Expectational Stability in Regime-Switching Rational Expectations Models*

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November 1, 2007

Abstract

Regime-switching rational expectations models, in which the parameters of the model evolve according to a finite state Markov process, have properties that differentiate them from linear models. Issues that are well understood in linear contexts, such as equilibrium determinacy and stability under adaptive learning, re-emerge in this new context. This paper outlines these issues and defines two classes of equilibria that emerge from regime-switching models. The distinguishing feature between the two classes is whether the conditional density of the endogenous state variables depends on past regimes. An assumption on whether agents condition their expectations on past regimes has important implications for determinacy and equilibrium dynamics. The paper addresses the stability properties of the different classes of equilibria under adaptive learning, extending the learning literature to a non-linear framework.

JEL Classifications: E52, E31, D83, D84

Key Words: E-stability, adaptive learning, regime switching, sunspots.

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1 Introduction

A given forward-looking macroeconomic model may admit different classes of rational expectations equilibria. Solutions can differ in terms of the set of state variables that agents use when forming expectations. For example, standard linear stochastic rational expectations models, depending on the precise parameterization, may admit minimal state variable solutions or solutions that depend on extrinsic random variables (i.e. sunspots). These existence issues are well understood in linear models with constant parameters; however, in a growing area of research that focuses on models with changing parameters, these issues are re-emerging.[1] In regime-switching models, which constitute the focus of this paper, parameters evolve according to a finite state Markov process. Since the parameters of the resulting expectational difference equation evolve stochastically, regime switching models incorporate – in a non-linear manner – state variables not present in their constant parameter counterparts. The method for characterizing solutions to models that take this form, and the appropriate class of solutions to consider, are open – and much debated – questions.

This paper addresses the existence and stability of different classes of rational expectations equilibria in regime-switching models. The non-linear structure of the model prevents a complete characterization of the full class of rational expectations solutions and so, motivated by the work of Davig and Leeper (2007) and Farmer, Waggoner and Zha (2007), we define and illustrate the properties of two classes of solutions. The distinguishing feature between the two classes is whether the resulting equilibrium's conditional distribution exhibits explicit dependence on *both* current and lagged regimes. To fix terminology, we define the class where lagged regimes are restricted from entering the state vector as *Regime-Dependent Equilibria* (RDE) and the other class, where lagged regimes enter the state vector, as *History-Dependent Equilibria* (HDE).

Each class of equilibria in this paper has been recently analyzed in the context of some 'off-the-shelf' macroeconomic models. For example, Davig and Leeper (2007) introduce a condition known as the Long Run Taylor Principle (LRTP) that ensures a unique RDE. Farmer, Waggoner, and Zha (2006, 2007) expand on this work by constructing an HDE admitting sunspot shocks even when the LRTP holds, implying that indeterminacy may be an even greater concern in models with regime-switching. This paper generalizes some of these earlier results in a multivariate setting. Adapting Davig-Leeper, whose LRTP is defined in the context of monetary models, we introduce the Conditionally Linear Determinacy Condition (CLDC) – whose meaning will become apparent below – as the condition that guarantees existence of a unique RDE.

[1]Some examples of work in this area include Leeper and Zha (2003), Andolfatto and Gomme (2003), Davig (2004), Zampolli (2006), Chung, Davig and Leeper (2007), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2006, 2007), and Svensson and Williams (2007). Brainard (1967) is an early example of work on parameter instability.

Also, adapting the main theorem of Farmer, Waggoner, and Zha (2007), we generalize the representation of HDE.

Since the essential distinction between RDE and HDE concerns the variables on which agents condition when forming expectations, it is natural to ask whether, and under what conditions, an equilibrium can emerge if agents formulate expectations using a reasonable learning algorithm. This bounded rationality perspective follows an extensive literature in dynamic macroeconomics that adopts learning as an equilibrium selection device: a rational expectations equilibrium is considered theoretically plausible if it can be obtained through a reasonable learning process on the part of agents.[2] This paper studies the stability under learning of RDE and HDE, thereby extending the learning literature, which usually focuses on linear models, to a non-linear framework.

This paper adopts the viewpoint that rational expectations equilibria are plausible only if they are attainable through a process of learning on the part of private-sector agents. As is standard in the learning literature, agents employ forecasting models consistent with the class of equilibria under consideration and update the parameters of the forecasting models as new data become available. Agents form expectations using their forecasting models, and we then determine whether the economy converges over time to a rational expectations equilibrium.

Of central interest to us is whether sunspot equilibria, within either the RDE or HDE class, can be stable under learning. In a constant parameters model, the stability under learning of a sunspot equilibrium may depend on the functional form of the forecasting model used by agents, and furthermore, a given sunspot equilibrium may be consistent with a number of natural forecasting models. For example, Evans and McGough (2005) show that, in a simple univariate model with a lag, if agents use a "general form representation" as their forecasting model – in which they condition on two lags of the endogenous variable and an *unforecastable* susnpot shock – then the associated equilibrium is not stable under learning. However, if agents adopt a "common factor representation" as their forecasting model – in which they condition on one lag of the endogenous variable and a *forecastable* sunspot shock – then the associated rational expectations equilibrium may be stable under learning.

Building on the insight that stability may depend on the particular forecasting model adopted by agents, this paper generalizes the notions of general form and common factor representations to regime-switching models. A primary result of this paper is that if there exists a unique RDE and simultaneously there exist HDE, then the unique RDE *and* the HDE are E-stable provided agents use a common factor representation. Conversely, in two examples, the general form representations of HDE are E-unstable.

[2]This view has been most forcefully advanced by Bray and Savin (1986), Marcet and Sargent (1989), Evans and Honkapohja (2001), and Bullard and Mitra (2002).

A theoretical contribution of this paper is that it provides a formal definition of rational expectations equilibria in regime switching models and illustrates the key dynamic and equilibrium selection properties of these equilibria. The results of the paper are distinct from constant parameter models and suggest that a priority for future research should be to test the distinct empirical implications of RDE and HDE and to study the design of rule based monetary policy in Markov switching environments.[3]

The paper is organized as follows : Section 2 introduces the techniques and approach we apply to regime switching models by first presenting results for the special case of constant parameters; Section 3 defines the two classes of equilibria and studies E-stability for each class of equilibria, Section 4 provides two examples that illustrate the general results, and considers real time learning; Section 5 presents further discussion; and, Section 6 concludes.

2 Equilibria and learning in a constant parameters model

To fix ideas, we begin by presenting results on equilibrium characterization and learning in a multivariate forward-looking linear model with an autoregressive shock. The analysis that follows is a generalization of results in Evans and Honkapohja (2001) and Evans and McGough (2005). Although the paper's main results are in the following sections, it is useful to begin with the constant parameter model to illustrate the techniques and approach to solving regime-switching models.

The model is given by

$$y_t = \beta E_t y_{t+1} + \gamma r_t, \tag{1}$$

$$r_t = \rho r_{t-1} + \varepsilon_t, \tag{2}$$

where y_t is an $(n \times 1)$ vector of random variables, β and γ are conformable matrices, with β invertible and having distinct eigenvalues, and r_t is a $(k \times 1)$ exogenous stationary VAR(1) process.[4]

A rational expectations equilibrium of the model is a solution to (1) that also satisfies a boundary condition. Often the definition of the boundary condition is somewhat

[3] Svensson and Williams (2007) derive a framework for designing optimal policy from a timeless perspective in general Markov-switching environments. Whether the instrument rules that implement those policies avoid the multiple equilibria problem is an open question. Though, it is well-known (e.g. Bernanke and Woodford (1997)) that, in constant parameter models, the nominal interest rate rules that implement optimal policy may render a model indeterminate.

[4] If β does not have distinct eigenvalues, the results presented here can be obtained using the Jordan decomposition.

vague, given as "non-explosiveness" and justified by appealing to a transversality condition, even though the usual transversality condition implies that solutions not explode "too quickly."As an alternative, sometimes the boundary condition requires the paths of variables in a rational expectations equilibrium remain conditionally uniformly bounded, such as in Evans and McGough (2005).

For our purposes, a strong notion of boundedness is useful. Specifically, we will focus on processes satisfying the following property:

Definition. A stochastic process y_t, with initial condition y_0 is *uniformly bounded* (almost everywhere) or UB if $\exists M(y_0)$ so that $\sup_t ||y_t||_\infty < M(y_0)$, where $||\cdot||_\infty$ is the L^∞ or "essential supremum" norm.

With this definition available, we may define a rational expectations equilibrium:

Definition. A *Rational Expectations Equilibrium* is any UB stochastic process satisfying (11).

While uniformly bounded (UB) may appear to be an *a priori* strong notion of boundedness, it is common in the linear rational expectations literature. In linear models with constant parameters, uniform boundedness is consistent with the usual notion of model determinacy, such as in Blanchard and Kahn (1980). Also, UB "bounds the paths" of all endogenous variables and is often desirable when using a first-order approximation to a nonlinear model around a fixed point, such as a steady state. Deviations too far from the steady state can render the dynamics from the first-order approximation invalid. Instead of bounding the paths, however, the following alternative assumption bounds expectations:

Definition. A stochastic process y_t is *conditionally uniformly bounded* (CUB) in expectation if $\sup_t E_t|y_{t+s}| < \infty$.

