inflation rate guarantees the E-stability. However, the endogenous fluctuations can occur, because multiple REE satisfy the E-stability. Honkapohja and Mitra (2004) recommend that the monetary policy rule

that the existence of information delays is a key to the discrepancy between the determinacy and the E-stability of REE.

should rule out this possibility. However, if the central bank avoids the emergence of E-stable sunspot equilibria, the REE must be E-unstable. In this sense, the central bank faces a serious trade-off.

In sum, the results indicate that, if private agents are learning from the central bank's forecast, the central bank's policymaking must be more restrictive than in the benchmark case in which both the central bank and private agents engage in homogeneous learning procedure. This means that, if the central bank is the leader of expectation formation, a forward-looking monetary policy rule has more serious problems than those pointed out in Bernanke and Woodford (1997).

## 4.3 A Remedy

As in the previous subsection, we find that a forward-looking policy rule has serious problems when private agents are learning from the central bank's forecast. A possible remedy for this problem is that the central bank additionally introduces the contemporaneous data of the inflation rate into a policy rule. Suppose that the central bank introduces the following monetary policy rule:

$$r_t = \phi_\pi E_t^{CB} \pi_{t+1} + \gamma \pi_t, \tag{44}$$

where $\gamma$ is the responsiveness to the contemporaneous data of the inflation rate. Then, the reduced model has the same form of (5). However, the coefficients are replaced by $A = 0$, $B = \frac{\kappa\sigma+\beta}{1+\kappa\sigma\gamma}$, $C = \frac{-\kappa\sigma\phi_\pi}{1+\kappa\sigma\gamma}$, and $D = \frac{\kappa\sigma}{1+\kappa\sigma\gamma}$.

As in Section 4.1, the determinacy condition is obtained as $|B + C| < 1$. This leads to the following proposition.

**Proposition 6** *The economy of (1), (2), (3), and (44) has a unique REE if and only if* *the following condition holds:*

$$1+\gamma+\frac{\beta+1}{\kappa\sigma}>\phi_\pi>1-\gamma+\frac{\beta-1}{\kappa\sigma}. \tag{45}$$

Thus, the central bank can relax the determinacy condition by increasing the value of $\gamma$. This is a natural consequence because previous studies (including Bullard and Mitra (2002)) have shown that the rule with contemporaneous data is more robust for the determinacy than the rule with forward-looking expectations. By responding to the contemporaneous data of the inflation rate, the central bank can reduce the sensitivity of the economic system to forward-looking expectations. This is why the determinacy is more easily satisfied under rule (44) than (4).

Next, we examine the E-stability condition under rule (44). Suppose that all private agents are learning from the central bank's forecast. Then, we can derive the E-stability condition, following the same steps in Section 3. The result is given by the following proposition.

**Proposition 7** *Suppose that the central bank is MSV learning and all private agents are learning from the central bank's forecast. Then, the REE of (1), (2), (3), and (44) is E-stable if and only if (46) and (47) hold.*

$$\phi_\pi+2\gamma \;>\; 2+\frac{2(\beta-1)}{\kappa\sigma}, \tag{46}$$

$$\phi_\pi+\gamma \;>\; 1+\frac{\beta-1}{\kappa\sigma}. \tag{47}$$

Thus, the E-stability condition is relaxed by introducing the coefficient $\gamma$. By increasing the value of $\gamma$, the central bank can easily attain the E-stability of the REE. The reason for this result is explained by the fact that, in the NKPC (3), the contemporaneous inflation rate is determined by private agents' expected inflation rate. Because of this property, the central bank can respond to the forecast errors of private agents, by responding to the contemporaneous data of the inflation rate.

Therefore, the central bank can simultaneously relax the conditions of determinacy and E-stability, by responding to the contemporaneous movements of the inflation rate. This result suggests that a more robust policy strategy for the central bank is to respond to the contemporaneous movements of the inflation rate to a certain degree.[18]

# 5 Further Analysis

In this section, we provide two additional analysis. First, we check the robustness of our main results by removing the restrictions on monetary policy rule (4). Second, we examine the E-stability of REE in the reverse situation in which private agents are learning from the central bank's forecast.

