Strongly rational expectations equilibria with endogenous acquisition of information

Gabriel Desgranges [1] Maik Heinemann [2]

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This paper analyzes conditions for existence of a strongly rational expectations equilibrium (SREE) in models with private information, where the amount of private information is endogenously determined. It is shown that the conditions for existence of a SREE known from models with exogenously given private information have to be qualified if private information is endogenously determined. As long as it is impossible to use the information transmitted through market prices, however, conditions for existence of a SREE change only if the noise terms of the private signals are correlated across agents. When there is such learning from prices, conditions for existence of a SREE known from models with exogenously given private information are too weak even if the noise terms are uncorrelated. It turns out that the properties of the function which describes the costs that are associated with the individual acquisition of information are important in this respect. In case of constant marginal costs, prices must be half as informative than private signals in order for a SREE to exist. An interpretation that relates our results to the famous Grossman–Stiglitz–Paradox is also given.

Key words: Eductive Learning, Private Information, Rational Expectations, Strongly Rational Expectations Equilibrium

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[1] University of Cergy-Pontoise, F–95011 Cergy-Pontoise CEDEX, France, gabriel.desgranges@eco.u-cergy.fr.
[2] University of Lüneburg, Institute of Economics, D–21332 Lüneburg, Germany, heinemann@uni-lueneburg.de.

1. Introduction

The concept of a rational expectations equilibrium (REE) is indeed quite ambitious if the underlying severe requirements on agent's information gathering and processing capabilities are considered. It is therefore not surprising that many attempts have been made in order to justify this concept and to state a clear set of assumptions that imply rational expectations on the side of the agents. One such attempt is the concept of a strongly rational expectations equilibrium (SREE) proposed by Guesnerie (1992, 2002). This concept asks, whether an REE can be educed by rational agents, meaning that the REE is the solution of some kind of mental process of reasoning of the agents. A SREE is then a REE that is learned by agents using this 'eductive' mental process (equivalently, the REE is said to be eductively stable). As shown by Guesnerie (1992, 2002), eductive learning of rational expectations is possible, if based on a suitably specified game–form of the model, agent's use an iterative process to eliminate non–best responses from their strategy sets and if this process converges to the REE. It turns out that an REE is not necessarily a SREE, but that additional restrictions have to met for a SREE to exist. Guesnerie (2002) provides an overview over the conditions for existence of SREE that have been derived in various economic contexts.

Among other things, the concept of a SREE has been successfully applied to models with private information, which usually exhibit quite complex rational expectations equilibria. Conditions for existence of a SREE have been derived for models, where agents are unable to use the information transmitted through current market prices (cf. Heinemann (2004)), as well as for models, where this information can be used (cf. Desgranges et al. (2003), Desgranges (1999), Heinemann (2002)). However, a common feature of all these studies is that they assume an exogenously given amount of private information. This means that so far not only the question how this private information comes into the market has been ignored. It also means that by now it has not been analyzed, whether the endogenization of private information acquisition causes additional restrictions an REE must fulfill in order to be strongly rational stable.

The present paper tries to fill this gap. We will introduce endogenous information acquisition into a simple market model and are able to derive conditions for existence of a SREE given this endogenously acquired information. Regarding the introduction of endogenous information acquisition, we follow the seminal work by Grossman and Stiglitz (1980) and more precisely Verrecchia (1982) who has analyzed rational expectations equilibria with endogenous acquisition of information in a quite similar economic environment. The present analysis considers two different equilibrium concepts that are both reasonable in the framework underlying our analysis. Initially, we will look at equilibria without learning from prices, where agents are not able to use the information transmitted trough current market prices for their own decisions. This model corresponds for example to a situation where every agent makes an irreversible production decision before he/she knows the price (this is the case of the cobweb models). After that, we will consider a more demanding equilibrium concept, where agents are able to use the information revealed by prices. This model is inspired from the well-known literature about REE under asymmetric information à la Grossman (1976).

The central results of the paper can be summarized as follows:

Independent of the underlying equilibrium concept, the opportunity to acquire private information endogenously leads conditions for existence of a SREE which are stronger than the respective conditions known for the case with exogenously given information. In the case without learning from prices, however, these conditions are actually relevant only if the noise terms of the private signals are correlated across agents. In our model such a correlation is induced by an aggregate stochastic component which contaminates the private signals. As a result, the precision of the agents' private signals is bounded from above by some exogenously given level. Given such a correlation, the curvature of the cost function associated with the acquisition of private information becomes important for existence of a SREE. If the noise terms are uncorrelated, endogenous acquisition of information, while changing the dynamics of the underlying best response mapping in a qualitative way, leads to no conditions for existence of a SREE beyond that known for the case with exogenously given information.

The additional coordination difficulties which arise from endogenously acquired information are more severe if learning from current prices is possible. With respect to equilibria where agents learn from current prices, we obtain conditions for existence of a SREE which turn out to be stronger than the respective conditions for the case with exogenously given information *even if* the noise terms of the private signals are uncorrelated across agents. Again, the properties of the cost function assiciated with the acquisition of information are relevant for existence of an SREE.

The difference between the stability results obtained for the two equilibrium concepts, which is striking if the case of uncorrelated noise terms is considered, is driven by the following intuition: Within the framework of the first concept where no information is extracted from the price by agents, endogenous acquisition of private information does not create additional difficulties of coordinating expectations. In other words, at the time where he/she makes his/her decision, every agent needs to guess the price to make an optimal choice. To this purpose, it is enough to guess the shape of the supply and demand curves.

In particular, no agent is concerned by the precision of the information acquired by others. The only new effect generated by the endogeneity of private information is a feedback effect between the weight agents give to their private information and the precision of this privately acquired information. However, as long as the noise terms of the private signals are uncorrelated, this feedback effect is unable to cause any additional stability problems. Hence, the conditions for stability of the REE (i.e. existence of a SREE) are not affected by endogenous acquisition of private information and they depend on the relative slope of the demand and supply curves only.

Within the framework of the second equilibrium concept where agents use the informational content of the price, the problem is quite different, and endogenous acquisition of private information does creates fundamental additional difficulties of coordinating expectations. Namely, every agent needs to know the precision of information acquired by others in order to correctly understand the informational content of the price. The condition for existence of a SREE in the model with information transmitted by the price deserves some more comments. The condition we derive states that the price must not be too informative, with respect to the informativeness of the private information acquired by agents. The underlying intuition is that the informational content of the price is determined by the correlation between private information and agents' decisions. As long as the price is not very informative, this correlation is easy to predict. But, when the price is very informative, agents' decisions mainly depend on the beliefs about the informational content of the price, and agents' decisions are therefore not easy to predict. While this condition for existence of a SREE is quite analogous to the condition in the case with exogenously given precision of private information, it is still a more demanding condition. The fact that endogenous information acquisition makes it more difficult for a SREE to exist can be explained as follows: As before, when the price is very informative, the REE is not strongly rational because every agent reacts less to his private information than to his beliefs about the information revealed by the price. In this case, given that private information is not very useful to agents, the precision of the private information acquired decreases. This last fact reinforces the stability problem. Namely, agents become much less reactive to their private information. Hence, agents' decisions depend more on their beliefs, which corresponds to a greater instability problem.

Lastly, an interesting feature of this stronger condition for existence of a SREE is that it ensures that the problem described by the the Grossman–Stiglitz–Paradox (cf. Grossman and Stiglitz (1980)) cannot occur. This famous paradox claims that existence of informationally efficient markets is impossible, since it is impossible to explain how information comes into the market in the first place. Namely, as long as the price publicly reveals all the relevant information, there is no incentive to acquire costly private information. But, if no one acquires information in order to make an accurate decision, the price cannot aggregate any information. In the model we consider, exogenous noise a priori prevents the market price from being fully informative, such that the Grossman–Stiglitz–Paradox in its original form does not appear. Nevertheless, our results regarding REE with learning from prices show that even if a REE exits where prices transmit some information, this REE might not be a SREE, i.e. it might suffer from coordination difficulties. In particular, if prices are too informative, no one can rule out that there is already so much information in the market that it becomes individually rational not to acquire any information at all. Our condition for existence of a SREE provides a solution which ensures that this problem which is in fact quite similar to the Grossman–Stiglitz–Paradox cannot occur: Existence of a SREE requires that the informativeness of the market price is bounded from above. As a consequence, each firm can educe that there is always a positive amount of private information in the market, because the incentive to free–ride on others' information will then be bounded from above. Consequently, the results of this paper show that the problem described by the Grossman–Stiglitz–Paradox is not only relevant for the question regarding existence of fully informative REE but also relevant for the question whether partially informative REE can be justified using the assumptions of individual rationality and common knowledge.

2. A competitive market model

The model that builds the framework of our analysis is a model of a competitive market with a continuum of risk neutral firms in $I = [0,1]$. Market demand X is random, but the inverse demand function is known to the firms:

$$p = \beta - \frac{1}{\phi} X + \varepsilon$$

Here, ε is a normally distributed demand shock with zero mean and precision τ_ε. $\beta > 0$ and $\phi > 0$ are known constants. Every firm faces increasing marginal costs that are affected by the parameter θ (this is a productivity shock unknown at the time where the production decision is made). With $x(i)$ denoting the output of firm i, its costs are $c(i) = \theta x(i) + \frac{1}{2}\frac{1}{\psi} x(i)^2$, where $\psi > 0$. The cost parameter θ is unknown to the firms. The firms, however, know that this parameter is drawn from a normal distribution with mean $\bar{\theta}$ and precision τ. For analytical simplicity, we assume $\bar{\theta} = 0$ in the sequel.

Private information on the side of the firms regarding the unknown parameter is introduced into the model by allowing for endogenous acquisition of information as in Verrecchia (1982) (generalizing the seminal framework of Grossman and Stiglitz (1980)). It is assumed that each firm is able to perform an experiment (independent from experiments of other firms) that reveals additional but costly information regarding the unknown parameter θ. In particular, it is assumed that each firm $i \in I$ can acquire a costly private signal $s(i)$ that reveals additional private information. The private signal is given by $s(i) = \theta + u(i)$, where the signal's noise $u(i)$ is normally distributed with mean zero and precision $\tau(i)_u$. The costs of acquiring a signal with precision $\tau(i)_u$ are given by $K(\tau(i)_u)$ and we let $K'(\tau(i)_u)$ denote the respective marginal costs. The objective of a firm is to maximize the expected profit, where profit $\pi(i)$ of firm i is given by:

$$\pi(i) = [p - \theta]\, x(i) - \frac{1}{2}\frac{1}{\psi}\, [x(i)]^2 - K(\tau(i)_u), \tag{1}$$

Costs are are assumed to be increasing and convex: $K'(\tau(i)_u) \geq 0$ and $K''(\tau(i)_u) \geq 0$ for all $\tau(i)_u \geq 0$.