Conditionally uniformly bounded (CUB) processes remain bounded in expectation, but may assume temporarily explosive trajectories. An alternative notion of boundedness, mean-square stability, has been advanced in the engineering literature and requires a particular bound placed on the first two unconditional moments of a stochastic process. Svensson and Williams (2007) adopt these bounds when solving linear-quadratic policy problems in forward-looking rational expectations models with regime-switching parameters. For an example of CUB solutions in a Markov-switching context see Farmer, Waggoner, and Zha (2006).

In many linear, constant-parameter models, the two definitions of boundedness are operationally equivalent.[5] In a regime-switching framework, expectations can remain bounded, but the path of a variable can temporarily diverge. So, the definitions have different implications for equilibrium dynamics in a switching model when assessing

[5] Of course, the literature on recurrent rational bubbles is a notable exception.

conditions delivering a unique "bounded" equilibrium. Throughout, we will define rational expectations solutions so that the equilibria are uniformly bounded. We will briefly discuss the implication of relaxing the UB assumption and demonstrate that sunspot equilibria abound if one is willing to accept CUB as the appropriate restriction on equilibrium processes.

2.1 Determinacy and Equilibrium Representation

The model (1) is often called *determinate* if it has a unique rational expectations equilibrium – that is, if there is a unique UB solution to (1) – and *indeterminate* if there are multiple rational expectations equilibria. Methods of assessing the determinacy properties of (1) are well-known – see e.g. Blanchard and Kahn (1980). This simple model is determinate if and only if all eigenvalues of β are inside the unit circle.

If the model is determinate, the unique rational expectations equilibrium may be written $y_t = br_t$ where[6]

$$\text{vec}(b) = (I_n - \rho' \otimes \beta)^{-1} \text{vec}(\gamma).$$

and b is $n \times k$. In the case of indeterminacy, there is still a solution of the form $y_t = br_t$ – often called the minimal state variable (MSV) solution – but there are also many others. Let $\Lambda_u \oplus \Lambda_s$ be the diagonal matrix of the eigenvalues of β^{-1} written in decreasing order of modulus so that the n_u eigenvalues outside the unit circle are those on the diagonal of Λ_u, and the n_s eigenvalues inside the unit circle are those on the diagonal of Λ_s. Write

$$S\left(\Lambda_u \oplus \Lambda_s \oplus \rho\right) S^{-1} = \begin{pmatrix} \beta^{-1} & -\beta^{-1}\gamma \\ 0 & \rho \end{pmatrix}.$$

It can be shown that y_t is a rational expectations equilibrium if and only if there is an n_s-dimensional martingale difference sequence ξ_t and $n_u \times (n_s + k)$ matrix Q so that

$$\begin{pmatrix} y_t \\ r_t \end{pmatrix} = S(0 \oplus \Lambda_s \oplus \rho)S^{-1} \begin{pmatrix} y_{t-1} \\ r_{t-1} \end{pmatrix} + \begin{pmatrix} Q \\ I_{(n_s+k)} \end{pmatrix} \begin{pmatrix} \xi_t \\ \varepsilon_t \end{pmatrix}. \tag{3}$$

Equivalently, since r_t is exogenous, we may take y_t to solve

$$y_t = by_{t-1} + cr_{t-1} + \tilde{\xi}_t \tag{4}$$

where $\tilde{\xi}_t$ is a linear combination of the exogenous shock ε_t and the sunspot ξ_t, and the coefficients b and c can be calculated from (3). Equation (4) is the *general form representation* of the sunspot equilibria to (1).

[6] Throughout the paper, $\otimes$ is the Kronecker product and $\oplus$ denotes direct summation.

The sunspot equilibrium represented by equation (4) can be *equivalently* expressed in terms of a common factor representation (Evans and McGough (2005)). To this end, define the $n_s + k$ dimensional exogenous process η_t as follows:

$$\eta_t = (\Lambda_s \oplus \rho)\eta_{t-1} + \begin{pmatrix} 0 & I_{n_s+k} \end{pmatrix} S^{-1} \begin{pmatrix} Q \\ I_{n_s+k} \end{pmatrix} \begin{pmatrix} \xi_t \\ \varepsilon_t \end{pmatrix}. \tag{5}$$

Now notice that

$$S^{-1} \begin{pmatrix} y_t \\ r_t \end{pmatrix} = \begin{pmatrix} 0 \\ I_{n_s+k} \end{pmatrix} \eta_t,$$

implying that y_t satisfies

$$y_t = br_t + c\eta_t, \tag{6}$$

for appropriate b and c. The equation (6) is the *common factor representation* of the sunspot equilibria to (1), and η_t are common factor sunspots. This alternate representation for sunspot equilibria is really quite natural since it consists of a term bearing resemblance to an MSV solution plus a common factor sunspot. MSV solutions are often stable under learning, so the resemblance of the common factor representation to the MSV solution gives insight into why equilibria in this form may be stable under learning.

2.2 Stability Under Learning: E-stability

Consider again the constant parameters model

$$y_t = \beta E_t^* y_{t+1} + \gamma r_t, \tag{7}$$

now written with a (possibly) boundedly rational expectations operator E^*. The learning approach is quite simple: instead of assuming agents in the economy are rational, they are taken to be boundedly rational in the sense that they forecast using a reduced-form model consistent with a rational expectations equilibrium except they do not know the parameters. Over time, agents update their parameter estimates; if these estimates settle down (asymptotically) to the equilibrium parameter values, then the rational expectations equilibrium is stable. Operationally, agents update the parameters of their forecasting model via a recursive algorithm as new data becomes available; they form expectations using the updated forecasting model; these expectations are imposed in (7), thereby generating new data.

We first consider the case where the model is determinate and then, below, we examine the indeterminate case. When the model is determinate, there exists a unique equilibrium that has the form $y_t = br_t$. Agents have a perceived law of motion (i.e. a forecasting model) whose functional form is consistent with the equilibrium representation

$$y_t = A + Br_t. \tag{8}$$

While there is no constant in the equilibrium representation $y_t = br_t$, it is standard to allow agents to consider the possibility that there may be a constant term so as to require agents to learn the steady-state values of y as well.

The parameters A and B capture agents' perceptions of the relationship between y and r and may be estimated using, for example, recursive least squares. Let A_t and B_t be the respective estimates using data up to time t. Agents form forecasts using the perceived law of motion $E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t$. Plugging these forecasts into (7) leads to the actual law of motion

$$y_t = \beta A_{t-1} + (\beta B_{t-1}\rho + \gamma) r_t.$$

Here we assume that agents know the true process governing r_t. The actual law of motion illustrates the manner in which time t endogenous variables are determined by perceptions (A_{t-1}, B_{t-1}) and realizations of r_t. Given new data on y_t agents then update the forecasting model to obtain (A_t, B_t). The unique rational expectations equilibrium $y_t = br_t$ is stable under learning if $(A_t, B_t) \to (0, b)$ almost surely. Stability under learning is non-trivial precisely because of the self-referential nature of rational expectations models. That is, the actual law of motion depends on the perceptions A_{t-1}, B_{t-1} and convergence is not obvious.

While assessing the asymptotic behavior of the non-linear stochastic process (A_t, B_t) is quite difficult, there is an extensive literature that demonstrates the technical requirements for convergence often reduce to a fairly simple and intuitive condition known as E-stability (see Evans and Honkapohja (2001)). To illustrate, suppose agents hold generic beliefs (A, B). The actual law of motion then defines a map $T : \mathbb{R}^n \oplus \mathbb{R}^{n\times k} \to \mathbb{R}^n \oplus \mathbb{R}^{n\times k}$ that takes perceived coefficients to actual coefficients

$$T(A, B) = (\beta A, \beta B\rho + \gamma).$$

Notice that the fixed point of the T-map identifies the unique rational expectations equilibrium of the model. The rational expectations equilibrium is said to be E-stable if it is a locally asymptotically stable fixed point of the ordinary differential equation (o.d.e.)

$$\frac{d(A, B)}{d\tau} = T(A, B) - (A, B). \tag{9}$$

The E-stability Principle states that if agents use recursive least squares – or, similar reasonable learning algorithms – then E-stable rational expectations equilibria are locally stable under learning.[7] In this simple example, if $(0, b)$ is a locally asymptotically stable fixed point of (9) then $(A_t, B_t) \to (0, b)$ almost surely.

The economic intuition behind the E-stability principle is that reasonable learning algorithms dictate that agents update their parameter estimates in the direction of

[7]The connection between E-stability of an rational expectations equilibrium and its stability under real time learning is quite deep: see Evans and Honkapohja (2001) for details.

forecast errors. This is evident in (9), as $T(A,B)-(A,B)$ is, in a sense, a forecast error. If the resting point of the o.d.e. is stable then adjusting parameters in the direction of the forecast error will lead the parameters toward the rational expectations equilibrium. Conveniently, conditions for local asymptotic stability are easily computed by examining the eigenvalues of the Jacobian matrix DT. If all eigenvalues of DT have real parts less than one then the rational expectations equilibrium is E-stable. For the case at hand, the derivatives are given by β and $\rho' \otimes \beta$.[8] Since the model is determinate by assumption, the eigenvalues of β are inside the unit circle and so the rational expectations equilibrium is stable under learning.