## 5.1 E-stability in the Absence of the Restrictions on Monetary Policy Rule

Until the previous section, we have imposed the restrictions of $E_t^{CB} x_{t+1} = E_t^P x_{t+1}$ and $\phi_x = \sigma^{-1}$ on monetary policy rule (4). These restrictions imply that the central bank has precise knowledge on these aspects. Here, we examine the robustness of our main results

[18] Apparently, this policy prescription requires that the central bank can obtain the contemporaneous data of the inflation rate in setting the current interest rate. However, we do not insist that responding to the contemporaneous data of the inflation rate is the sole remedy. For example, introducing the interest rate smoothing into the monetary policy rule will be another possible prescription because previous studies, such as Bullard and Mitra (2007), point out that it is useful to avoid the problem of indeterminacy of REE.

by removing these restrictions. In particular, we numerically check whether the Taylor principle is sufficient condition for the E-stability of REE even if we do not impose that the central bank has precise knowledge on $E_t^P x_{t+1}$ and $\sigma$.

In doing so, we restrict our attention to the responsiveness to the expected inflation ($\phi_\pi$) because our analysis up to the previous section has focused on this parameter. In this regard, we introduce the following monetary policy rule which responds only to the expected inflation rate:

$$r_t = \phi_\pi E_t^{CB} \pi_{t+1}. \tag{48}$$

Although this rule imposes another restriction of $\phi_x = 0$ in monetary policy rule (4), it does not require the central bank to have any information on private agents' expectation for future output gap and the slope of IS equation. In this sense, the analysis below does not depend on the central bank's precise knowledge on $E_t^P x_{t+1}$ and $\sigma$.

When we introduce the monetary policy rule (48) into the model which consists of (1) and (2), the model is summarized as the following system:

$$y_t = A^* + B^* E_t^P y_{t+1} + C^* E_t^{CB} y_{t+1} + D^* r_t^n, \tag{49}$$

where $y_t = (x_t,\ \pi_t)'$, $A^* = (0,\ 0)'$, $B^* = \begin{pmatrix} 1 & \sigma \\ \kappa & \kappa\sigma + \beta \end{pmatrix}$, $C^* = \begin{pmatrix} 0 & -\sigma\phi_\pi \\ 0 & -\kappa\sigma\phi_\pi \end{pmatrix}$, and $D^* = (\sigma,\ \kappa\sigma)'$.

We assume that the central bank's PLM is given as follows:

$$y_t = \widetilde{a}^* r_t^n, \tag{50}$$

where $\widetilde{a}^* = \left(\widetilde{a}_1^*,\ \widetilde{a}_2^*\right)'$. For simplicity, we assume that the central bank knows the steady state level of output and the inflation rate. Therefore, the PLM does not include constant term. Based on this PLM, the central bank calculates the forward-looking expectations as follows:

$$E_t^{CB} y_{t+1} = \widetilde{a^*} \rho r_t^n. \tag{51}$$

All private agents are learning from the central bank's forecast. We assume that their PLM is given as follows:

$$y_t = \widetilde{b^*} E_{t-1}^{CB} y_t, \tag{52}$$

where $\widetilde{b^*} = \begin{pmatrix} \widetilde{b_1^*} & 0 \\ 0 & \widetilde{b_2^*} \end{pmatrix}$. (52) implies that private agents also know the steady state level of output and the inflation rate. The reason why the matrix $\widetilde{b^*}$ is diagonal is explained by the assumption that private agents evaluate the historical performance of the central bank's forecast concerning each variable separately.

Based on the PLM, private agents form their forward-looking expectations as follows:

$$E_t^P y_{t+1} = \widetilde{b^*} E_t^{CB} y_{t+1} = \widetilde{b^*} \widetilde{a^*} \rho r_t^n. \tag{53}$$

By substituting (51) and (53) into (49), we obtain the ALM as follows:

$$y_t = \left( (B^* \widetilde{b^*} + C^*) \widetilde{a^*} \rho + D^* \right) r_t^n. \tag{54}$$