Throughout the following analysis it will be assumed that the noise terms $u(i)$ are correlated so that the collection $(s(i))_{0 \leq i \leq 1}$ of the private signals does not reveal exactly the value of the unknown parameter. Recall that a sufficient statistic for the collection of private signals is the average signal (given that the variables are normally distributed). We assume, by the law of large numbers, that $\int_0^1 u(i)\, di = \bar{u}$, where $\bar{u}$ is a normally distributed stochastic variable with zero mean and (exogenous) precision $\tau_{\bar{u}}$. It follows, by the law of large numbers again, that the average $\bar{s} = \int_0^1 s(i)\, di$ of the firm's private signals is the variable $(\theta + \bar{u})$. Notice that $\tau(i)_u \leq \tau_{\bar{u}}$: the maximum available information precision to firm i is $\tau_{\bar{u}}$. Namely, $s(i) = \bar{s} + (u(i) - \bar{u})$, where the $(u(i) - \bar{u})$ are i.i.d. with zero mean and precision $1/(1/\tau(i)_u - 1/\tau_{\bar{u}})$.

In what follows, we will first consider equilibria of this simple market model, where the firms are unable to use the information transmitted trough prices. This simply means, that every firm must decide on her profit maximizing output, before the actual market price becomes known and is unable to condition her supply decision on the market price. An equilibrium concept, where such learning from prices is possible because the information transmitted through prices can be used, will be analyzed in section 4.

3. SREE without learning from prices

3.1. *Description of the linear REE*

We will start here with a brief description of the kind of REE that appears, when decisions are made before the actual market price becomes known. Because of the distributional assumptions made above, this REE takes a quite simple form: In equilibrium, each firm's supply decision $x(i)$ will be a linear function of the estimator for the unknown parameter θ based on public information and — if the firm chooses to acquire private information — the private signal $s(i)$ the firm observes. The decision to acquire information altogether, in turn depends on the marginal costs and benefits associated with private information acquisition.

Focusing on the linear REE is a quite common restriction in this kind of model. Actually, existence of non linear REE in this kind of model is, up to our knowledge, an open question (DeMarzo and Skiadas (1998) give a negative answer to this question in a non noisy setting with no demand shock). As our main interest is REE stability, we will not tackle the non linear problem.

Denote $x(i) = \psi\left[\gamma_0(i) + \gamma_1(i)\, s(i)\right]$ the linear supply of firm i. The decision of firm i consists then in three parameters $(\gamma_0(i), \gamma_1(i), \tau_u(i))$. The linear REE is standardly defined: it simply consists in three parameters $\left(\gamma_0^*, \gamma_1^*, \tau_u^*\right)$ that are self-fulfilling (with the exact meaning that $\left(\gamma_0^*, \gamma_1^*, \tau_u^*\right)$ is a fixed point of the best response map defined below in Equations (4a) to (4c)).

The next result summarizes the properties of the REE:

Proposition 1. *Let $\alpha = \psi/\phi > 0$. The model then possesses an unique linear REE with the following properties:*

(i) Each firm $i \in I$ will acquire the same level of precision $\tau(i)_u^* = \tau_u^* = \min\left\{\max\left\{0, \tilde{\tau}_u\right\}, \tau_{\bar{u}}\right\}$ of her private signal $s(i)$. $\tilde{\tau}_u$ is the unique solution of the equation:

$$\frac{\psi}{2}\frac{1}{\left(\tau + \left[1 + \alpha\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right)\right]\tilde{\tau}_u\right)^2} = K'(\tilde{\tau}_u) \tag{2}$$

A positive amount of information is acquired in equilibrium, i.e., $\tau_u^* > 0$, iff $\frac{\psi}{2\tau^2} > K'(0)$. Furthermore, $\tau_u^* < \tau_{\bar{u}}$ if in addition

$$\frac{\psi}{2}\frac{1}{\left((1+\alpha)(\tau_{\bar{u}} + \tau)\right)^2} < K'(\tau_{\bar{u}}) \tag{3}$$

(ii) Each firm $i \in I$ will use the same supply function $x(i) = \psi\left[\gamma_0^* + \gamma_1^*\, s(i)\right]$, where the weights γ_0^* and γ_1^* are functions of the model parameters:

$$\gamma_0^* = \frac{\beta}{1+\alpha}, \qquad \gamma_1^* = -\frac{\tau_u^*}{\tau + \left[1 + \alpha\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right)\right]\tau_u^*}$$

Proof. See Appendix. □

Existence and uniqueness of a linear equilibrium in various cases of CARA/Gaussian settings is a very common result, and this result deserves few comments only. A market equilibrium with private information acquisition (i.e., $\tau_u^* > 0$) will therefore exist only if the marginal benefit of information acquisition at zero (i.e., $\frac{\psi}{2\tau^2}$), is greater than the marginal cost of information acquisition at zero (i.e., $K'(0)$). In what follows, we assume that this condition is satisfied. Thus, there always exists a nontrivial REE, where individual acquisition of information takes place. For simplicity, we will also sometimes make the assumption that the marginal costs of information acquisition are constant, such that $K'(\tau(i)_u)) = \bar{\kappa} > 0$ for all $\tau(i)_u > 0$. Under this assumption, a REE with information acquisition (i.e., $\tau(i)_u^* = \tau_u^* > 0$ for all $i \in I$) exists if and only if $Q \equiv \sqrt{\frac{\psi}{2\bar{\kappa}}} > \tau$. From the equilibrium condition (2) we obtain that in this case the equilibrium amount of information acquisition is $\tau_u^* = \frac{Q-\tau}{1+\alpha}$.

3.2. *Existence of a SREE*

Since detailed descriptions of of the concept of a SREE are already available in the literature (cf. Guesnerie (2002)), it is adequate to limit the present analysis to an informal and pragmatic treatment of this concept and the game–theoretical issues that are involved here.

An informal presentation of this stability concept of a REE goes as follows. The fundamental question associated with the concept of a SREE is whether or not common knowledge of individual rationality and model is sufficient to predict an unique outcome. In general, the answer is no and the set of outcomes predicted by the common knowledge assumptions is called the set of ”rationalizable” solutions. It includes the Nash/REE outcomes [3], but it typically includes other outcomes as well. Still, under some conditions, the set of rationalisable solutions reduces to one element, and the unique outcome compatible with the common knowledge assumptions is then the (unique) REE. The question adressed here is to find the conditions under which the linear REE described above is the unique rationalizable solution, that is: the linear REE is the only outcome surviving to a process of infinitely repeated elimination of strategies that are non best responses: at the first step, eliminate decisions that are not rational (i.e. best response to none of others' possible behavior); at the second step, eliminate decisions that are not best response to some rational decisions of others (i.e. eliminate decisions not compatible with the fact that ”everyone knows that everyone is rational”); at the third step, eliminate decisions that are not best response to some decisions of others that have gone through the second step (i.e. eliminaite decisions not compatible with the fact that ”everyone knows that everyone knows that everyone is rational”); ... Any further step is analogously defined.

Here, we consider local stability only: we restrict *a priori* the set of firms' decisions to a neighborhood of the REE $\left(\gamma_0^*, \gamma_1^*, \tau_u^*\right)$. This restriction is common knowledge.

[3] Recall that Nash equilibrium and REE share the same property (and, in this model, define the same outcome): it is generally not optimal to play Nash/to form RE when others do not play Nash/form REE.

Therefore it is necessary to look at a suitable game–form of the model and to analyze the best responses of the individual firms to actions taken by other firms in order to derive conditions for existence of a SREE. If we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i) = \psi\,[\gamma(i)_0 + \gamma(i)_1\, s(i)]$, the respective best response mapping can be summarized by the equations listed in the following Lemma: [4]

Lemma 1. *If aggregate behavior is summarized by the coefficients* $\gamma_0 = \int_0^1 \gamma(j)_0\, dj$ *and* $\gamma_1 = \int_0^1 \gamma(j)_1\, dj$*, the best response* $\gamma(i)_0$*,* $\gamma(i)_1$ *as well as* $\tau(i)_u$ *of a firm* $i \in I$ *is uniquely defined by the following equations:*

$$\gamma(i)_0 = \beta - \alpha\gamma_0 \tag{4a}$$

$$\gamma(i)_1 = -\left[\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right]\frac{\tau(i)_u}{\tau+\tau(i)_u} \tag{4b}$$

$$\frac{\psi}{2}\left[\frac{\gamma_1(i)}{\tau(i)_u}\right]^2 = K'(\tau(i)_u) \quad \textit{if there is a solution} \quad 0<\tau(i)_u<\tau_{\bar{u}} \tag{4c}$$

Proof. See Appendix. □

The last equation admits a solution $0<\tau(i)_u<\tau_{\bar{u}}$ unless:

- $\frac{\psi}{2}\left[\frac{\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau+\tau_{\bar{u}}}\right]^2 > K'(\tau_{\bar{u}})$. In this case, $\tau(i)_u = \tau_{\bar{u}}$.
- or $\frac{\psi}{2}\left[\frac{\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau}\right]^2 < K'(0)$. In this case, $\tau(i)_u = 0$.

This Lemma (that is central to the study of stability, as will soon be clear) calls for several comments:

(i) The best response mapping defined by the above Equations (4a)–(4c) maps the three real parameters $(\gamma_0,\gamma_1,\tau_u)$ into the three real parameters $(\gamma(i)_0,\gamma(i)_1,\tau(i)_u)$ characterizing the best response of firm i (where the aggregate value $\tau_u = \int_0^1 \tau(j)_u\, dj$ is defined analogously to γ_0 and γ_1).

(ii) By definition, the linear REE $\left(\gamma_0^*,\gamma_1^*,\tau_u^*\right)$ is the unique fixed point of the best response mapping (4a)–(4c). In particular, equation (4b) implies that $\gamma_1^* \leq 0$.