Now take the case where the model is indeterminate and there exists a continuum of equilibria. Each equilibrium has a general form representation and a common factor representation. Suppose first that agents observe a common factor sunspot η_t and forms expectations using a forecasting model of the form

$$y_t = A + Br_t + C\eta_t, \tag{10}$$

where η satisfies (5). The T-map is precisely the same as the determinate case for the perceived parameters A and B, so the common factor representations often have the same stability properties as MSV solutions. The additional component of the T-map is $C \to \beta C(\Lambda_s \oplus \rho)$. The derivatives DT are as above and $(\Lambda_s \oplus \rho) \otimes \beta$ so that stability hinges on the eigenvalues of β. Because the model is now indeterminate, some of the eigenvalues of β will have modulus larger than one. However, as long as all these eigenvalues have real part less than one, then the sunspot equilibrium will be stable under learning. Thus, E-stability obtains for eigenvalues of β sufficiently negative.

Now suppose agents observe a martingale difference sequence sunspot ξ_t and form expectations using a forecasting model consistent with the general form representation (4):

$$y_t = A + By_{t-1} + Cr_t + Dr_{t-1} + F\xi_t.$$

Computing the T-map provides the following derivatives

$$\begin{aligned} DT_A &= \beta\,(I+b) \\ DT_B &= b' \otimes \beta + I \otimes (\beta b) \\ DT_C &= \rho' \otimes \beta + I \otimes (\beta b) \\ DT_D &= DT_F = \beta b \end{aligned}$$

where b is the coefficient obtained from (4). The complicated nature of the derivatives DT makes general statements difficult, but the central insight emerges clearly in the univariate case (i.e. $n=1$). In this case, $b = \beta^{-1}$ implying that $DT_B = 2$. So if agents use general form representations to form expectations, then for $n=1$, the

[8]Here, and below, we exploit that when the T-map decouples, we can compute derivatives separately. Also, recall that the eigenvalues of the Kronecker product are the products of the eigenvalues.

sunspot equilibria are never stable under learning. Also, in the univariate case, if $\beta < -1$ then the model is indeterminate and, moreover, if agents use common factor representations to form expectations, then the susnpot equilibria are stable under learning. These results in the constant parameter case underscore that stability may depend on the functional form of the forecasting model. The next Section expands on the implications for learning of the different representations in a regime-switching framework.

3 Equilibria and Learning In A Regime Switching Model

We now relax the assumption of constant parameters in rational expectations models and characterize two classes of equilibria. We focus on models whose reduced form consists of a system of non-linear expectational difference equations such as

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t, \tag{11}$$

$$r_t = \rho r_{t-1} + \varepsilon_t, \tag{12}$$

where y_t is an $(n \times 1)$ vector of random variables, β_t and γ_t are conformable matrices that follow an m state Markov process with $(\beta_t = \beta_i, \gamma_t = \gamma_i) \Leftrightarrow s_t = i, i = 1, 2, ..., m$, and r_t is a $(k \times 1)$ exogenous stationary VAR(1) process independent of s_j for all j. The stochastic matrix P governs the evolution of the state, s_t, and contains elements

$$p_{ij} \equiv \Pr\left[s_t = j | s_{t-1} = i\right],$$

for $i, j \in \{1, 2, ..., m\}$. P is taken to be recurrent and aperiodic, so that it has a unique stationary distribution Π. For simplicity, β_i is taken to be invertible for all i. Davig and Leeper (2007) consider a version of this model in the context of a univariate monetary model and a bivariate New Keynesian model. Most macroeconomic models feature expectational structures similar to (11) – albeit with constant parameters – making (11) a natural laboratory to study the existence and stability of rational expectations equilibria in regime-switching models. In Section 4, we present a univariate example and New Keynesian example.

Analogous to the definition in Section 2, a rational expectations equilibrium is any UB process satisfying (11). However, an important difference that arises in regime-switching rational expectations models, versus constant-parameter models, is that agents incorporate the probability of a regime change into their expectations. The resulting non-linear structure that is inherent in such models presents difficulties when seeking to characterize the full class of rational expectations equilibria. However, two classes naturally emerge, which we define as *Regime-Dependent Equilibria* (RDE) and *History-Dependent Equilibria* (HDE).

3.1 Regime-Dependent Equilibria

3.1.1 Definition and Existence

The first class follows Davig and Leeper (2007) and focuses on state-contingent solutions that allow the current realization of the regime, s_t, to enter the state vector, but are otherwise independent of its history. The state vector also includes current realizations of the exogenous shocks, as well as (possibly) sunspot variables. Formally, the definition for an RDE is as follows:

Definition. Let s_t be the Markov process governed by P and taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (11). Then y_t is a *Regime Dependent Equilibrium* (RDE) if it is uniformly bounded and there exist uniformly bounded stochastic processes $y_{1t}, y_{2t}, ..., y_{mt}$, with y_{it} independent of s_{t+j} for all integers j, such that $y_t = y_{it} \Leftrightarrow s_t = i$.

In an RDE, depending on the realization of s_t, y_t can take on realizations from m stochastic processes, with each process being independent of the Markov state. To compute these equilibria, condition (11) on each regime to get the following system

$$\begin{aligned}
y_{1t} &= \beta_1 p_{11} E_t y_{1t+1} + \beta_1 p_{12} E_t y_{2t+1} + \cdots + \beta_1 p_{1m} E_t y_{mt+1} + \gamma_1 r_t, \\
y_{2t} &= \beta_2 p_{21} E_t y_{1t+1} + \beta_2 p_{22} E_t y_{2t+1} + \cdots + \beta_2 p_{2m} E_t y_{mt+1} + \gamma_2 r_t, \\
\vdots \;\; &\;\; \vdots \qquad\qquad\qquad\qquad \vdots \\
y_{mt} &= \beta_m p_{m1} E_t y_{1t+1} + \beta_m p_{m2} E_t y_{2t+1} + \cdots + \beta_m p_{mm} E_t y_{mt+1} + \gamma_m r_t,
\end{aligned}$$

which governs dynamics for y_{it} for $i = 1, 2, ..., m$.[9] The system can be recast in the form of a 'stacked system', which has a more compact representation.

Definition. The *Stacked System* associated with the switching model (11) is the system of multivariate linear expectational difference equations

$$\hat{y}_t = (\oplus_{j=1}^m \beta_j)(P \otimes I_n) E_t \hat{y}_{t+1} + \gamma r_t \tag{13}$$

where $\widehat{y}_t = [y'_{1t}, y'_{2t}, ..., y'_{mt}]'$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_m)'$.

The following proposition summarizes the relationship between solutions to the stacked system (13) and UB solutions to (11).

Proposition 1 *There is a bijection between the RDE of (11) and UB solutions to (13). Specifically,*

1. *If $\hat{y}_t$ solves (13) and is UB then $y_t = \hat{y}_{it} \Leftrightarrow s_t = i$ is an RDE.*

[9]The reason history independence is needed in the definition of an RDE should now be clear. Independence allows us to write $E_t y_{it+1}$ in each row of the stacked system independently of s_t.

2. If $y_t = y_{it} \Leftrightarrow s_t = i$ is an RDE then $\hat{y}_t = (y'_{1t}, y'_{2t}, ..., y'_{mt})'$ is a UB solution to (13).

All proofs are contained in the Appendix. The power of this proposition is that all the results and techniques reviewed in Section 2 can be brought to bear on the stacked system (13). For example, necessary and sufficient conditions for a unique RDE are as follows:

Corollary 2 *There is a unique RDE if and only if the eigenvalues of $(\oplus_{j=1}^{m}\beta_j)(P \otimes I_n)$ are inside the unit circle.*

In the context of monetary analysis, Davig and Leeper (2007) refer to the uniqueness condition as the *Long Run Taylor Principle* (LRTP). In this respect, Davig and Leeper (2007) completely characterize the uniqueness conditions for RDE in some standard monetary models. Davig and Leeper, however, restrict attention to models with monetary policies with positive feedback so that the eigenvalues of $\oplus\beta_j$ are positive. The condition in Corollary 2 is more general and so we refer to it as the Conditionally Linear Determinacy Condition (CLDC), since it is the necessary and sufficient condition for existence of a unique RDE.

In a monetary model, the CLDC permits a central bank to briefly engage in 'dovish' monetary policy without inducing sunspot equilibria, though such policy does result in greater volatility in response to exogenous shocks. To illustrate intuition, Davig and Leeper (2007) present a simple example of a univariate monetary model with a closed form solution and an analytically tractable uniqueness condition. The example assumes $s_t \in \{1, 2\}$ and y_t is univariate. The following corollary gives the CLDC for this special case.

Corollary 3 *For $s_t \in \{1, 2\}$, $\beta_i^{-1} > 1$ for some i and $\beta_i^{-1} > p_{ii}$ for all i, there exists a unique RDE if and only if*

$$\beta_1\beta_2 + (1 - \beta_2)\beta_1 p_{11} + (1 - \beta_1)\beta_2 p_{22} < 1. \tag{14}$$

Subsequent sections show a close connection between the conditions for unique RDE and E-stable rational expectations equilibria, and so the CLDC takes on added importance below.