To examine the E-stability of equilibrium, we derive the T-maps. From (50) and (54), the T-map about $\widetilde{a}$ is given as follows:

$$T_{a^*} = (B^* \widetilde{b^*} + C^*) \widetilde{a^*} \rho + D^*, \tag{55}$$

where $T_{a^*} = \left( T_{a_1^*},\ T_{a_2^*} \right)'$. $T_{a_1^*}$ and $T_{a_2^*}$ are given as follows:

$$T_{a_1^*} = \rho \widetilde{b_1^*} \widetilde{a_1^*} + \rho \sigma (\widetilde{b_2^*} - \phi_\pi) \widetilde{a_2^*} + \sigma, \tag{56}$$

$$T_{a_2^*} = \kappa \rho \widetilde{b_1^*} \widetilde{a_1^*} + \kappa \rho \sigma (\widetilde{b_2^*} - \phi_\pi) \widetilde{a_2^*} + \beta \rho \widetilde{b_2^*} \widetilde{a_2^*} + \kappa \sigma. \tag{57}$$

From (51) and (52), private agents' projected ALM is given as follows:

$$y_t = T_{b^*} \widetilde{a^*} \rho r^n_{t-1}, \tag{58}$$

where $T_{b^*}(\widetilde{b^*}) = \begin{pmatrix} T_{b_1^*} & 0 \\ 0 & T_{b_2^*} \end{pmatrix}$. The orthogonality conditions are given by

$$\begin{aligned} E\left[\widetilde{a_1^*} \rho r^n_{t-1} \left(x_t - T_{b_1^*} \widetilde{a_1^*} \rho r^n_{t-1}\right)\right] &= 0, & (59) \\ E\left[\widetilde{a_2^*} \rho r^n_{t-1} \left(\pi_t - T_{b_2^*} \widetilde{a_2^*} \rho r^n_{t-1}\right)\right] &= 0. & (60) \end{aligned}$$

From (54), (59), and (60), $T_{b_1^*}$ and $T_{b_2^*}$ are calculated as follows:

$$\begin{aligned} T_{b_1^*} &= \widetilde{a_1^*}^{-1} T_{a_1^*}, & (61) \\ T_{b_1^*} &= \widetilde{a_2^*}^{-1} T_{a_2^*}. & (62) \end{aligned}$$

From (56), (57), (61), and (62), the coefficients at the equilibrium (the fixed point of the T-maps) are calculated as follows:

$$\begin{aligned} \overline{a_1^*} &= \frac{\sigma(1-\beta\rho)}{(1-\rho)(1-\beta\rho) - \kappa\sigma\rho(1-\phi_\pi)}, \\ \overline{a_2^*} &= \frac{\kappa\sigma}{(1-\rho)(1-\beta\rho) - \kappa\sigma\rho(1-\phi_\pi)}, \ \overline{b_1^*} = \overline{b_2^*} = 1. \end{aligned}$$

To examine the stability of coefficients, we define the ODEs as follows:

$$h\begin{pmatrix} \widetilde{a_1^*} \\ \widetilde{a_2^*} \\ \widetilde{b_1^*} \\ \widetilde{b_2^*} \end{pmatrix} \equiv \begin{pmatrix} T_{a_1^*} - \widetilde{a_1^*} \\ T_{a_2^*} - \widetilde{a_2^*} \\ T_{b_1^*} - \widetilde{b_1^*} \\ T_{b_2^*} - \widetilde{b_2^*} \end{pmatrix} = \begin{pmatrix} \rho\widetilde{b_1^*}\widetilde{a_1^*} + \rho\sigma(\widetilde{b_2^*} - \phi_\pi)\widetilde{a_2^*} + \sigma - \widetilde{a_1^*} \\ \kappa\rho\widetilde{b_1^*}\widetilde{a_1^*} + \kappa\rho\sigma(\widetilde{b_2^*} - \phi_\pi)\widetilde{a_2^*} + \beta\rho\widetilde{b_2^*}\widetilde{a_2^*} + \kappa\sigma - \widetilde{a_2^*} \\ \rho\widetilde{b_1^*} + \widetilde{a_1^*}^{-1}(\rho\sigma(\widetilde{b_2^*} - \phi_\pi)\widetilde{a_2^*} + \sigma) - \widetilde{b_1^*} \\ ((\kappa\sigma + \beta)\rho\widetilde{b_2^*} - \kappa\sigma\phi_\pi\rho) + \widetilde{a_2^*}^{-1}(\kappa\rho\widetilde{b_1^*}\widetilde{a_1^*} + \kappa\sigma) - \widetilde{b_2^*} \end{pmatrix}. \tag{63}$$