(iii) The best response mapping is defined for "point beliefs" only and not for "stochastic beliefs", that is: $(\gamma(i)_0,\gamma(i)_1,\tau(i)_u)$ is a best response to a given aggregate behavior $(\gamma_0,\gamma_1,\tau_u)$, and not to a distribution of parameters $(\gamma_0,\gamma_1,\tau_u)$. Carefully reading the

[4] Optimal output of a firm is given by $x(i) = \psi\,\mathrm{E}[p-\theta\,|\,s(i)]$. Hence, this linear supply rule assumes that $\mathrm{E}[p-\theta\,|\,s(i)] = \gamma(i)_0 + \gamma(i)_1\, s(i)$. It amounts to assume that the the stochastic variables $(p,\theta,s(i))$ are jointly normally distributed.

proof of the Lemma shows that extending the best response mapping to the case of "stochastic beliefs" is straightforward (given that the profit of firm i is linear in price).

(iv) Notice that $(\gamma(i)_0, \gamma(i)_1, \tau(i)_u)$ is not affected by τ_u, i.e., the average precision of the information acquired by others. Intuitively, firm i makes its supply decision considering (1) its information on θ (that is s_i only as there is no learning from the price), and (2) its information on the price, that consists in the market clearing equation $p = \beta - \alpha[\gamma_0 + \gamma_1 \bar{s}] + \varepsilon$, where ε and $\bar{s}$ are unknown. Thus, given that the precision of the aggregate information $\bar{s}$ on θ does not depend on the individual precisions $\tau(j)_u$ (it is $\tau_{\bar{u}}$), the decision made by firm i does not depend on the $\tau(j)_u$ either.

We can now turn attention to the question of the strong rationality (or stability) of the REE. By definition, the REE γ_0^*, γ_1^* and τ_u^* is a fixed point of the best response mapping (4a)–(4c). Again, a detailed account of the analytical characterization of SREE is given in Guesnerie (2002). We just recall here that this REE is a SREE (or, equivalently is "eductively stable") if and only if it is a locally stable stationary point of the dynamical system made up from this best response mapping. Now, with respect to this dynamical system, the eigenvalues λ_1, λ_2 and λ_3 of the Jacobian matrix at the equilibrium point can be computed as follows: [5]

$$\lambda_1 = 0, \quad \lambda_2 = -\alpha \le 0, \quad \lambda_3 = -\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\left(K''(\tau_u^*)\tau_u^{*3} + \psi\gamma_1^{*2}\right)}{\psi\gamma_1^{*2} + (\tau+\tau_u^*)K''(\tau_u^*)\tau_u^{*2}} \le 0,$$

where $K''(\tau_u^*) \ge 0$ has been assumed. Thus, the conditions for strong rationality of the linear REE are fully described in the next proposition:

Proposition 2. *Consider a linear REE with private information acquisition where the information precision satisifes $0 < \tau_u^* < \tau_{\bar{u}}$. The REE is locally strongly rational if and only if*[6] $\alpha < 1$ *and*

$$\frac{K''(\tau_u^*)}{K'(\tau_u^*)} > 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau_u^*\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau}. \tag{5}$$

In particular, we have:

(i) A sufficient condition for stability is

$$\alpha < \frac{1}{1+\frac{\tau}{\tau_{\bar{u}}}}.$$

If the marginal cost is constant ($K''=0$), then this condition is necessary as well.

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[5] Details on the computation of the eigenvalues of this dynamical system are given in the appendix at the end of the proof of Lemma 1.

[6] Recall that $\alpha > 0$, since $\alpha = \psi/\phi$, where ψ and ϕ are positive constants.

(ii) A necessary condition for stability is $\alpha < 1$. *If the average signal* $\bar{s}$ *reveals exactly the value* θ *of the unknown parameter,* $\bar{s} = \theta$ *(i.e.* $\tau_{\bar{u}} = +\infty$*), then this condition is sufficient as well.*

The conditions $\alpha < 1$ and (5) are both necessary. For instance, in the case $K'' = 0$ (Point (i) above), condition (5) becomes

$$1 < \frac{1 + \frac{\tau}{\tau_u^*}}{1 + \frac{\tau}{\tau_{\bar{u}}}} < \alpha \quad \text{or} \quad \alpha < \frac{1}{1 + \frac{\tau}{\tau_{\bar{u}}}} < 1$$

Hence, condition (5) neither implies, nor is implied by condition $\alpha < 1$. Interestingly, the necessary and sufficient condition for stability is expressed as the combination of 2 conditions:

- the stability condition for the case with exogenously given private information, that is: $\alpha < 1$ (cf. Heinemann (2004)),
- an additional condition (5) involving the cost of acquiring private information (more precisely, the variation rate $\frac{K''(\tau_u^*)}{K'(\tau_u^*)}$ of the equilibrium marginal cost).

It follows that *even when no learning from prices is considered, allowing for endogenous acquisition of information sometimes creates additional coordination difficulties.* The further question is then: when does this happen? Proposition 2 states that $\alpha > 1$ implies instability and $\alpha < 1/\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right)$ implies stability. It follows that the information cost matters only in the case $1/\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right) < \alpha < 1$ (through Condition (5)). Still, the RHS of Condition (5) depends on the endogenous τ_u^*, making the condition difficult to interpret. The next corollary makes more explicit the role played by the cost function:

Corollary 1. *Assume* $1/\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right) < \alpha < 1$. *Consider a linear REE where the information precision satisfes* $0 < \tau_u^* < \tau_{\bar{u}}$.

(i) If the cost function satisfies: $\frac{K''(\tau_u^*)}{K'(\tau_u^*)} > 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{(1-\alpha)(\tau_{\bar{u}}+\tau)}$, *then stability of the REE obtains.*

(ii) If the cost function satisfies: $\frac{K''(\tau_u^*)}{K'(\tau_u^*)} < 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau}$, *then stability of the REE does not obtain.*

(iii) Otherwise, consider a given value vr such that $2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau} < vr < 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{(1-\alpha)(\tau_{\bar{u}}+\tau)}$. *Consider all the function costs such that* $\frac{K''(\tau_u^*)}{K'(\tau_u^*)} = vr$ *at the REE. Among these functions, stability obtains iff* $K'(\tau_u^*)$ *is large enough.*

Proof. See Appendix. □

Hence, *allowing for endogenous acquisition of information sometimes creates additional coordination difficulties when the variation rate* $\frac{K''(\tau_u^*)}{K'(\tau_u^*)}$ *of the equilibrium marginal cost is small enough (Points (ii) and (iii)). Otherwise, endogenous information acquisition creates no additional coordination difficulties (Point (i)).*

To understand why a small curvature of the cost function K results in nonexistence of a SREE, consider that $\tau(i)_u$ is determined by equating K' to marginal benefits of information. Hence, a given change in marginal benefits of information implies a large change $d\tau(i)_u$ when K''/K' is small. This large $d\tau(i)_u$ is detrimental to stability as shown by looking at the best response dynamics in the case with endogenous information (as described in the proof of Lemma 1). These dynamics involves one more effect than the respective dynamics in case of exogenous information. To see this, consider the problem faced by firm i. If firm i expects a change $d\gamma_1$ in the aggregate supply, then the variation $d\gamma(i)_1$ of the optimal $\gamma(i)_1$ is due to 2 effects:

(1) an elementary "price" effect: an increase (decrease) in the expected supply implies a decrease (increase) in the expected price and then a decrease (increase) in i's supply. This effect exists when acquisition of information is exogenous as well.

(2) a "feedback" effect following from $\gamma(i)_1$ being the weight given to private information in the individual supply decision: a decrease (increase) in $\gamma(i)_1$ results in a decrease (increase) in the acquired precision $\tau(i)_u$, which in turn results in a further decrease (increase) of $\gamma(i)_1$. This effect (that goes through the choice of $\tau(i)_u$) cannot exist when acquisition of information is exogenous.

Thus, *the endogeneity of* $\tau_u(i)$ *affects the sensitivity of the individual supply decision on to the supply decisions of other firms*. Existence of this feedback effect explains that endogeneous information acquisition can create additional coordination difficulties.

To give a formal account of the 2 effects, let us differentiate (4b) and (4c) at the equilibrium:

$$d\gamma(i)_1 = -\alpha \frac{1+\frac{\tau}{\tau_{\bar{u}}}}{1+\frac{\tau}{\tau_u^*}} d\gamma_1 - \left[\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right] d\left(\frac{\tau(i)_u}{\tau+\tau(i)_u}\right),$$

$$d\tau(i)_u = W d\gamma(i)_1 \text{ where } W \text{ is a negative parameter (see Appendix).}$$

A change in other firms' decisions $d\gamma_1$ creates a change $d\gamma(i)_1$ in i's supply decision $\gamma(i)_1$ and a change $d\tau(i)_u$ in i's information precision $\tau(i)_u$. The first equation shows that $d\gamma(i)_1$ is the sum of two effects: the (negative) "price" effect $-\alpha\frac{1+\frac{\tau}{\tau_{\bar{u}}}}{1+\frac{\tau}{\tau_u^*}}$ and the "feedback" effect $-\left[\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right] d\left(\frac{\tau(i)_u}{\tau+\tau(i)_u}\right)$. This latter effect is induced by the variation $d\tau(i)_u$ (characterized in the 2nd equation). Indeed, if $\tau(i)_u$ was exogenous, the 2 equations would reduce to the following one:

$$d\gamma(i)_1 = -\alpha \frac{1+\frac{\tau}{\tau_{\bar{u}}}}{1+\frac{\tau}{\tau_u^*}} d\gamma_1$$

Only the "price" effect would be present.

Lastly, when the noise terms of the agents' signals are uncorrelated (i.e. $\tau_{\bar{u}} = +\infty$), the condition for existence of a SREE is the same as the one obtained for the case with exogenous information: it does not matter for existence of a SREE whether or not information is endogenous. This result obscures the role played by the information cost, and this is precisely why we introduce $\tau_{\bar{u}} < +\infty$ in the model.