3.1.2 E-Stability of RDE

We now consider representations of RDE and their stability properties under learning. Suppose conditions for a unique RDE are satisfied then the unique RDE will have the following minimal state variable representation

$$y_t = B(s_t)r_t. \tag{15}$$

To solve for $B(s_t)$ for $s_t \in \{1, 2, ..., m\}$, use the stacked system and set $B = (B(1)', ..., B(m)')'$, which yields $\hat{y}_t = Br_t$, where

$$\text{vec}(B) = \left(I_{nm} - \rho' \otimes \left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right)^{-1} \text{vec}(\gamma).$$

It is worth remarking at this point that the class of RDE includes the MSV solution to the regime-switching model.

When the uniqueness conditions are not satisfied, the RDE may depend on extraneous sunspot variables. In this case, we may proceed to analyze the stacked system using precisely the methods described in Section 2. In particular, if the matrix $\left(\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right)^{-1}$ has n_s eigenvalues inside the unit circle then for each n_s-dimensional martingale difference sequence ξ_t there is a martingale difference sequence $\tilde{\xi}_t$ and an equilibrium $\hat{y}_t$ with a general form representation given by

$$\hat{y}_t = b\hat{y}_{t-1} + cr_{t-1} + \tilde{\xi}_t, \tag{16}$$

There is a sense, though, in which the general form sunspot equilibria are not 'in the spirit' of regime dependent equilibria. In (16) the reduced-form expression depends explicitly on $\hat{y}_{t-1}$ and so to forecast using this equation agents will condition on both s_t and s_{t-1}. It is more natural instead to focus on common factor sunspots. It is straightforward to verify that there also exists an $n_s + k$-dimensional martingale difference sequence $\hat{\xi}_t$ so that if[10]

$$\eta_t = (\Lambda_{n_s} \oplus \rho)\eta_{t-1} + \hat{\xi}_t \tag{17}$$

then $\hat{y}_t$ has the common factor representation

$$\hat{y}_t = br_t + c\eta_t. \tag{18}$$

Here, Λ_s has the stable eigenvalues of $\left(\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right)^{-1}$ on its diagonal. Finally, if (i.) $\hat{\xi}_t$ is any $n_s + k$-dimensional martingale difference sequence, (ii.) η_t satisfies (17), and (iii.) $\hat{y}_t$ satisfies (18) then $\hat{y}_t$ is an RDE.

Using the representations (15) and (18) as our guide to specifying a perceived law of motion, we now turn to the stability of RDE under learning. Throughout, we assume that agents observe the current state s_t and know the true transition probabilities. This is consistent with the conventions of the adaptive learning literature that assumes agents observe contemporaneous exogenous variables, but not current values of endogenous variables.

[10] $\hat{\xi}_t$ is also a linear combination of ε_t and ξ_t.

First, assume the CLDC is satisfied so that there is a unique RDE – that is, the eigenvalues of $\left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)$ are inside the unit circle. Agents have a perceived law of motion (PLM) of the following form, which is consistent with the MSV solution,

$$y_t = A(s_t) + B(s_t)r_t \tag{19}$$

where $A(j)$ is $(n\times 1)$, and $B(j)$ is $(n\times k)$. Notice that, just as in Section 2, we assume that agents do not know that in equilibrium the $A_i = 0$.[11]

Given the PLM in (19), expectations are state contingent, where $s_t = j$ implies

$$\begin{aligned} E_t\left[y_{t+1}|s_t = j\right] &= p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m) + \qquad (20)\\ &\quad \left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho r_t. \qquad (21)\end{aligned}$$

This produces a state-contingent ALM, or, equivalently, a state-contingent T-map

$$\begin{aligned} A(j) &\rightarrow \beta_j\left(p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m)\right)\\ B(j) &\rightarrow \beta_j\left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho + \gamma_j.\end{aligned}$$

Conveniently, this state-contingent T-map may be stacked, and becomes the T-map associated to the stacked system under the PLM $\hat{y}_t = A + Br_t$, where, as before, $B = (B(1)', ..., B(m)')'$, and also $A = (A(1)', ..., A(m)')'$. The T-map is given by

$$T(A,B)' = \left(\left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)A, \left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)B\rho + \gamma\right),$$

and the RDE is a fixed point of $T(A,B)$. Here $T:\mathbb{R}^{(nm\times 1)}\oplus\mathbb{R}^{(nm\times k)}\longrightarrow\mathbb{R}^{(nm\times 1)}\oplus\mathbb{R}^{(nm\times k)}$.

The eigenvalues of the Jacobian matrices

$$\begin{aligned} DT_A &= \left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)\\ DT_B &= \rho'\otimes\left[\left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)\right]\end{aligned}$$

govern E-stability. Thus, we obtain the following result:

Proposition 4 *If the eigenvalues of $(\oplus_{j=1}^{m}\beta_j)(P\otimes I_n)$ are inside the unit circle (i.e. the CLDC holds), then the unique RDE is E-stable.*

[11] In the univariate case below, we consider real time learning based on an alternative perceived law of motion of the following form

$$y_t = A + \hat{A}(s_t - 1) + Br_t + \hat{B}(s_t - 1)r_t,$$

where s_t acts as a dummy variable. While this formulation is more natural for real time learning, the E-stability results are identical in both cases and so we focus on the more parsimonious (19) As an additional alternative, agents can have a PLM with the same form as the stacked system.

This result states that an economy described by the main expectational difference equation (11), with expectations formed using (20) and updated using least squares, will converge to the unique RDE.

Now we turn to the case of multiple RDE. To examine the stability of RDE sunspot solutions we assume agents have a PLM consistent with the common factor representation of y_t. As discussed above, examining E-stability based on the general form representation of the RDE is not natural because it involves a lagged term, which will make the perceived law of motion depend on s_{t-1}, which is not in the spirit of regime dependent equilibria.

Assume $\left(\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right)^{-1}$ is diagonalizable and has n_s eigenvalues inside the unit circle. For a given n_s-dimensional martingale difference sequence ξ_t let

$$\eta_t = (\Lambda_{n_s} \oplus \rho)\eta_{t-1} + \hat{\xi}_t. \tag{22}$$

be the associated common factor sunspot. Agents are given the PLM

$$\hat{y}_t = A(s_t) + B(s_t)r_t + C(s_t)\eta_t, \tag{23}$$

where the dimension of $C(s_t)$ is $n \times (n_s + k)$. Setting $C = (C(1)', ..., C(m)')'$, we find that the T-maps for A and B are the same as in case of unique RDE, and the T-map for C is given by

$$C \to \left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\, C(\Lambda_{n_s} \oplus \rho),$$

so

$$DT_C = (\Lambda_{n_s} \oplus \rho)' \otimes \left(\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right).$$

The following proposition summarizes the stability properties for the common factor representation of sunspot RDE.

Proposition 5 *Common factor RDE sunspot solutions are E-stable provided the real parts of the diagonal entries in* $\Lambda_{n_s}^{-1}$ *are less than* -1.

This proposition indicates that common factor sunspot representations of the RDE will not be E-stable unless the economic model generates sufficiently strong negative feedback. The canonical New Keynesian model, augmented with a nominal interest rate rule that responds to inflation and output, features positive expectational feedback. Other interest rate rules – for example, a rule that responds to expected inflation and output – may generate negative feedback.

3.2 History Dependent Equilibria

3.2.1 Definition and Existence

Farmer, Waggoner, and Zha (2007) consider whether UB solutions exist that are not within the class of RDE. They show that UB solutions admitting sunspots do exist, even when there is a unique RDE. The central element in Farmer, Waggoner, and Zha (2006, 2007) (FWZ), is they allow lagged states to enter the state vector. That is, FWZ have agents conditioning their expectations on an expanded state vector that includes s_{t-1}. For this reason, we call the class of solutions *History Dependent Equilibria* (HDE). By assuming agents condition on current and past realizations of the state variable s_t, the class of bounded equilibria now include solutions that depend on arbitrary sunspot variables.

Definition. Let s_t be the Markov process governed by P, taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (11). Then y_t is a *History Dependent Equilibrium* (HDE) if it is uniformly bounded and its distribution conditional on s_t differs from its distribution conditional on s_t and s_{t-1}; that is, $y_t|s_t \nsim y_t|(s_t, s_{t-1})$.

Remark. The definition of a History Dependent Equilibrium restricts solutions to the class of uniformly bounded stochastic processes whose conditional density exhibits dependence on s_t and s_{t-1}. Notice that if y_t is an RDE then $y_t|s_t \sim y_t|(s_t, s_{t-1})$. Then by definition and Proposition 1, HDE can not solve the stacked system.

We now turn to representations of HDE, which we use to conduct stability analysis. We illustrate representations and stability results assuming $\gamma_t = 0$ for all t, as the presence of exogenous shocks do not alter the results and do distract from the presentation. Note that if ξ_t is any m.d.s., then $y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t$ is a solution to (11). Farmer, Waggoner, and Zha (2007) show that there exist multiple uniformly bounded HDE that have the following representation

$$y_t = \left(\frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) y_{t-1} + v_{s_t} \xi_t, \tag{24}$$

provided there exists $c_1, \ldots, c_m$ and $v = (v'_1, \ldots, v'_m)' \neq 0$ so that $|c_j| \leq 1$ and c and v solve

$$\left[\left(\oplus_{j=1}^m \beta_j \right)^{-1} - \left(\left(\oplus_{j=1}^m c_j \right) P \right) \otimes I_n \right] v = 0. \tag{25}$$

Here ξ_t is independent of s_{t+n} for all n. The condition (25) is essentially derived from the method of undetermined coefficients. When (25) is satisfied, solutions to the representation (24) are solutions to (11).[12] The construction of the autoregressive parameter in the representation (24) is chosen so that, regardless of the history of realizations of s_t, these parameters are bounded in matrix norm and, hence, the solutions are uniformly bounded.