The Jacobian of (63) is given as follows:

$$Dh\begin{pmatrix}\widetilde{a_1^*}\\ \widetilde{a_2^*}\\ \widetilde{b_1^*}\\ \widetilde{b_2^*}\end{pmatrix}|_{\widetilde{a_1^*}=\overline{a_1^*},\ \widetilde{a_2^*}=\overline{a_2^*},\ \widetilde{b_1^*}=\overline{b_1^*},\ \widetilde{b_2^*}=\overline{b_2^*}}$$

$$=\begin{pmatrix}\rho\overline{b_1^*}-1 & \sigma\rho(\overline{b_2^*}-\phi_\pi) & \rho\overline{a_1^*} & \sigma\rho\overline{a_2^*}\\ \kappa\rho\overline{b_1^*} & (\kappa\sigma+\beta)\rho\overline{b_2^*}-\kappa\sigma\rho\phi_\pi-1 & \kappa\rho\overline{a_1^*} & (\kappa\sigma+\beta)\rho\overline{a_2^*}\\ -\overline{a_1^*}^{-2}(\sigma\rho(\overline{b_2^*}-\phi_\pi)\overline{a_2^*}+\sigma) & \overline{a_1^*}^{-1}\sigma\rho(\overline{b_2^*}-\phi_\pi) & \rho-1 & \overline{a_1^*}^{-1}\sigma\rho\overline{a_2^*}\\ \overline{a_2^*}^{-1}\kappa\rho\overline{b_1^*} & -\overline{a_2^*}^{-2}(\kappa\rho\overline{b_1^*a_1^*}+\kappa\sigma) & \overline{a_2^*}^{-1}\kappa\rho\overline{a_1^*} & (\kappa\sigma+\beta)\rho-1\end{pmatrix}. \tag{64}$$

The E-stability of the REE means that all of the eigenvalues of (64) have negative real parts. This can be numerically examined by setting the values of structural parameters. Specifically, we set these values as $\sigma = 1.0$, $\beta = 0.99$, $\kappa = 0.10$, and $\rho = 0.8$. These values are fairly standard in the literature of New Keynesian models (for example, Ireland (2004)).

Here, our question is whether the Taylor principle is a sufficient condition for the E-stability of the REE or not. To examine this issue, we set the policy responsiveness to the expected inflation rate as $\phi_\pi = 1.5$. This value satisfies the Taylor principle[19]. Under this parameterization, we calculate the eigenvalues of (64) as $0.0231 \pm 0.5173i$ and $-0.4111 \pm 0.3678i$. Therefore, the E-stability condition is violated even though the value of $\phi_\pi$ satisfies the Taylor principle.

Next, we alternatively use the value of $\phi_\pi = 5.0$. Then we calculate the eigenvalues of (64) as $-0.0065 \pm 0.9994i$ and $-0.5215 \pm 0.3002i$[20]. Therefore, this result suggests that a

[19] Note that, in the case of this subsection, the Taylor principle, which is originally defined as $\phi_\pi + \frac{1-\beta}{\kappa}\phi_x > 1$, corresponds to $\phi_\pi > 1$ since we assume $\phi_x = 0$.

[20] We have confirmed that all eignvalues are negative in the case of still larger value of $\phi_\pi$.

stronger response to the expected inflation rate guarantees the E-stability condition. These results suggest that the Taylor principle is not a sufficient condition for the E-stability when private agents are learning from the central bank's forecast, even if we do not impose the central bank's precise knowledge on $E_t^P x_{t+1}$ and $\sigma$.

## 5.2 E-stability when the Central Bank is Learning from Private Agents' Forecast

In our main analysis, we have examined the situation in which private agents are learning from the central bank's forecast. Readers may be interested in the E-stability condition in the reverse situation in which the central bank is learning from private agents' forecast.