Case with constant marginal costs ($K'' = 0$): In order to understand the intuition for this result, it is useful to look at the specific case with constant marginal costs. Under the assumption of constant marginal costs of information acquisition, it is quite easy to give a graphical representation of the stability condition and the iterative process that leads to the REE. Under the assumption that $K'(\tau_u) = \bar{\kappa}$ such that $K'' = 0$, Equation (4c) gives $\tau(i)_u = -Q\gamma(i)_1$, where Q was already defined above. Then, the best response mapping (4a)-(4b) rewrites as the following linear system:

$$\gamma(i)_0 = \beta - \alpha\gamma_0 \tag{6a}$$

$$\gamma(i)_1 = -\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\gamma_1 - 1 + \frac{\tau}{Q} \equiv g(\gamma_1) \tag{6b}$$

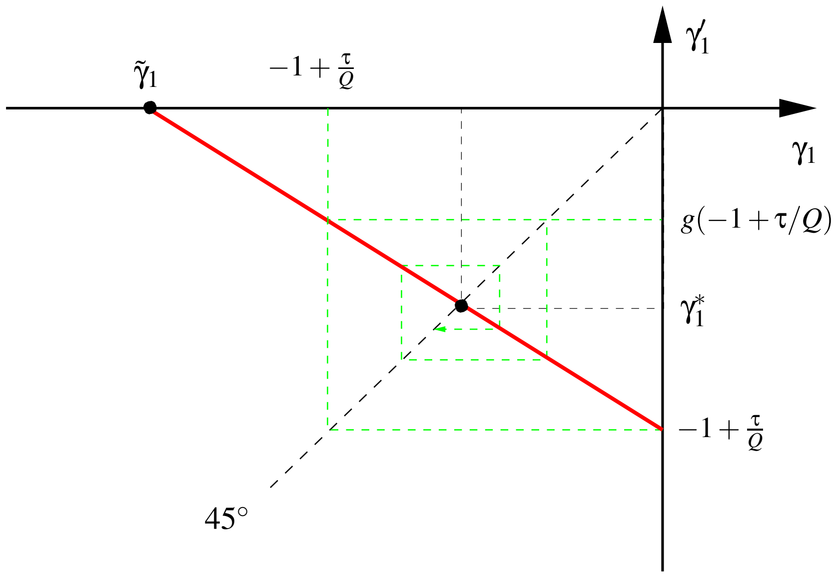
where the variables τ_u and $\tau(i)_u$ do not appear, as explained above. In particular, the second equation characterizes the dynamics of γ_1.

Consider the dynamics of γ_1 as depicted in figure 1. To draw the figure, denote γ_1^* the equilibrium weight of private information (such that $\gamma_1^* = g(\gamma_1^*)$) and $\tilde{\gamma}_1 = \frac{1}{1+\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)}\left(\frac{\tau}{Q}-1\right)$ the root of g, and notice that existence of an REE with $\tau_u^* > 0$ implies $Q > \tau$ such that $g(0) < 0$. Thus, whenever the stability condition stated in Proposition 2 is satisfied, we have $\tilde{\gamma}_1 < \frac{\tau}{Q} - 1$. Figure 1 can then be used to describe the iterative process of elimination of non-best responses that converges to this REE. This process is illustrated in the figure, starting from the assumption that it is common knowledge that no firm uses a weight $\gamma(i)_1$ greater than zero.[7] This necessarily implies that $\gamma_1 \leq 0$ and from the figure it can be seen that in this case no firm will ever choose a weight $\gamma(i)_1$ which smaller than $\frac{\tau}{Q} - 1$.[8] From this, however,

[7] From equation (4c) it follows that this is equivalent to the assumption that it is common knowledge that $\tau(i)_u \geq 0$ for all i.

[8] Using equation (4c) it can be shown that with respect to the amount of information that is acquired this means that no firm will acquire information with precision greater than $\tau_u = (Q - \tau)$.

Fig. 1. *Graphical representation of the SREE condition*



it in turn follows that γ_1 must be greater than $\frac{\tau}{Q}-1$, which implies that no firm i will use a weight $\gamma(i)_1 > g(\tau/Q-1)$. It is easily verified that this process converges to the equilibrium γ_1^*, whenever $0 < \alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right) < 1$.

4. SREE with learning from prices

4.1. *The case of exogenously given information*

Let us now turn to the second equilibrium concept, where learning from current prices is possible. It is reasonable to start this analysis with a brief discussion of a version of the model, where the amount of private information is given. This enables us to build on some known results and to illustrate, where these known results have to be modified if endogenous acquisition of information is allowed for. The analysis is based on the initially considered model with risk neutral firms and it is assumed that each firm's signal has precision $\tau_u > 0$. Furthermore, for analytical simplicity we now consider only the case where the noise in the private signals is uncorrelated across agents such that $\tau_{\bar{u}} = +\infty$. The reason for this is that, unlike the previuos analysis, endogenous acquisition of information creates even in this case severe coordination difficulties, which lead to stronger conditions for existence of a SREE.

When there is learning from prices, the firms are able to use the information transmitted through the actual market price for their own decisions. Hence, profit maximizing output for a firm $i \in I$ is now given by $x(i) = \psi\left[p - \mathrm{E}[\theta \,|\, s(i), p]\right]$. In analogy to the financial market models considered by Desgranges (1999) and Heinemann (2002), it can then be established that there exists an unique linear REE in this model with learning from prices.

Proposition 3. *Let again $\alpha = \psi/\phi > 0$. The model with learning from prices then possesses an unique linear REE, where every firm uses a linear supply function $x(i) = \psi[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$. The coefficient γ_1^* is the unique solution of the polynomial*

$$H(\gamma_1^*) \equiv \gamma_1^* \left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right] = \tau_u,$$

and the coefficients γ_0^* and γ_2^* are given by:

$$\gamma_0^* = \frac{\beta}{\alpha}\frac{\tau_p}{\tau_u + \tau_p}, \quad \gamma_2^* = \frac{1+\alpha}{\alpha}\frac{\tau_p}{\tau_u+\tau_p},$$

where $\tau_p = \alpha^2\gamma_1^{*2}\tau_\varepsilon$ ($\tau_p = 1/Var(\theta|p) - 1/Var(\theta)$ reflects the precision of the information revealed by p).

Conditions for existence of a SREE in this model with exogenously given private information are derived by Heinemann (2004). For convenience the respective conditions are reproduced in the following proposition:

Proposition 4. *The rational expectations equilibrium $\{\gamma_0^*, \gamma_1^*, \gamma_2^*\}$ is a locally SREE if and only if*

$$\tau_p < \tau_u$$

where $\tau_p = \alpha^2\gamma_1^{*2}\tau_\varepsilon$. This condition rewrites: $\tau + \tau_p < \tau + \tau_u$, that is: the precision of the information revealed by p is smaller than the precision of the information revealed by a private signal.

It is easy to check (see the proof of Corollary 2 in Appendix) that existence of a SREE is favored by a large α and τ_ε (τ_p decreases in α and τ_ε). The effect of a change in τ_u is not a priori obvious because an increase of τ_u increases both the precision of the information revealed by p and the precision of the information revealed by a private signal. The corollary below gives a necessary and sufficient for stability that makes explicit the fact that the impact of τ_u is not monotonic: both a large and a small τ_u are compatible with existence of a SREE.

Corollary 2. *If $\alpha^2\tau_\varepsilon < 8\tau$, then the REE is a SREE. Otherwise, $\alpha^2\tau_\varepsilon > 8\tau$, and the necessary and sufficient condition for existence of a SREE is*

$$\tau_u < \frac{\alpha^2\tau_\varepsilon - 4\tau - \sqrt{\alpha^2\tau_\varepsilon\,(\alpha^2\tau_\varepsilon - 8\tau)}}{8} \text{ or } \tau_u > \frac{\alpha^2\tau_\varepsilon - 4\tau + \sqrt{\alpha^2\tau_\varepsilon\,(\alpha^2\tau_\varepsilon - 8\tau)}}{8}$$

Proof. See Appendix. □

This non monotonic effect of τ_u is analogous to the one exhibited in Desgranges (1999) in a CARA/Gaussian model à la Grossman. This is due to the fact that τ_p is increasing but not linear in τ_u.

4.2. *Conditions for existence of a SREE with endogenous acquisition of information*

Starting from the above described rational expectations equilibrium with exogenously given information, it is quite easy to derive the respective equilibrium conditions for the model with endogenous information acquisition. The reason is, that all the conditions stated in Proposition 3 remain essentially valid. The only modification consists in an additional condition requiring that the marginal costs of the acquired information are equal to the marginal benefits from this information. This additional condition leads to the following characterization of the REE:

Proposition 5. *In the model with learning from prices and endogenous information acquisition exists an unique linear REE, where every firm uses a linear supply function* $x(i) = \psi[(1-\gamma_2^*)\,p - \gamma_0^* - \gamma_1^*\,s(i)]$.

(i) If $K'(0) > \frac{\psi}{2\tau^2}$, *then each firm* $i \in I$ *acquires the same level of precision* $\tau_u^* = 0$ *and its supply is 0.*

(ii) If $K'(0) < \frac{\psi}{2\tau^2}$, *then each firm* $i \in I$ *acquires the same level of precision* $\tau_u^* > 0$. τ_u^* *is the solution of the equation:*

$$\sqrt{\frac{2K'(\tau_u^*)}{\psi}}\left[\frac{2K'(\tau_u^*)}{\psi}\tau_u^{*2}\alpha^2\tau_\varepsilon + \tau + \tau_u^*\right] = 1,$$

The coefficients γ_0^* *and* γ_1^* *and* γ_2^* *are given as in Proposition 3:*

$$\gamma_0^* = \frac{\beta}{\alpha}\frac{\tau_p}{\tau_u^* + \tau_p}, \quad \gamma_1^* = \sqrt{\frac{2K'(\tau_u^*)}{\psi}}\tau_u^*, \quad \gamma_2^* = \frac{1+\alpha}{\alpha}\frac{\tau_p}{\tau_u^* + \tau_p},$$

where $\tau_p = \alpha^2 \frac{2K'(\tau_u^*)}{\psi}\tau_u^{*2}\tau_\varepsilon$.

Proof. See Appendix. □

Once the optimal τ_u^* is computed, the equilibrium supply is identical to the one in the case with exogenous information precision (described in Proposition 3). The τ_p is the same as above ($\tau_p = \alpha^2\gamma_1^{*2}\tau_\varepsilon$).