The HDE also have a common factor representation, which we take to have the

[12]If one were to literally use the method of undetermined coefficients, the v in (25) would be y_t. However, if v is taken to be a vector of initial conditions chosen to lie on the stable manifold, and if (25) is satisfied at $t = 1$, then it will be satisfied for all t.

form

$$y_t = \eta_t \tag{26}$$

where

$$\eta_t = \left(\frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) \eta_{t-1} + v_{s_t} \xi_t.$$

The common factor representation here is different in nature than in the constant parameters model and the RDE where the extrinsic noise was independent of s_t. Here the dependence is explicit in the autoregressive parameter. This has important implications for stability under learning of HDE.

Farmer, Waggoner, and Zha (2007) illustrate that in a multivariate model, it is possible for there to exist HDE even when the RDE is unique. In this case, conditions ensuring uniqueness within the class of RDE, such as the CLDC, does not ensure equilibrium determinacy in the regime-switching rational expectations framework.

By defining HDE as rational expectations equilibria representations that exhibit dependence on both s_t and s_{t-1}, it is possible to generalize the existence of equilibria even further than (26). Assume HDE take the form[13]

$$y_t = B(s_{t-1}, s_t) y_{t-1} + C(s_{t-1}, s_t) \xi_t. \tag{27}$$

It is straightforward to verify, for this posited solution, that HDE must satisfy

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B_{jk} \right) \right) B(i,j) = 0 \tag{28}$$

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B_{jk} \right) \right) C(i,j) = 0 \tag{29}$$

Notice that provided non-zero $B(i,j)$ satisfy (28), then $C(i,j)$ is arbitrary. We have the following result.

Proposition 6 *Let y_t be a solution to (11). Then y_t is an HDE if it is also a solution to (27) such that $B(i,j), C(i,j)$ satisfy (28)-(29) for all i,j and there exists real numbers $M_1, ..., M_m$ so that $\|B(i,j)\| < \frac{M_i}{M_j}$.*

Remark. It is straightforward to verify that (24) is a solution to (27). Also, Farmer, Waggoner, and Zha (2007) prove a theorem providing sufficient conditions for the existence of the numbers M_i mentioned in the proposition and thereby establish a fairly general existence result.

[13] Adopting the earlier notation, since $\gamma_t = 0$ it follows that $\tilde{\xi}_t = \xi_t$.

3.2.2 E-Stability of HDE

We begin by considering the stability of the general form representation. In this case, the PLM takes the following form:

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t \tag{30}$$

where ξ_t is the m.d.s. sunspot variable independent of the Markov states. The PLM makes clear the primary distinction of HDE from the class of RDE solutions, since coefficients depend explicitly on s_t and s_{t-1}, whereas coefficients in the PLM for the RDE only depend on s_t.

Taking expectations conditional on the PLM given by (30) and values of (s_{t-1}, s_t) yields

$$\begin{aligned} E_t(y_{t+1}|s_{t-1} &= i, s_t = j) = \sum_{k=1}^{m} p_{jk}A(j,k) + \\ &\left(\sum_{k=1}^{m} p_{jk}B(j,k)\right)\left(A(i,j) + B(i,j)y_{t-1} + C(i,j)v_t\right). \end{aligned}$$

The T-map is given by

$$A(i,j) \rightarrow \beta_j\left[\sum_{k=1}^{m} p_{jk}A(j,k) + \left(\sum_{k=1}^{m} p_{jk}B(j,k)\right)A(i,j)\right], \tag{31}$$

$$B(i,j) \rightarrow \beta_j\left(\sum_{k=1}^{m} p_{jk}B(j,k)\right)B(i,j), \tag{32}$$

$$C(i,j) \rightarrow \beta_j\left(\sum_{k=1}^{m} p_{jk}B(j,k)\right)C(i,j). \tag{33}$$

E-stability is determined by the Jacobian $DT(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A}, \bar{B}, \bar{C}$ are the HDE parameters found by solving (28)-(29). Given the complexity of the Jacobian, we are not able to obtain general E-stability results for the general form representation of HDE. The next Section presents results for a univariate and New Keynesian example.

We now analyze the common factor representation of HDE. Assume agents hold a PLM consistent with (26)

$$y_t = A(s_{t-1}, s_t) + B\eta_t,$$

where η_t is a sunspot that follows an AR(1) process with regime switching parameters

$$\eta_t = \phi(s_{t-1}, s_t)\eta_{t-1} + \theta(s_{t-1}, s_t)\xi_t. \tag{34}$$

This is the FWZ common factor representation for appropriately defined ϕ, θ. Notice that the perceived coefficient on the sunspot shock η_t in (34) is not state-dependent. Due to its construction, the common factor sunspot η_t is state dependent whereas in equilibrium, its coefficient is not. Taking expectations conditional on this PLM leads to the T-map

$$A(i,j) \to \beta_j \sum_{k=1}^{m} p_{jk} A(j,k), \tag{35}$$

and $T(B) = B$.[14]

The following proposition provides E-stability conditions for common factor representations of HDE.

Proposition 7 *Assume HDE exist. If there exists a unique E-stable RDE, then the common factor representation of the HDE is E-stable.*

Remark. When the CLDC does not hold, and there exists HDE and indeterminate RDE, then the common factor representations of HDE and RDE are E-stable under the same set of conditions. We, thus, conclude that the same conditions govern E-stability of rational expectations equilibria in regime-switching models regardless of the class of equilibria that is of particular interest.

That HDE are E-stable, provided they exist, when there exists unique E-stable RDE is a primary result of this paper. In particular, if the model is parameterized to ensure uniqueness within the class of RDE (perhaps via a policy rule), then this is not sufficient to guarantee agents' learning process will not settle down on sunspot equilibria. Such a result does not arise in constant parameter models where conditions for ruling out sunspot equilibria are more straightforward, and the expectational stability of sunspot equilibria is often elusive.

4 Examples

In this Section we illustrate the results of this paper by examples. Using a simple univariate model, we consider HDE, its properties, and real time learning of RDE. In a New Keynesian example, we consider the stability of the RDE and the HDE based on an empirically realistic calibration.

[14]The result $T(B) = B$ is standard in models with sunspots and reflects the fact that multiples of sunspots are typically also sunspots. The same type of result obtains for general form representations.

4.1 Univariate Model: RDE and HDE

To provide a concrete illustration of the class of solutions, we consider the special case where y_t is univariate, s_t takes values in $\{1, 2\}$, and $\gamma_t = 0$. Then

$$y_t = \beta_t E_t y_{t+1}. \tag{36}$$

Note that in this case, if there is a unique RDE, it is, trivially, $y_t = y_{it} \Leftrightarrow s_t = i$, where $y_{it} = 0$ for $i = 1, 2$.

To compute an HDE, recall that a rational expectations equilibrium is a process y_t such that

$$y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t \tag{37}$$

where ξ_t is an m.d.s. that satisfies $E_{t-1}\xi_t = 0$. Of particular interest is the case in which "one regime is determinate and one regime is indeterminate," or, formally, for example, $|\beta_1| < 1 < \beta_2$. In this case, the non-degenerate case is where the exploding regime is not absorbing so that $p_{22} > 0$. Define

$$\xi_t = \begin{cases} -\beta_1^{-1} y_{t-1} + \delta_{11} v_t & (s_{t-1}, s_t) = (1,1) \\ \frac{P_{11}}{P_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1,2) \\ -\beta_2^{-1} y_{t-1} + \delta_{21} v_t & (s_{t-1}, s_t) = (2,1) \\ \frac{P_{21}}{P_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2,2) \end{cases}$$

where $\delta_{ij} \in \mathbb{R}$ is arbitrary, and v_t is any martingale difference sequence with uniformly bounded support. Dynamics for y_t follow

$$y_t = \begin{cases} \delta_{11} v_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{P_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21} v_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{P_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2,2) \end{cases}, \tag{38}$$

Note that provided $|\beta_2 P_{22}| > 1$, y_t is UB. The process given by (38) is an HDE, since dynamics explicitly depend on s_t and s_{t-1}. Notice that the indeterminacy of region 2 spills over across regimes so that there is sunspot dependence in both regimes. It should be clear from this representation of an HDE that it is not possible to represent this class of equilibria in terms of a stacked system. In an RDE y_t switches between two stochastic processes that are independent of the underlying Markov state. In an HDE the value of y_t depends on the current state s_t and also explicitly on the Markov state in the previous period. This dependence is self-fulfilling in the sense that it exists only because agents expect it.

We now turn to the learnability of the univariate HDE given in (38). In this case, the parameters A, B, C in the PLM (30) are elements of the real line. Computing conditional forecasts using this PLM, we obtain the following T-map for B:

$$B(i,j) \longrightarrow \beta_j \left(P_{j1} B(j,1) + P_{j2} B(j,2) \right) B(i,j) \tag{39}$$

Ignoring the boundedness requirement, a fixed point of this map identifies an HDE. The only restrictions, then, are the following:

$$1 = \beta_1 \left(P_{11} B(1,1) + P_{12} B(1,2) \right) = \beta_2 \left(P_{21} B(2,1) + P_{22} B(2,2) \right). \tag{40}$$

In particular, there is a two dimensional continuum of coefficients on lagged y providing fixed points.