To examine this point, we again impose the restrictions of $E_t^{CB} x_{t+1} = E_t^P x_{t+1}$ and $\phi_x = \sigma^{-1}$ on monetary policy rule (4). In addition, we assume that private agents are the leaders and the central bank is the follower of expectation formation. Furthermore, we assume that private agents are MSV learning and the central bank is learning from private agents' forecast by following the PLM analogous to (18).

The derivation of the E-stability condition is just the same as in Section 3. Following similar steps, we obtain the following proposition.

**Proposition 8** *Suppose that all private agents are MSV learning and the central bank is learning from private agents' forecast. Then, the REE of (5) is E-stable if and only if (15) holds.*

***Proof.*** *See Appendix C.* ■

Thus, in this reverse situation, the E-stability condition corresponds to the Taylor principle. Intuitively, this result can be interpreted as follows. In this situation, the central bank's forecast errors exceed the forecast errors of private agents. In order to offset private agents' forecast errors, the central bank's reaction to its own forecast need not to be as large as the Taylor principle suggests (i.e. $\phi_\pi$ can be smaller than unity). However, to offset the central bank's own forecast errors, the Taylor principle is still required. This is why the E-stability condition is given by the Taylor principle.

Therefore, if private agents are the leaders and the central bank is the follower, the E-stability condition is just the same as in the benchmark case, which is investigated in Section 2. In this environment, the central bank can guarantee both the determinacy and the E-stability of the REE by satisfying the Taylor principle. This implies that the central bank can more easily ensure macroeconomic stability in a case in which the central bank is the follower, rather than the leader of expectation formation.

# 6 Conclusion

In this study, we have examined the E-stability of the REE in a standard New Keynesian model in which private agents are learning from the central bank's forecast. More specifically, we have investigated the situation in which private agents introduce the central bank's forecast in their PLM and they determine how to use the forecast information based on their evaluation of the historical performance of the central bank's forecast.

We find that, in contrast to a situation in which both the central bank and private agents homogeneously (or simultaneously) engage in adaptive learning, such as the case of Bullard and Mitra (2002), the E-stability is not attained solely by the so-called Taylor principle. To ensure convergence to the REE, the central bank must respond more strongly to the expected inflation rate than the Taylor principle suggests.

On the other hand, we show that the central bank's strong reaction to the expected inflation rate raises the possibility of indeterminacy of the REE, as pointed out by Bernanke and Woodford (1997). This means that the central bank's policymaking must be more restrictive when the central bank is the leader and private agents are followers in the expectation formation mechanism. In this situation, we find that a robust policy strategy is to respond to the contemporaneous data of the inflation rate to a certain degree because it helps to ensure both of the determinacy and the E-stability of REE.

**Appendix A: Proof of Proposition 1**

The local asymptotic stability of $\widetilde{b}$ and $\widetilde{d}$ is satisfied if and only if all the eigenvalues of the Jacobian of (29) at the REE, which is expressed in (A1), have negative real parts:

$$Dh\begin{pmatrix} \overline{b} \\ \overline{d} \end{pmatrix} |_{\widetilde{b}=\overline{b},\widetilde{d}=\overline{d}}= \begin{pmatrix} \rho(B+C)-1 & \rho B(1-\rho(B+C))^{-1}D \\ -(1-\rho(B+C))^2 D^{-1} & \rho B-1 \end{pmatrix}. \tag{A1}$$

The characteristic polynomial of (A1) is given as follows:

$$\lambda^2+(2-2\rho B-\rho C)\lambda+1-\rho B-\rho C=0. \tag{A2}$$

All the eigenvalues of (A1) have negative real parts if and only if $(2-2\rho B-\rho C)>0$ and $1-\rho B-\rho C>0$. From the definition of $B$ and $C$, it corresponds to the following conditions:

$$\phi_\pi > 2+\frac{2(\beta-\rho^{-1})}{\kappa\sigma}, \tag{A3}$$

$$\phi_\pi > 1+\frac{\beta-\rho^{-1}}{\kappa\sigma}. \tag{A4}$$

Next, we examine the local asymptotic stability of $\widetilde{a}$ and $\widetilde{c}$. The Jacobian of (30) is derived as follows:

$$Dh\begin{pmatrix} \widetilde{a} \\ \widetilde{c} \end{pmatrix} |_{\widetilde{b}=\overline{b},\widetilde{d}=\overline{d}}= \begin{pmatrix} B+C-1 & B \\ B+C-1 & B-1 \end{pmatrix}. \tag{A5}$$