We now again ask, whether the assumptions of individual rationality and common knowledge are sufficient for a justification of this REE. In order to derive the respective conditions for existence of a SREE, we have again to look a the best responses of the individual firms to actions taken by other firms. As in the preceeding section, we confine our analysis to linear supply functions, such that an individual firm's supply is given by $x(i) = \psi[(1-\gamma(i)_2)\,p - \gamma(i)_0 - \gamma(i)_1\,s(i)]$. The respective best response mapping is then as summarized in the following Lemma:

Lemma 2. *Let $\gamma_0 = \int_0^1 \gamma(j)_0\, dj$, $\gamma_1 = \int_0^1 \gamma(j)_1\, dj$ and $\gamma_2 = \int_0^1 \gamma(j)_2\, dj$.*[9] *Aggregate demand is then*

$$\int_0^1 x(j)dj = \psi[(1-\gamma_2)p - \gamma_0 - \gamma_1\theta].$$

(so that aggregate behavior is summarized by the coefficients γ_0, γ_1 and γ_2). Then, the best response of a firm $i \in I$ to $(\gamma_0, \gamma_1, \gamma_2)$ is characterized by the coefficients $(\gamma'_0(i), \gamma'_1(i), \gamma'_2(i), \tau'_u(i))$ defined by:

$$\gamma'(i)_0 = -\frac{\alpha\gamma_1\tau_\varepsilon(\beta + \alpha\gamma_0)}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{7}$$

$$\gamma'(i)_1 = \frac{\tau'_u(i)}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{8}$$

$$\gamma'(i)_2 = \frac{\gamma_1\alpha(1 + \alpha(1-\gamma_2))\tau_\varepsilon}{\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon} \tag{9}$$

where $\tau'_u(i) = 0$ if $K'(0) > \frac{\psi}{2(\tau+\alpha^2\gamma_1^2\tau_\varepsilon)^2}$ and $\tau'_u(i)$ is the unique solution of

$$\frac{\psi}{2}\frac{1}{(\tau + \tau'_u(i) + \alpha^2\gamma_1^2\tau_\varepsilon)^2} = K'(\tau'_u(i)) \tag{10}$$

otherwise.

Notice that equation (10) rewrites

$$\frac{\psi}{2}\left(\frac{\gamma'(i)_1}{\tau'_u(i)}\right)^2 = K'(\tau'_u(i)) \tag{11}$$

The REE is locally strongly rational if the map defined by equations (7)–(11) is contracting at the REE values $\left(\gamma_0^*, \gamma_1^*, \gamma_2^*, \tau_u^*\right)$. The next technical Lemma characterizes locally SREE:

Lemma 3. *Assume $K'(0) < \frac{\psi}{2\tau^2}$ (so that $\tau_u^* > 0$ at the REE). The REE is a locally SREE if and only if*

$$\tau_p < \tau_u^* \text{ and}$$

$$\tau_p < \tau_u^* + \tau - (1-\gamma_1^*)\frac{\psi\frac{\gamma_1^*}{\tau_u^{*2}}}{K''(\tau_u^*) + \psi\frac{\gamma_1^{*2}}{\tau_u^{*3}}}$$

where $\tau_p = \alpha^2\gamma_1^{*2}\tau_\varepsilon$.

Proof. See Appendix. □

[9] All the measurability assumptions required are made. In particular, we assume that $\int_0^1 \gamma(j)_0\, dj$, $\int_0^1 \gamma(j)_1\, dj$ and $\int_0^1 \gamma(j)_2\, dj$ exist.

Recall that $\tau_p < \tau_u^*$ is the condition for existence of a locally SREE when information precision is exogenous. Straighforwardly, from the above lemma, endogeneity of information precision makes existence of a SREE more requiring when

$$\tau - (1-\gamma_1^*)\frac{\psi\frac{\gamma_1^*}{\tau_u^{*2}}}{K''(\tau_u^*)+\psi\frac{\gamma_1^{*2}}{\tau_u^{*3}}} < 0.$$

This condition rewrites:

$$K''(\tau_u^*) < (1-\gamma_1^*)\psi\frac{\gamma_1^*}{\tau_u^{*2}}\left(\frac{1}{\tau}-\frac{\gamma_1^*}{\tau_u^*}\right)$$

Notice that the above RHS is always positive ($0 < \gamma_1^* < 1$ - see proof of Lemma 3 and it follows from $H(\gamma_1^*) = \tau_u^*$ that $\frac{\gamma_1^*}{\tau_u^*} < \frac{1}{\tau}$). The RHS is endogenous and interpretation is then delicate. Still, this inequality mainly says that K'' must not be too large. This can be easily understood: a small K'' implies that this is not very costly to adjust $\tau_u(i)$ for firm i so that this quantity cannot be easily predicted by others. As in the previous case with exogenous information precision, a small K'' is detrimental to existence of a SREE.

The implications of this Lemma for existence of a locally SREE are summarized in the next Proposition:

Proposition 6.

(i) If private information is endogenously acquired, a sufficient condition for the rational expectations equilibrium $\{\gamma_0^*, \gamma_1^*, \gamma_2^*, \tau_u^*\}$ to be a SREE is $\tau_p < \frac{1}{2}\tau_u^*$.

(ii) If marginal costs K' are constant, this sufficient condition is necessary as well.

(iii) The condition for existence of a SREE rewrites:

$$\frac{1}{Var(\theta|p)} - \frac{1}{Var(\theta)} < \frac{1}{2}\left(\frac{1}{Var(\theta|s_i)} - \frac{1}{Var(\theta)}\right),$$

the equilibrium market price p is at most half as informative regarding θ than the private signals.

Proof. See Appendix. □

The condition stated in Proposition 6 is obviously stronger than the respective condition for existence of a SREE with exogenously given information which is stated in Proposition 4. As already emphasized in the previous result, contrary to the above considered case without learning form prices, the presence of endogenous information acquisition in the model with learning from prices implies that conditions for existence of a SREE have to be qualified even if the noise terms in the private signals are uncorrelated across agents.

4.3. *The set of rationalizable solutions*

As usual, our condition for existence of a SREE is based on local stability of the best response mapping. Thus, without further restrictions on the set of strategies used by the firms, even this condition might not be sufficient for convergence of the eductive process towards the REE if the process starts from arbitrary but reasonable initial conditions.

In the remainder of the paper, we describe the set of rationalizable outcomes (that is the set of outcomes compatible with common knowledge of rationality and model) in the case of constant marginal costs of information acquisition. In particular, the question of global stability is solved (global stability corresponds to the case of a unique rationalizable outcome).

We first define the set R of rationalizable outcomes. To this purpose, we explain in greater details the eductive process. This process is defined by means of the best response mapping defined in Lemma 2. Denote $\mathcal{T}$ this best response mapping, that is: the best response of firm i to an aggregate behavior $(\gamma_0,\gamma_1,\gamma_2,\tau_u)$ is $(\gamma(i)_0,\gamma(i)_1,\gamma(i)_2,\tau(i)_u) = \mathcal{T}(\gamma_0,\gamma_1,\gamma_2,\tau_u)$ (notice that $\mathcal{T}$ is constant w.r.t. $\tau_u = \int_0^1 \tau_u(j)dj$, τ_u serves only notational purposes). We define R as the set of the limits of the iterates of $\mathcal{T}(\gamma_0,\gamma_1,\gamma_2,\tau_u)$ for an arbitrary $(\gamma_0,\gamma_1,\gamma_2,\tau_u)$. Formally, a vector $(\gamma_0',\gamma_1',\gamma_2',\tau_u')$ belongs to R whenever there is a vector $V_0 = (\gamma_0,\gamma_1,\gamma_2,\tau_u)$ in $\mathbb{R}^3 \times \mathbb{R}_+$ such that $(\gamma(i)_0,\gamma(i)_1,\gamma(i)_2,\tau(i)_u)$ is the limit of the sequence V_n defined by $V_{n+1} = \mathcal{T}(V_n)$ (shortly, $R = \mathcal{T}^\infty(\mathbb{R}^3 \times \mathbb{R}_+)$).

The assumption of constant marginal costs makes it possible to cut into two separate questions the problem of describing the set R: first, the best response dynamics of τ_u can be analyzed independently of any other variables, with the help of a single equation, as stated in the following Lemma (second, the whole set R can be described, based on the study of the dynamics of τ_u).

Lemma 4. *Consider the case with constant marginal costs. Denote $K'(\tau_u(i)) = \bar{\kappa}$ and $Q \equiv \sqrt{\frac{\psi}{2\bar{\kappa}}}$.*

(i) If the 'amount of information in the market' *is $\tau_u = \int_0^1 \tau_u(j)dj$ (that is the average precision of information), then the optimal information precision of firm $i \in I$ is characterized by:*

$$\tau'(i)_u = T(\tau_u) = \max\left\{0,\, Q - \tau - \frac{\alpha^2\tau_\varepsilon}{Q^2}\tau_u^2\right\} \tag{12}$$

(ii) The limit set $T^\infty(\mathbb{R}_+)$ is the set of the rationalizable information precisions (that is: τ_u is the last component of some element of R iff $\tau_u \in T^\infty(\mathbb{R}_+)$. More formally, $T^\infty(\mathbb{R}_+)$ is the projection of R on the τ_u-axis).

Proof. See Appendix. □

A precise proof is given in the Appendix. The main idea is to substitute equation (8) into (11). Equation (11), which implies that $\gamma_1^2 = \frac{2\bar{\kappa}}{\psi}\tau_u^2$, can then be used to eliminate γ_1.

Equation (12) describes the best response dynamics for the endogenously acquired amount of private information (the nonnegativity constraint $\tau(i)_u \geq 0$ is taken into account). Recall from Proposition 6 that, when marginal costs are constant, a SREE exists iff $\tau_p < \frac{1}{2}\tau_u^*$. In this case, some computations show that $\tau_p < \frac{1}{2}\tau_u^*$ rewrites $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} < Q-\tau$. Proposition 7 below describes the rationalizable precisions based on the map $\tau(i)_u = T(\tau_u)$:

Proposition 7. *Consider the case with constant marginal costs of information acquisition. Let S^* denote the set of rationalizable precisions which therefore represent outcomes of an eductive learning process on the side of the firms (formally, $S^* = T^\infty(\mathbb{R}_+)$):*

(a) If a SREE exists (that is $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} < Q-\tau$), $S^* = \{\tau_u^*\}$, i.e., τ_u^* is the unique and globally stable fixed point of the mapping $\tau_u' = T(\tau_u)$.

(b) If no SREE exits, one of the following two cases applies:

(b.1) $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} \leq Q-\tau < \frac{Q^2}{\alpha^2\tau_\varepsilon}$ such that $S^* = [\underline{\tau}_u, \overline{\tau}_u]$, where $\underline{\tau}_u$ and $\overline{\tau}_u$ satisfy $0 < \underline{\tau}_u < \tau_u^* < \overline{\tau}_u < Q-\tau$.