Farmer, Waggoner, and Zha (2006) focus on particular fixed points, given by

$$B(1,1) = B(2,1) = 0, \;\; B(1,2) = \frac{\beta_1^{-1}}{P_{12}}, \;\; B(2,2) = \frac{\beta_2^{-1}}{P_{22}},$$

and $d(i,j) = \delta_{ij}$.

To analyze stability we compute the eigenvalues of DT. The first four equations decouple, and, when evaluated at the fixed point, provide the following Jacobian:

$$DT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta_1^{-1}\beta_2 P_{21}/P_{12} & \beta_1^{-1}\beta_2 P_{21}/P_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & P_{21}/P_{22} & 2 \end{pmatrix}.$$

The Jacobian has an eigenvalue of 2, which implies that the general form HDE is E-unstable. Importantly, this computation is fixed point specific, so we can only conclude that this particular equilibrium is unstable.

However, the HDE also has a common factor representation given by

$$y_t = \eta_t$$

where

$$\eta_t = \begin{cases} \delta_{11} v_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{P_{12}} \beta_1^{-1} \eta_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21} v_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{P_{22}} \beta_2^{-1} \eta_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2,2) \end{cases}$$

The PLM consistent with the common factor representation is

$$y_t = a(s_{t-1}, s_t) + b\eta_t, \tag{41}$$

yielding

$$a(i,j) \to \beta_j \left(p_{j1} a(j,1) + p_{j2} a(j,2) \right), \tag{42}$$

and, as in the general case, $b \to b$. The eigenvalues of the Jacobian DT_a are a pair of zeros and the eigenvalues of $(\beta_1 \oplus \beta_2) P$. Therefore,

Proposition 8 *In the univariate model where s_t takes values in $\{1,2\}$, if HDE exist then*

1. *The general form representation of the HDE given in (38) is E-unstable.*
2. *Given β_i there exists $\bar{\beta}_j < 0$, $j \neq i$, so that $\beta_j < \bar{\beta}_j$ implies that the common factor representation of the HDE given in (38) is E-stable.*

4.2 Real Time Learning of RDE

So far we have identified E-stability with stability under real-time learning. This connection is made in constant parameter models by Evans and Honkapohja (2001). However, it is not clear that the results in Evans and Honkapohja (2001) apply to the regime-switching framework. To address this issue, we present a real time learning formulation of regime dependent equilibria.

We again take y_t to be univariate, and assume s_t takes values in $\{1,2\}$, but now we allow γ_t to be non-trivial. The model is given by

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t. \tag{43}$$

Assume $(\beta_1 \oplus \beta_2)P$ has eigenvalues inside the unit circle, so that there is a unique RDE. To consider the stability under learning of this RDE, we provide agents with the following forecasting model

$$y_t = A + \hat{A}\hat{s}_t + Br_t + \hat{B}\hat{s}_t r_t,$$

where $\hat{s}_t = s_t - 1$. To simplify notation, let $\theta = \left(A, \hat{A}, B, \hat{B}\right)'$ and $X = (1, \hat{s}_t, r_t, \hat{s}_t r_t)'$. Agents estimate θ by regressing y_t on X_t. Letting θ_t be the time t estimate of θ, the recursive formulation of this estimation procedure is given by

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} R_t^{-1} X_t \left(y_t - \theta_{t-1}' X_t\right) \\ R_t &= R_{t-1} + t^{-1}\left(X_t X_t' - R_{t-1}\right). \end{aligned} \tag{44}$$

The matrix R consists of the sample second moments of the regressors. The agents use these estimates, together with their forecasting model, to form expectations. These expectations are embedded into the expectational difference equation to obtain the actual law of motion and associated T-map: the ALM may then be written $y_t = T(\theta_{t-1})X_t$. The T-map is given by

$$\begin{aligned} A &\to \beta_1(A + \hat{A}(1 - P_{11})) \\ \hat{A} &\to \beta_2(A + \hat{A}P_{22}) - \beta_1(A + \hat{A}(1 - P_{11})) \\ B &\to \beta_1(B + \hat{B}(1 - P_{11}))\rho \\ \hat{B} &\to \beta_2(B + \hat{B}P_{22})\rho - \beta_1(B + \hat{B}(1 - P_{11}))\rho. \end{aligned}$$

Imposing this into the algorithm (44) identifies a dynamic system that can be analyzed using the theory of stochastic recursive algorithms. Letting θ^* be the fixed point of the T-map identifying an RDE, the learning question is, does θ_t converge to θ^* almost surely? We have the following proposition.

Proposition 9 *If $\gamma_t = 0$, then, locally, $\theta_t \to \theta^*$ almost surely.*[15]

The restriction γ_t is needed to simplify the proof, though we feel it is very likely that the proposition holds for $\gamma_t \neq 0$. The difficulty raised by non-zero γ reflects the fact that the state dynamics are not conditionally linear and so the usual theorems of stochastic recursive theory do not apply.

To illustrate this result for $\gamma \neq 0$, we use simulations. We parameterize the model so that the CLDC is satisfied. This ensures the existence of a unique rational expectations equilibrium that is also an RDE. We set $\beta_1 = 1/1.5, \beta_2 = 2, p_{11} = .95, p_{22} = .2, \rho = 0, \gamma_1 = 1, \gamma_2 = .5$. For these parameter values the unique RDE coefficients are $A_1 = A_2 = 0, B_1 = 1, B_2 = .5$. We draw initial conditions for θ randomly and simulate the model for 5000 time periods. Figure 1 plots a typical simulation. As the figure makes clear, the RDE is stable under least squares learning.

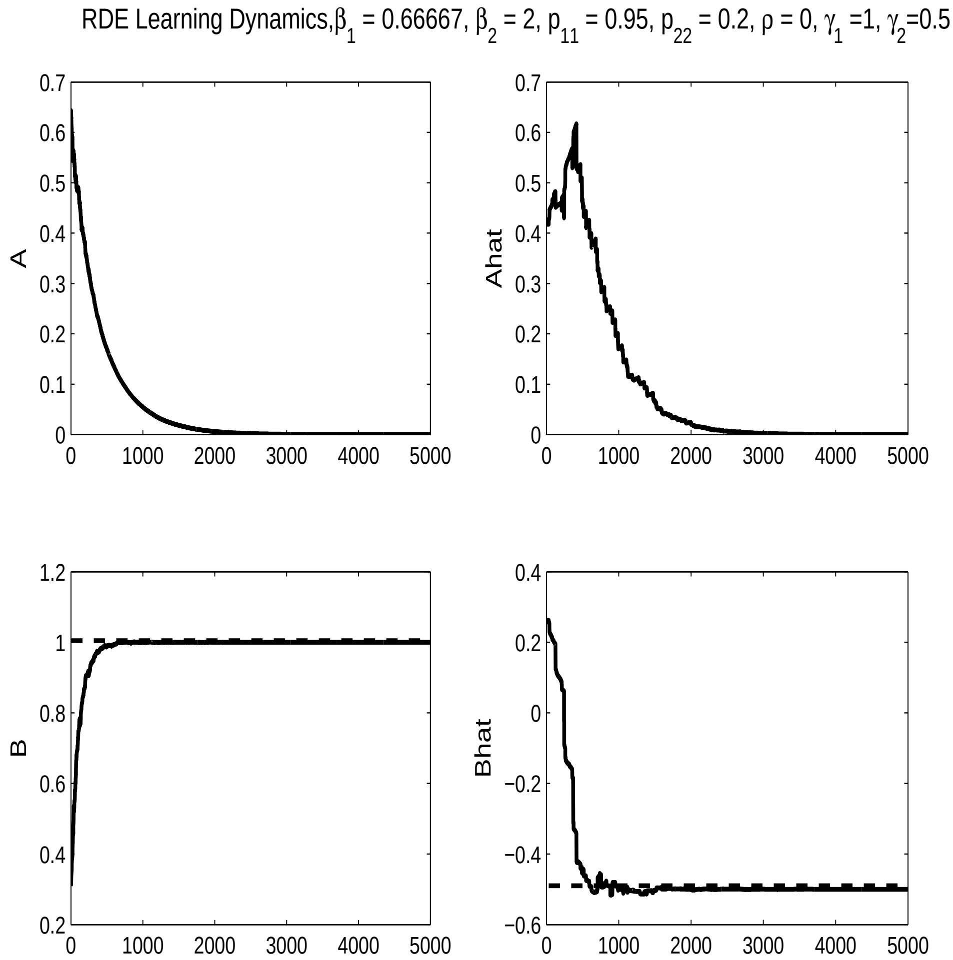
4.3 A New Keynesian Model

Farmer, Waggoner, and Zha (2007) illustrate the CLDC is necessary for determinacy, but not sufficient, as policymakers focusing on obeying the CLDC may not bring about an equilibrium that is immune from sunspots. Thus, whether the resulting HDE can arise in a setting where the CLDC holds and agents update their expectations using a reasonable learning algorithm, such as recursive least squares, is of particular interest.