The characteristic polynomial of (A5) is as follows:

$$\lambda^2+(2-2B-C)\lambda+1-B-C=0. \tag{A6}$$

Therefore, the local asymptotic stability of (A5) at REE is satisfied if and only if $2-2B-C>0$ and $1-B-C>0$. These correspond to the following conditions:

$$\phi_\pi > 2+\frac{2(\beta-1)}{\kappa\sigma}, \tag{A7}$$

$$\phi_\pi > 1+\frac{\beta-1}{\kappa\sigma}. \tag{A8}$$

Note that, since $1 > \rho > 0$, (A3) holds if (A7) holds. Similarly, (A4) holds if (A8) holds. Therefore, the E-stability condition corresponds to (A7) and (A8).

**Appendix B: Proof of Proposition 2**

If the central bank and a proportion $1-\mu$ of private agents are MSV learning and a proportion $\mu$ of private agents are learning from the central bank's forecast, the relevant characteristic polynomials are given as follows:

$$\lambda^2 + (2 - 2\rho\widehat{B} - \rho\widehat{C})\lambda + 1 - \rho\widehat{B} - \rho\widehat{C} = 0, \tag{B1}$$

$$\lambda^2 + (2 - 2\widehat{B} - \widehat{C})\lambda + 1 - \widehat{B} - \widehat{C} = 0. \tag{B2}$$

Then, the E-stability condition corresponds to that in which all of $2 - \rho\widehat{B} - 2\rho\widehat{C}$, $1 - \rho\widehat{B} - \rho\widehat{C}$, $2 - \widehat{B} - 2\widehat{C}$, and $1 - \widehat{B} - \widehat{C}$ are strictly positive. These are equivalent to the following conditions:

$$\phi_\pi > 1 + \mu + \frac{(1+\mu)\beta - 2\rho^{-1}}{\kappa\sigma}, \tag{B3}$$

$$\phi_\pi > 1 + \frac{\beta - \rho^{-1}}{\kappa\sigma}. \tag{B4}$$

$$\phi_\pi > 1 + \mu + \frac{(1+\mu)\beta - 2}{\kappa\sigma}, \tag{B5}$$

$$\phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{B6}$$

Since $1 > \rho > 0$, (B3) holds if (B5) holds. Similarly, (B4) holds if (B6) holds. Therefore, the E-stability condition corresponds to (B5) and (B6).

**Appendix C: Proof of Proposition 8**

If all private agents are MSV learning and the central bank is learning from private agents' forecast, the relevant characteristic polynomials are given as follows:

$$\lambda^2 + (2 - \rho B - 2\rho C)\lambda + 1 - \rho B - \rho C = 0, \tag{C1}$$

$$\lambda^2 + (2 - B - 2C)\lambda + 1 - B - C = 0. \tag{C2}$$

Then, the E-stability condition corresponds to that in which all of $2-\rho B-2\rho C$, $1-\rho B-\rho C$, $2 - B - 2C$, and $1 - B - C$ are strictly positive. These are equivalent to the following conditions:

$$\phi_\pi > \frac{1}{2} + \frac{\beta - 2\rho^{-1}}{2\kappa\sigma}, \tag{C3}$$

$$\phi_\pi > 1 + \frac{\beta - \rho^{-1}}{\kappa\sigma}, \tag{C4}$$

$$\phi_\pi > \frac{1}{2} + \frac{\beta - 2}{2\kappa\sigma}, \tag{C5}$$

$$\phi_\pi > 1 + \frac{\beta - 1}{\kappa\sigma}. \tag{C6}$$

Since $1 > \rho > 0$, (C3) holds if (C5) holds. Similarly, (C4) holds if (C6) holds. Furthermore, (C5) holds if (C6) holds. Therefore, the E-stability condition is given by (C6).

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