(b.2) $Q-\tau \geq \frac{Q^2}{\alpha^2\tau_\varepsilon}$ such that $S^* = T(\mathbb{R}_+) = [0, Q-\tau]$.

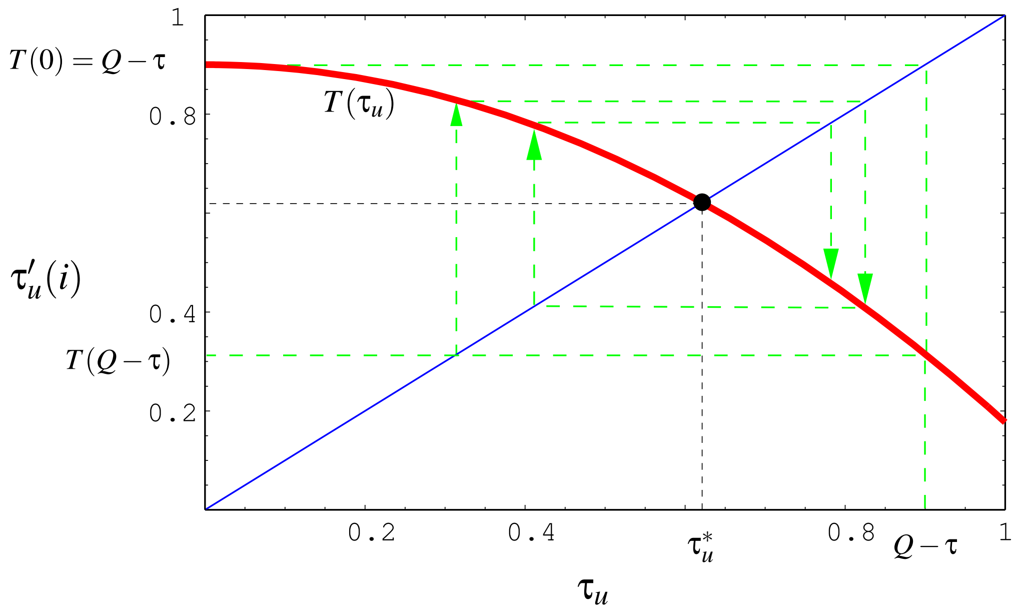
Proof. See Appendix. □

Point (a) in Proposition 7 states that, whenever a REE is locally stable, the REE precision τ_u^* is the globally unique rationalizable precision. Proposition 8 below extends this result and states that, whenever a REE is locally stable, there is a globally unique rationalizable demand as well, i.e. the REE is globally stable.

We now illustrate the three cases in Proposition 7 and the properties of the best response mapping (12) with three examples, bearing in mind that the 'amount of information in market' τ_u is necessarily non–negative and that $Q-\tau$ represents the maximum precision ever acquired ($Q-\tau = \sup_{\tau_u \geq 0} T(\tau_u)$). Thus, we can restrict the formal analysis of the best response dynamics described by the mapping $T(\tau_u)$ to the set $S = T(\mathbb{R}_+) = [0, Q-\tau]$ without loss of generality.

Example 1 (illustrating case (a)): Consider a numerically specified version of the model where $\alpha = -0.85$, $\psi = 1$, $\tau = 0.1$, $\tau_\varepsilon = 1$ and $\bar{\kappa} = 0.5$. From equation (8) and (11), equilibrium values can be computed as: $\gamma_1^* = 0.621$, $\tau_u^* = 0.621$ and $\alpha^2\tau_\varepsilon\gamma_1^{*2} = 0.279$. A SREE exists both when the amount of private information is exogenously given and equal to τ_u^*, and when it

Fig. 2. *Best response mapping $T(\tau_u)$ for example 1*



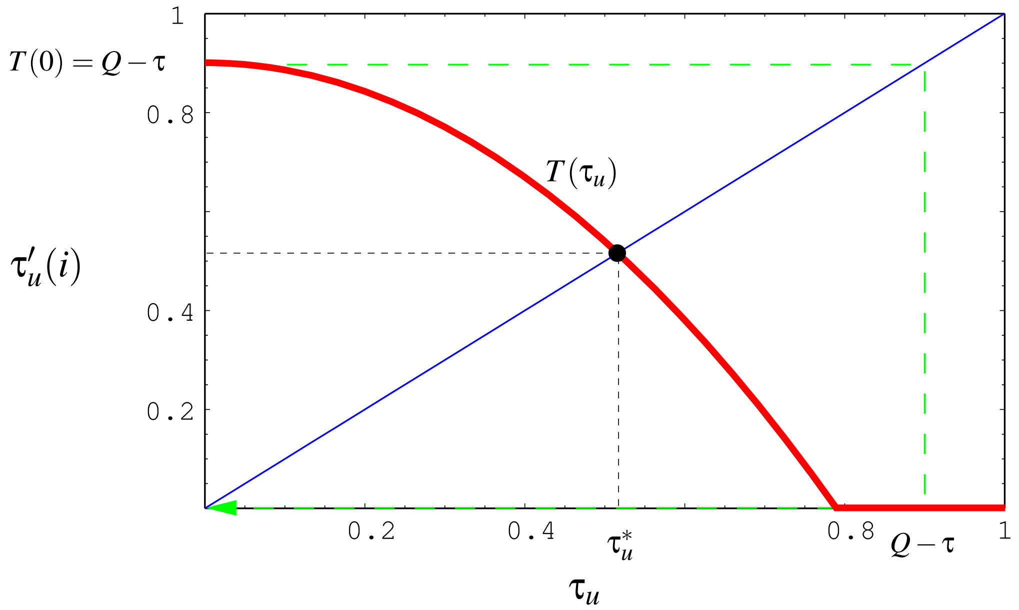
is endogenous (the two conditions $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \tau^*_u$ of Proposition 4 and $\alpha^2 \tau_\varepsilon \gamma_1^{*2} < \frac{1}{2}\tau^*_u$ in Proposition 6 are satisfied). Thus, in this example, the fact that information acquisition is endogenously determined is not relevant for existence of a SREE.

Figure 2 shows how the function $T(\tau_u)$ looks like. The eductive process proceeds similarly to the well known cobweb–dynamics.[10] The first step of the process is to consider that $\tau_u \geq 0$ is common knowledge. Given that T is decreasing, this fact implies that the maximum amount of private information a firm will ever acquire is given by $T(0) = Q - \tau > 0$. Since T and rationality are common knowledge, it is therefore also common knowledge that $\tau_u \leq T(0)$. A further step of the process shows then that no firm will ever choose $\tau(i)_u < T(T(0)) = T(Q - \tau)$. Thus, this second step restricts the set of possible precision to $[T(T(0)), T(0)]$. As indicated in the figure, the dynamics that result if this kind of reasoning is iterated converges to the REE precision τ^*_u (because the condition stated in Proposition 6 is satisfied): each firm can educe that only the precision REE $\tau^*_u = 0.621$ constitutes a possible solution under the assumptions of common knowledge of individual rationality and model.

Example 2 (illustrating case (b1)): The precision of the noise is now $\tau_\varepsilon = 1.3$, which is larger than in example 1. From equations (8) and (11), equilibrium values can be computed as $\gamma_1^* = 0.582$, $\tau^*_u = 0.582$ and $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.318$. We have then $\frac{1}{2}\tau^*_u < \alpha^2 \tau_\varepsilon \gamma_1^{*2} < \tau^*_u$: a SREE

[10] This description of the process originates in Guesnerie (1992).

Fig. 3. *Best response mapping $T(\tau_u)$ for example 2*



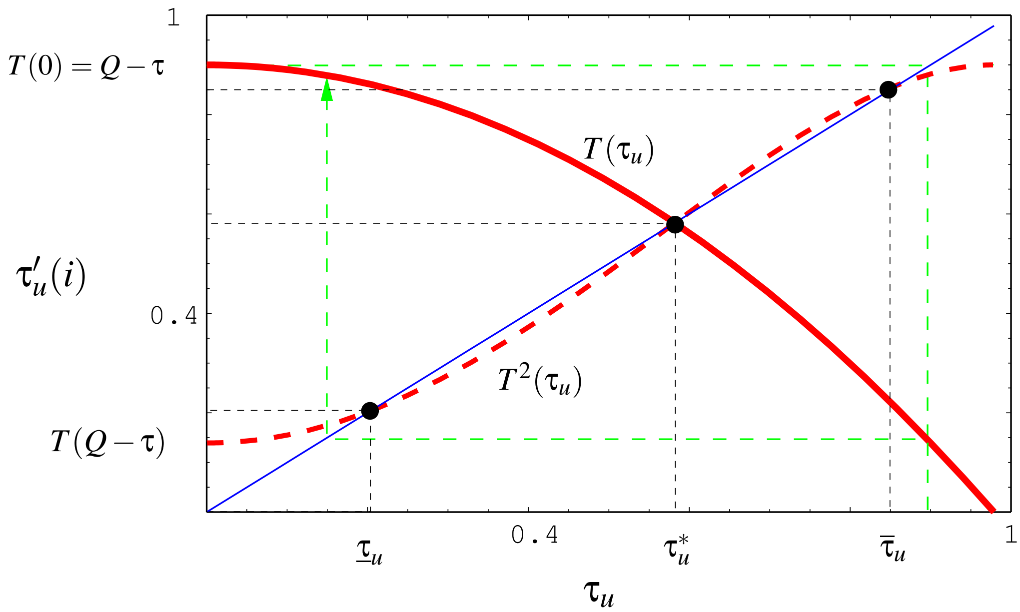
exists if the amount τ^*_u of private information is exogenously given, but does not exist if information is endogenously acquired.

On figure 4, we have now also plotted the function $T(\tau_u)$ and the second iterate of this function $T^2(\tau_u) \equiv T(T(\tau_u))$. As can be seen, this function possesses two additional fixed points, denoted $\underline{\tau}_u$ and $\overline{\tau}_u$. Notice too that the associated 2–cycle is stable. If we repeat the argumentation used in the discussion of the first example, we therefore get a process which converges to this 2–cycle: the first step of the process shows that $\tau_u \leq T(0) = Q - \tau$, a second step shows that $\tau_u \geq T(Q-\tau) = T^2(0)$,... Clearly, iterating this argument eliminate the precisions outside the interval $[\underline{\tau}_u, \overline{\tau}_u]$, but not precisions in $[\underline{\tau}_u, \overline{\tau}_u]$. It follows that all precisions in the set $[\underline{\tau}_u, \overline{\tau}_u]$ constitute possible solutions under individual rationality and common knowledge.