This section follows Farmer, Waggoner, and Zha (2007) who consider an empirically plausible specification of a benchmark New Keynesian model and its stability properties. The example uses parameter values from Farmer, Waggoner, and Zha (2007), which resemble estimates from those in Lubik and Schorfheide (2004), to construct a sunspot HDE. This example is of particular interest because the parameter values are empirically plausible, imply the CLDC holds and the RDE is unique, yet a sunspot HDE exists.

[15] As is standard in the learning literature, in order to apply the theory of stochastic recursive algorithms requires imposing a "projection facility" on the recursive least squares algorithm. See Evans and Honkapohja (2001) for details.

Figure 1: Real time learning of an RDE.



The model is given by

$$\begin{aligned}\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + u_t^S \\ x_t &= E_t x_{t+1} - \sigma^{-1}\left(i_t - E_t \pi_{t+1}\right) + u_t^D \\ i_t &= \alpha_t \pi_t + \gamma_t x_t,\end{aligned}$$

where

$$\alpha_t = \begin{cases} \alpha_1 \text{ for } s_t = 1 \\ \alpha_2 \text{ for } s_t = 2 \end{cases}$$

and

$$\gamma_t = \begin{cases} \gamma_1 \text{ for } s_t = 1 \\ \gamma_2 \text{ for } s_t = 2 \end{cases}$$

The random variable s_t follows a finite-state Markov chain with transition probabilities $p_{ij} \equiv \Pr[s_t = j | s_{t-1} = i]$ for $i, j = 1, 2$.

Parameters values are from Table 3 in Farmer, Waggoner, and Zha (2007) and are

β	σ	κ	α_1	γ_1	α_2	γ_2	p_{11}	p_{22}
.9949	1.655	.675	.77	.17	2.19	.30	.8577	.99

In this calibrated example, FWZ compute the HDE as[16]

$$\begin{aligned} c_1 &= 0.999795 \\ c_2 &= .738137 \\ v_1 &= \begin{pmatrix} -.977509 \\ -.210551 \end{pmatrix} \\ v_2 &= \begin{pmatrix} -0.010062 \\ 0.0065658 \end{pmatrix} \end{aligned}$$

where the reduced-form autocorrelation coefficients are computed by plugging into (27). It is straightforward to verify that the resulting stochastic process is uniformly bounded.

The PLM is given by (30), where Farmer, Waggoner, and Zha (2007) provide values for the coefficients for the HDE. Evaluating the T-map given by (31) – (33) at the above rational expectations parameter values leads to the Jacobian of the T-map relevant for E-stability. Because the DT_B block de-couples it is sufficient to examine only this portion of the Jacobian

$$\begin{bmatrix} DT_B^1 & DT_B^2 \\ DT_B^3 & DT_B^4 \end{bmatrix}$$

where DT_B^1, DT_B^4 are given, respectively, by

$$\begin{bmatrix} p_{11}B(1,1)' \otimes \beta_1 + I \otimes \beta_1 (p_{11}B(1,1) + p_{12}B(1,2)) & p_{12}B(1,1)' \otimes \beta_1 \\ 0 & I \otimes \beta_2 (p_{21}B(2,1) + p_{22}B(2,2)) \end{bmatrix}$$

$$\begin{bmatrix} I \otimes \beta_1 (p_{11}B(1,1) + p_{1,2}B(1,2)) & 0 \\ p_{21}B(2,2)' \otimes \beta_2 & p_{22}B(2,2)' \otimes \beta_2 + I \otimes \beta_2 (p_{21}B(2,1) + p_{22}B(2,2)) \end{bmatrix},$$

and

$$\begin{aligned} DT_B^2 &= \begin{bmatrix} 0 & 0 \\ p_{21}B(1,2)' \otimes \beta_2 & p_{22}B(1,2)' \otimes \beta_2 \end{bmatrix} \\ DT_B^3 &= \begin{bmatrix} p_{11}B(2,1)' \otimes \beta_1 & p_{12}B(2,1)' \otimes \beta_1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

[16] See Farmer, Waggoner, and Zha (2007) for details on the numerical procedure for computing these values.

Evaluating this Jacobian leads to repeated eigenvalues of $2.6323, 2.0755, .6635, .4197, 0$. Therefore, the general form HDE is not E-stable. However, we know from above that the RDE and common factor HDE are E-stable.

This example highlights a central point of Farmer, Waggoner, and Zha (2007) that determinacy conditions may be elusive in empirically plausible New Keynesian models. These calibrated values reflect the conventional wisdom of monetary policy that the Federal Reserve did not adhere to the 'Taylor Principle' during the 1970's but did so subsequently. Under these parameter values the CLDC is satisfied and there exists a unique E-stable RDE. However, if agents anticipate that future monetary policy rules may be less active against inflation then it is possible for there to exist multiple equilibria that are stable under learning. These results imply that monetary policy needs to not only act aggressively to pin down inflation expectations they also need to take actions that will minimize the anticipation of future dovish policy. That is, credibility is even more crucial in these models.

5 Further Discussion

There are a number of ways in which the E-stability results of this paper are distinct from the results in constant parameter models. First, HDE sunspot equilibria may be E-stable provided agents' perceived law of motion is consistent with the common factor representation of HDE. This E-stability condition does not require any of the eigenvalues in the feedback matrices β_t have real part less than -1, a condition that is typically required in constant parameter models and is the case for the RDE in the indeterminate case. This result suggests that sunspot equilibria are more likely to be stable in regime-switching models. Second, the condition for E-stability is identical across both classes of rational expectations solutions. Finally, E-stability as an equilibrium selection device does not select one class of equilibrium for another. In fact, if agents were endowed with a perceived law of motion that nests the unique RDE and common factor HDE as special cases, depending on initial beliefs, the learning process may settle down at either class of rational expectations equilibrium.

A central result of this paper is that E-stable HDE exist even when the CLDC is satisfied. This suggests that HDE are not only theoretically plausible equilibria, but that they may also be realistic outcomes which modelers and policy makers should take seriously. However, this observation does come with a caveat that, for some, may call in to question whether the common factor representation of an HDE is natural; and, if it is not, then HDE are not stable under learning – at least not the ones analyzed in this paper. To illustrate this point, first recall the general form representation of an HDE:

$$y_t = B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t.$$

As this representation makes clear, the characteristic feature of an HDE is the explicit dependence of y_t on s_{t-1}, and, through the self-fulfilling nature of the model, this explicit dependence is present precisely because agents believe it will be present. Now recall the common factor representation: $y_t = \eta_t$, where η_t is an exogenous process depending explicitly on s_t and s_{t-1}. Coordination on this equilibrium requires only that agents believe that η_t governs the economy: in particular, y_t depends explicitly on s_{t-1} not because agents believe this dependence exists, but rather because this dependence has been exogenously imposed on the associated coordinating sunspot. In this sense, stable HDE and their explicit dependence on lagged Markov states are not the outcome of naturally evolving beliefs that these lagged states are important; rather, stable HDE require a coordinating sunspot with fortuitous stochastic properties.[17] If agents attempt to explicitly capture the lagged state dependence, using, for example, a general form representation, then the HDE will not be stable.

From an applied standpoint, the empirical investigation of indeterminacy in the U.S. time-series, which includes an extensive literature, is an important issue. While there is some debate within the literature whether the 1970's era inflation was driven by sunspot fluctuations, there is clear evidence for a switch in the Federal Reserve's response to inflation fluctuations. So U.S. time-series provide a natural place to look for evidence of RDE and HDE. Further, if the parameters of the model satisfy the CLDC, then a sharp distinction between RDE and HDE arises: either the economy was never driven by sunspot fluctuations or always was. Also, the stochastic properties of these equilibria are very different, since there is additional serial correlation in HDE that is not present in the unique RDE. Future research can shed light on the debate of HDE versus RDE by providing evidence of which equilibrium is more consistent with the data.

So, this begs the question, what is one to take away from these results? First, and from a technical standpoint, comparison of RDE and HDE highlights that in nonlinear models there often exist many classes of equilibrium. In nonlinear models, assumptions about the set of variables agents use to condition their expectations is important and uniqueness of equilibrium within one class does not necessarily imply uniqueness in another. Second, in the context of monetary analysis, where the application of the results has clear relevance, if policy makers feel that the results on the stability of common factor representations make HDE a serious concern, then a new, and rather stringent, set of requirements for monetary policy rules is suggested: a rule should be chosen to yield a unique RDE that is E-stable, yet rule out E-stable

[17] A similar point arises in common factor representations of sunspot equlbria in constant parameters models. These equilibria require that the associated common factor sunspot, which typically follows a VAR(1) process, satisfies a knife-edge "resonance frequency" property, that explicitly pins down the sunspot's serial correlation. However, the knife-edge nature of these equilibria is an artifact of the linearization; in non-linear models, more general sunspots are allowed: see Evans and Honkapohja, (2003).

HDE. Strict adherence to a constant-parameter rule that reacts strongly to inflation is one prescription; however, this "tying of one's hands" may not be reasonable. Events often arise that result in central banks deviating from a constant parameter rule – that is, the systematic response to inflation and output changes, even if temporarily. For example, a monetary authority may wish to act less aggressively in response to inflation during a recession or financial crisis. If private agents place positive probability on recurring changes in such systematic components of policy, then a monetary authority should be fully aware of the dangers of "indeterminacy spillovers," and modify how it acts in 'normal' times. In other words, a central bank may want to retain the discretion to change its systematic actions, but also eliminate the possibility of sunspot fluctuations. Thus, a monetary authority that wishes to occasionally change its systematic actions, should implement a rule that implies a unique RDE that is E-stable and at the same time, rules out HDE. The design of such rules should be a priority of future research.