Example 3 (illustrating case (b2)): The precision of noise is $\tau_\varepsilon = 2.0$ and, hence, larger than in examples 1 and 2. At the REE, $\tau^*_u = 0.512$ and $\alpha^2 \tau_\varepsilon \gamma_1^{*2} = 0.384$. The REE is still strongly rational, if information precision τ^*_u is assumed to be exogenously given, but not (since $\tau^*_u/2 = 0.258$), when information acquisition is endogenous. The best response function $T(\tau_u)$ depicted in figure 3 reveals that in this example we have $T(Q-\tau) = 0$, i.e., now the nonnegativity constraint on $\tau'(i)_u$ becomes relevant.

Again, we repeat the argumentation used in the discussion of the first example. However, the process will here immediatly converges to the whole interval $[0, Q-\tau]$. Indeed, the first step of the process still shows that $\tau_u \leq T(0) = Q - \tau$. If, however, each firm acquires this

Fig. 4. *Best response mapping $T(\tau_u)$ for example 3*



maximum amount $T(0)$ of private information such that $\tau_u = T(0)$, there is so much information in the market, that it is individually optimal to stop the acquisition of information, i.e., $T(Q-\tau) = 0$. In other words, the second step of the process shows that $\tau_u \geq T(Q-\tau) = 0$, no additional restriction is created by this second step... Clearly, iterating this argument does not eliminate any precision: all the precisions in $[0, Q-\tau]$ constitute possible solutions under individual rationality and common knowledge.

We now describe the set R of rationalizable outcomes, that is: we analyze the consequences of nonexistence of a SREE for the remaining weights γ_0, $\gamma_1$and γ_2 of the linear supply function. The question is which predictions regarding the linear supply functions used by the firms can be made if no SREE exists and if only the assumptions of individual rationality and common knowledge are imposed. The next proposition establishes the respective result:

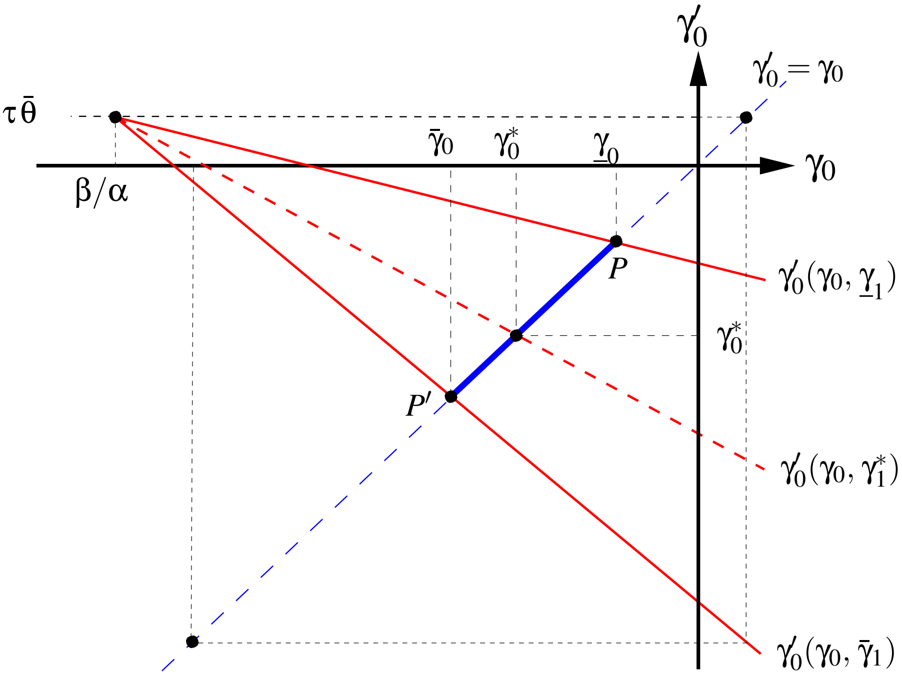
Proposition 8. *Consider the case with constant marginal costs of information acquisition.*

(a) If a SREE exists (that is $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} < Q - \tau$), the REE is the only rationalizable outcome: $(\gamma_0^*, \gamma_1^*, \gamma_2^*, \tau_u^*)$ is the unique and globally stable fixed point of the mapping $\mathcal{T}$.

(b) If no SREE exits, one of the following two cases applies:

(b.1) Assume $\frac{3}{4}\frac{Q^2}{\alpha^2\tau_\varepsilon} \leq Q - \tau < \frac{Q^2}{\alpha^2\tau_\varepsilon}$, such that $S^* = [\underline{\tau}_u, \bar{\tau}_u]$. The set R of rationalizable outcomes is exactly the product set $[\underline{\gamma}_0, \bar{\gamma}_0] \times [\underline{\gamma}_1, \bar{\gamma}_1] \times [\underline{\gamma}_2, \bar{\gamma}_2] \times S^*$, where $[\underline{\gamma}_1, \bar{\gamma}_1] = [\underline{\tau}_u/Q, \bar{\tau}_u/Q]$, and $[\underline{\gamma}_0, \bar{\gamma}_0]$ as well as $[\underline{\gamma}_2, \bar{\gamma}_2]$ denote the set of fixed points of equations (7) and (9) given $\gamma_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$ and $\tau_u \in S^*$.

Fig. 5. *Best response dynamics for the weight* γ_0.



(b.2) Assume $Q-\tau \geq \frac{Q^2}{\alpha^2\tau_\varepsilon}$, *such that* $S^* = [0, Q-\tau]$. *The set R of rationalizable outcomes is exactly the product set* $I\!R \times [0, \bar{\gamma}_1] \times I\!R \times S^*$, *where* $[\underline{\gamma}_1, \bar{\gamma}_1] = [0, (Q-\tau)/Q]$.

Proof. See Appendix. □

Thus, given the restriction of constant marginal costs of information acquisition, we get the result stated in Point a: if the price in the REE is at most half as informative as private signals, common knowledge of individual rationality and model justifies the REE without the need of any initial restriction to the set of linear strategies. Otherwise, if this condition for existence of an SREE fails to hold, individual rationality and common knowledge alone are not sufficient to justify the REE. The set of rationalizable outcomes is described in Point (b).

A detailed proof is given in Appendix. Characterizing the rationalizable γ_1 is an easy task, since $\gamma_1 = \tau_u/Q$. Characterizing the rationalizable γ_0 and γ_2 requires to analyze the dynamical properties of the two equations (7)–(9) for all values of τ_u that are rationalizable, i.e., for all $\tau_u \in S^*$ according to Proposition 7. This must be true because equations (7) and (9) are linear in $\gamma(i)_0$ and $\gamma(i)_2$ respectively and because $\gamma(i)_1$ is proportional to $\tau(i)_u$.

Figure 5 serves to illustrate the result concerning $\gamma(i)_0$ and $\gamma(i)_2$. The figure shows the best response function for the weight γ_0 according to equation (7) in case of a two–cycle for all values of γ_1 within the set $[\underline{\gamma}_1, \bar{\gamma}_1]$. Recall that equations (7) and (9) are linear in $\gamma(i)_0$ and

$\gamma(i)_2$ respectively and $\gamma(i)_1$ is proportional to $\tau(i)_u$ (see the proofs of Propositions 7 and 8). The result stated in Proposition 8 builds on the fact that the maximum of slopes of these best responses (i.e., the slope of the straight line $\gamma_0'(\gamma_0, \bar{\gamma}_1)$) is less then one in absolute value, whenever a stable two–cycle exists. Given this it is possible to restrict the set of weights γ_0 that are compatible with rationality and common knowledge to values corresponding to the line segment between the points P and P' in the figure. [11] In a similar fashion it can be shown that regarding the weight γ_2 there exists a set $[\underline{\gamma}_2, \bar{\gamma}_2]$ of weights such that all γ_2 within this set are compatible with rationality and common knowledge. Thus, even if there exists no SREE, the assumptions of rationality and common knowledge allow us to restrict the set of possible supply functions that will be used by rational firms, when a stable two–cycle exists. If even this is not the case, that is, if not even a stable two–cycle exists, it is still possible to restrict the weights γ_1 and the precisions τ_u of the privately acquired information, but the best response mappings (7) and (9) are unstable for some of the reasonable values for γ_1 and τ_u. This means any values for γ_0 and γ_2 are compatible with rationality and common knowledge in this case.

4.4. *Nonexistence of an SREE and the Grossman–Stiglitz–Paradox*

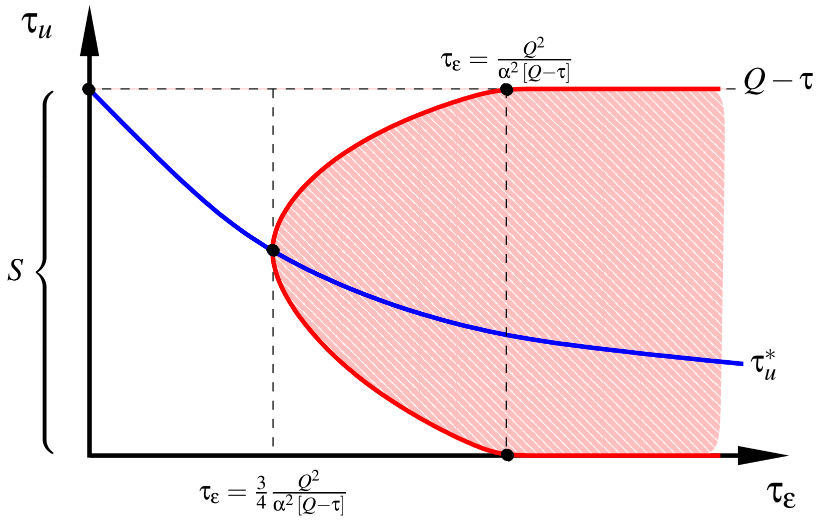
While nonexistence of an SREE implies that the assumptions of individual rationality and common knowledge are not sufficient to predict the REE as a reasonable outcome of our model, Proposition 7 also reveals that these can still lead to restrictions on the set of rationalizable precisions. Figure 6 displays the implications of this Proposition. The figure shows the set of rationalizable precisions τ_u dependent on the precision of the noise τ_ε. Notice, that an increase in the precision of the noise implies an increase in the precision of the market price even though the amount of information acquired in the REE becomes smaller when τ_ε grows. [12] Hence, moving along the horizontal axis in the figure, τ_ε increases and the informativeness τ_p^* of the market price in the REE increases too. The solid line in the figure represents the amount of information acquisition in the REE, which decreases as τ_ε increases.