6 Conclusion

This paper studies the existence and stability of two classes of rational expectations equilibria in a regime-switching rational expectations model under adaptive learning, extending the literature on learning to a non-linear framework. Building on the work of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2006, 2007), the two classes are:

- *Regime Dependent Equilibria*: An RDE is a uniformly bounded process that satisfies the regime-switching expectational difference equation and imposes the restriction that agents do not condition their expectations on lagged regimes (i.e. only the current regime enters the state vector).
- *History Dependent Equilibria*: An HDE is a process that satisfies the regime-switching expectational difference equation, where agents condition expectations on current and lagged values of the regime (i.e. current and past regimes enter the state vector).

The Conditionally Linear Determinacy Condition (CLDC), a generalization of the Long Run Taylor Principle of Davig and Leeper (2007), ensures the existence of a unique RDE that is also E-stable. When the CLDC is satisfied, there may still exist sunspot equilibria as demonstrated by Farmer, Waggoner, and Zha (2007). These equilibria can be represented by two different stochastic difference equations. Depending on which representation agents adopt for their reduced-form model, the HDE may be E-stable. Thus, whether sunspots are troubling or not in regime switching models depends on the perceived law of motion adopted by agents. The adoption of a perceived law of motion is a behavioral assumption and there has been little research that studies how agents might coordinate on a PLM.

Ultimately, which rational expectations equilibrium – a unique RDE or a common factor HDE – is the appropriate solution methodology is an empirical question. Both classes of equilibria are rational expectations solutions, uniformly bounded, and expectationally stable. The indeterminacy of HDE introduces additional serial correlation that is not present in RDE, and so it is a reasonable question to wonder whether extension of the various approaches to testing for indeterminacy such as Lubik and Schorfheide (2003) and Clarida, Gali, and Gertler (among many others) to Markov switching parameter models might shed light on many of these issues. Such an approach, of course, is beyond the scope of the present paper.

7 Appendix

Proof of Proposition 1

First let $\hat{y}_t$ be a uniformly bounded solution the stacked system, and construct y_{it} as indicated. Because r_t is independent of s_{t+n} for all n, it follows that y_{it} is as well. That $y_t = y_{it} \Leftrightarrow s_t = i$ provides a solution to the original model is then trivial.

To go the other way, let y_{it} identify an RDE. Denote by f_t the time t density functions; for example, $f_t(y, s|s_{t-1} = i, \Omega_{t-1})$ is the joint density of y_t and s_t conditional on $s_{t-1} = i$ and on all other time $t-1$ information, not including current and past s_{t-1}, as captured by Ω_{t-1}. Also, let $f_t^i(y|\Omega_{t-1})$ be the density for y_{it} conditional on Ω_{t-1}, and $f(s|s_{t-1} = i)$ be the conditional density of s_t given $s_{t-1} = i$ (course, $f(s = j|s_{t-1} = i) = P_{ij}$). With this notation, we may compute expectations as follows:

$$\begin{aligned}
E(y_{t+1}|s_t = i, \Omega_t) &= \int\int y f_{t+1}(y, s|s_t = i, \Omega_t)\, ds dy \\
&= \int\int y f_{t+1}(y|s, s_t = i, \Omega_t) f(s|s_t = i)\, ds dy \\
&= \int\int y f_{t+1}^s(y|\Omega_t) f(s|s_t = i)\, ds dy \\
&= \sum_j P_{ij} E_t y_{jt+1},
\end{aligned}$$

where the third equality precisely follows from the facts that $y_t = y_{it} \Leftrightarrow s_t = i$ and that y_{it} is independent of s_{t+n} for all n. Now we may simply use this formula for the expectations of y_t to verify that the stacked system is satisfied.

Proof of Proposition 5

Notice that stability obtains provided that the eigenvalues of $\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n) = \Lambda_u^{-1} \oplus \lambda_s^{-1}$ have real part less than one. The result follows from the fact that the eigenvalues of Λ_u are inside the unit circle.

Proof of Proposition 7

To examine the stability of the common factor representation, note that DT_B has unit eigenvalues and DT_A has (repeated) eigenvalues of zero and the eigenvalues of $\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)$, which is the uniqueness condition for RDE. Hence, if there exists a unique RDE, and there exists an HDE, then the common factor RDE are E-stable.

Proof of Proposition 8

The eigenvalues of $(\beta_1 \oplus \beta_2)P$ are given by

$$D^{\pm} = \frac{1}{2}\left(\beta_1 P_{11} + \beta_2 P_{22} \pm \Delta\right)$$

where

$$\Delta^2 = \left(\beta_1 P_{11} + \beta_2 P_{22}\right)^2 + 4\beta_1\beta_2(1 - P_{11} - P_{22}).$$

Then it is straight-forward to verify that

$$\lim_{\beta_i \to -\infty} D^{\pm} \in \{0, \beta_i P_{ii}\}.$$

Proof of Proposition 9

Using the notation from the body of the paper, we may write the recursive algorithm as

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} S_{t-1}^{-1} X_t \left(y_t - \theta_{t-1}' X_t\right) \\ S_t &= S_{t-1} + t^{-1}\left(X_t X_t' - S_{t-1}\right) - \frac{1}{t^2}\frac{t}{t+1}\left(X_t X_t' - S_{t-1}\right), \end{aligned}$$

where $X_t = (1, \hat{s}_t, r_t, \hat{s}_t r_t)$ and $S_{t-1} = R_t$. If X_t could be written as a linear difference equation in i.i.d. noise conditional on values of θ and S, we could immediately apply the main results of the learning literature; however, X_t is not conditionally linear, so we must work harder: we must verify conditions M in Chapter 7.3 of Evans and Honkapohja (2001).

First, notice that the evolution of X_t is independent of θ and S, simplifying our task. Let $Q^n(x, \cdot)$ be the distribution of X_{t+n} given that $X_t = x$. We must demonstration the following:

1. For all n, m there exists K so that $\int (1 + \|y\|^m)\, Q^n(x, dy) \le K(1 + \|x\|^m)$

2. For all p there exist K and δ so that for all $g \in \mathrm{Li}(p)$, for all n and for all x_1,x_2, we have

$$\left| \int g(y) Q^n(x_1, dy) - \int g(y) Q^n(x_2, dy) \right| \leq K\rho^n \|x_1 - x_2\| (1 + \|x_1\|^p + \|x_2\|^p)$$

Here $\mathrm{Li}(p)$ is a space of functions from $\mathbb{R}^2$ to itself, defined in Evans and Honkapohja (2001): it turns out, as will be seen shortly, the simplicity of our set-up allows us to ignore the special properties of $\mathrm{Li}(p)$, so we may simply take any $g : \mathbb{R}^2 \to \mathbb{R}^2$.

3. For all $q \geq 1$ there exist n, $\alpha < 1$ and β so that for all x we have

$$\int \|y\| Q^n(x, dy) \leq \alpha \|x\|^q + \beta.$$

The simplicity of our state dynamics allows these items to be easily demonstrated. Indeed, X_t is uniformly bounded a.s. by some number M, so $\int (1 + \|y\|^m) Q^n(x, dy) \leq 1 + M^m$, thus demonstrating item 1. Item 2, which would be quite difficult to demonstrate if $\gamma_t \neq 0$ follows here because there only two states: the left-hand-side is

$$\begin{pmatrix} (1,0)P^n \begin{pmatrix} g_1(x_1) \\ g_1(x_2) \end{pmatrix} \\ (0,1)P^n \begin{pmatrix} g_2(x_1) \\ g_2(x_2) \end{pmatrix}, \end{pmatrix}$$

which goes to zero exponentially because P is a stochastic matrix (here g_i is the $i-th$ coordinate of g, and it is because there are only a finite number of states that we do not have to worry about the special properties of g). Finally, item 3 follows in a fashion similar to item one because X_t is uniformly bounded.

Because the Markovian state dynamics satisfy the correct conditions, we may proceed as usual: stack the estimators S and θ into a matrix ϕ and write the recursive system as

$$\phi_t = \phi_{t-1} + \frac{1}{t} H(\phi_{t-1}, X_t) + \frac{1}{t^2} q(t, \phi_{t-1}, X_t)). \tag{45}$$

The linearity of the T-map makes it straight-forward to verify that this recursion satisfies the necessary properties. Now set

$$h(\phi) = \lim_t E\left(H(\phi, X_t)\right).$$

The possible convergence points of (45) are the locally asymptotically stable fixed points of the differential equation $\dot{\phi} = h(\phi)$. Computing $h(\phi)$ yields the decoupled system

$$\begin{aligned} \frac{d\theta}{dt} &= S^{-1} E(X_t X_t')(T(\theta) - \theta) \\ \frac{dS}{dt} &= E(X_t X_t') - S. \end{aligned}$$

We conclude that locally asymptotic stability obtains provided the eigenvalues of DT have negative real part. The proof is completed by noting that the eigenvalues of DT are the eigenvalues of $(\beta_1 \oplus \beta_2)P$.

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