Now, as long $\tau_\varepsilon < \frac{3}{4}\frac{Q^2}{\alpha^2\,[Q-\tau]}$, which is equivalent to $\tau_p^* < \tau_u^*/2$, a SREE exists. Thus, the corresponding part of the solid line also represents the set of rationalizable precisions, which coincides with the REE precision. If $\tau_\varepsilon \geq \frac{3}{4}\frac{Q^2}{\alpha^2\,[Q-\tau]}$, no SREE exists. The shaded area in the figure represents all precisions that are rationalizable in such a case. As can be seen, whenever the precision of prices is not too large, i.e. if $\tau_\varepsilon < \frac{Q^2}{\alpha^2\,[Q-\tau]}$, the best response

[11] The underlying argument is quite similar to the one presented in section 3 in the discussion of figure 1.

[12] From Proposition 5 we get that in a REE $\gamma_1^* = \tau_u^*/Q$, such that τ_p^* is given by $\tau_p^* = \alpha^2\,(\tau_u^*)^2\,\tau_\varepsilon$.

Fig. 6. *Set of rationalizable strategies.*



dynamics exhibit a two–cycle and the assumptions of individual rationality and common knowledge lead to restrictions on the set of rationalizable precisions. Even this is impossible, when prices become too informative, i.e. if $\tau_\varepsilon \geq \frac{Q^2}{\alpha^2\,[Q-\tau]}$.

As prices become fully informative regarding the unknown parameter, which happens if $\tau_\varepsilon \longrightarrow \infty$, the famous Grossman–Stiglitz–Paradox appears: In such a case, no firm has an incentive to acquire costly the information, prices will reveal anyway. If, however, no firm acquires any information, the market price cannot be revealing. In this case no REE exists. This problem cannot arise in our model, since the precision τ_ε of the noise is bounded from above. In this case the precision of the market price is bounded from above too and, thus, a REE always exists. Notice, that the Grossman–Stiglitz–Paradox relies on an argument quite similar to our discussion of the map $\tau(i)_u = T(\tau_u)$ in the preceeding section. The Grossman–Stiglitz–Paradox says that with $\tau_\varepsilon \longrightarrow \infty$ any $\tau_u > 0$ implies that $\tau(i)_u = 0$ is a best response, while with $\tau_u = 0$, a best response might be to acquire information, i.e. $\tau(i)_u > 0$. If τ_ε in our model is greater than $\frac{Q^2}{\alpha^2\,[Q-\tau]}$ but still bounded from above, the best response map regarding the privately acquired precision doesn't take such an extreme form, but a similar problem arises: Individual rationality and common knowledge are not sufficient to exclude the possibility that it is individually optimal to acquire no private information because there might already be much information in the market. Since the corresponding best response mapping implies that it is optimal to acquire information individually if no other firm does it, but to stop the acquisition of information if every firms behaves like this, it is not possible to restrict the set S of precisions in any way. This problem disappears only if prices are not too informative. If for instance $\tau_\varepsilon < \frac{Q^2}{\alpha^2\,[Q-\tau]}$, the assumptions of individual rationality and common knowledge indeed provide further restrictions on the set of precisions and agents are at least able to exclude the possibility that there might be no information at all in the market, but unless $\tau_\varepsilon < \frac{3}{4}\frac{Q^2}{\alpha^2\,[Q-\tau]}$ the set of rationalizable precisions will not coincide with the REE precision.

Viewed from this perspective, our condition for existence of a SREE in the model with learning from prices implies that the problem underlying Grossman–Stiglitz–Paradox cannot arise. The reason is that this condition simply results in an upper bound for the informativeness of the market price. Moreover, our result shows that the fundamental problem described by the Grossman–Stiglitz–Paradox is not exclusively related to the existence of fully informative REE. As we have shown, even REE where prices are only partially informative are subject to coordination difficulties and it might be impossible to justify these equilibria with the assumptions of individual rationality and common knowledge. It is not enough to have prices which are not fully informative in order to get rid of these problems. The informativeness of prices must be below a well defined upper bound in order to achieve this.

5. Conclusions

In the present paper, we have shown how known results for existence of SREE must be modified, if models with endogenously acquired private information are considered. Generally, endogenous acquisition of information leads to stronger conditions for existence of a SREE than the respective conditions known for the case with exogenously given information. When there is no learning from prices, however, conditions for existence of a SREE are the same as with exogenously given information if the noise terms of the private signals are uncorrelated across agents. While the dynamics of the best response mapping change due the endogeneity of private information, this causes no additional coordination difficulties if these noise terms are uncorrelated.. When there is learning from prices, we arrive at stronger conditions for existence of a SREE than known for the case with exogenously given information even if these noise terms are uncorrelated. In particular, it was shown that prices in a REE need to be half as informative than private signals for a SREE to exist in case of learning from prices, whereas it is sufficient for prices to be less informative than private signals without such learning. It was also possible to give an interpretation of the result that falls back on the well known Grossman–Stiglitz–Paradox of the impossibility of informationally efficient markets. Viewed from this perspective, our result says that for existence of a SREE markets have to show a minimum level of informational inefficiency.

Future work on this subject will analyze the case of increasing marginal costs of information acquisition in more detail in order to check the robustness of the results obtained for the case of constant marginal costs. Moreover, it should be analyzed whether the results carry over to financial market models with learning from current prices, where risk aversion of traders is allowed for.

References

DeMarzo, P. and Skiadas, C., (1998), Aggregation, determinacy, and informational efficiency for a class of economies with asymmetric information, Journal of Economic Theory 80, 123–152.

Desgranges, G., (1999), Common knowledge and information revealed by prices: An example, mimeo.

Desgranges, G., Geoffard, P. and Guesnerie, R., (2003), Do prices transmit rationally expected information?, Journal of the European Economic Association 1, 124–153.

Grossman, S., (1976), On the efficiency of competitive stock markets where traders have diverse information, Journal of Finance 31, 573–585.

Grossman, S. and Stiglitz, J., (1980), On the impossibility of informationally efficient markets, American Economic Review 70, 393–408.

Guesnerie, R., (1992), An exploration of the eductive justifications of the rational–expectations hypothesis, American Economic Review 82, 1254–1278.

Guesnerie, R., (2002), Anchoring economic predictions in common knowledge, Econometrica 70, 439–480.

Heinemann, M., (2002), Existence of strongly rational expectations equilibria on asset markets with asymmetric information, Schumpeter Working Paper 2002-2, Humboldt–University Berlin, Berlin.

Heinemann, M., (2004), Are rational expectations equilibria with private information eductively stable?, Journal of Economics 82, 169–194.

Verrecchia, R., (1982), Information acquisition in a noisy rational expectations economy, Econometrica 50, 1415–1430.

Appendix

Proof of Proposition 1. We first derive the best response mapping. After that, we show that the best response mapping has a unique fixed point, that is: there is a unique linear REE. Let us assume $\bar{\theta} = 0$.
Profit $\pi(i)$ of firm i in an equilibrium is given by:

$$\pi(i) = [p-\theta]\,x(i) - \frac{1}{2}\frac{1}{\psi}\,[x(i)]^2 - K(\tau(i)_u)$$
$$= \psi\left[\beta - \alpha\,[\gamma_0 + \gamma_1\,\bar{s}] + \varepsilon - \theta\right]\left[\gamma(i)_0 + \gamma(i)_1\,s(i)\right]$$
$$-\frac{1}{2}\psi\left[\gamma(i)_0 + \gamma(i)_1\,s(i)\right]^2 - K(\tau(i)_u)$$

Taking expectations then yields:

$$\mathrm{E}[\pi(i)] = \psi E\left[\,[\beta - \alpha\,[\gamma_0 + \gamma_1\,\bar{s}] + \varepsilon - \theta]\,[\gamma(i)_0 + \gamma(i)_1\,s(i)]\right]$$
$$-\frac{1}{2}\psi E\left[\,[\gamma(i)_0 + \gamma(i)_1\,s(i)]^2\right] - K(\tau(i)_u)$$
$$= \psi\left(\beta\gamma(i)_0 - \alpha\left[\gamma_0\gamma(i)_0 + \gamma_1\,\gamma(i)_1\left(\frac{1}{\tau} + \frac{1}{\tau_{\bar{u}}}\right)\right] - \gamma(i)_1\,\frac{1}{\tau}\right)$$
$$-\frac{1}{2}\psi\left(\gamma(i)_0^2 + \gamma(i)_1^2\,\frac{\tau + \tau(i)_u}{\tau(i)_u\,\tau}\right) - K(\tau(i)_u)$$

The three partial derivatives with respect to $\gamma(i)_0$, $\gamma(i)_1$ and $\tau_u(i)$ of the expected profit are then:

$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_0} = \psi\,(\beta - \alpha\gamma_0 - \gamma(i)_0)$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_1} = -\psi\left(\alpha\gamma_1\left(\frac{1}{\tau} + \frac{1}{\tau_{\bar{u}}}\right) + \frac{1}{\tau} + \gamma(i)_1\frac{\tau + \tau(i)_u}{\tau(i)_u\,\tau}\right)$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} = \frac{1}{2}\psi\frac{\gamma(i)_1^2}{\tau(i)_u^2} - K'(\tau(i)_u)$$

The first order conditions are then:

$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_0} = 0$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\gamma(i)_1} = 0$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} \le 0 \text{ and } \tau(i)_u\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} = 0 \text{ (given the constraint } \tau(i)_u \ge 0)$$
$$\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} \ge 0 \text{ and } (\tau(i)_u - \tau_{\bar{u}})\frac{\partial\mathrm{E}[\pi(i)]}{\partial\tau(i)_u} = 0 \text{ (given the constraint } \tau(i)_u \le \tau_{\bar{u}})$$

$\gamma(i)_0$, $\gamma(i)_1$ and $\tau_u(i)$ are then implictly defined by the following conditions:

$$\gamma(i)_0 = \beta - \alpha\gamma_0 \tag{A.1a}$$
$$\gamma(i)_1 = -\left(\alpha\gamma_1\left(1 + \frac{\tau}{\tau_{\bar{u}}}\right) + 1\right)\frac{\tau(i)_u}{\tau + \tau(i)_u} \tag{A.1b}$$

and, concerning $\tau(i)_u$:

$$\tau(i)_u = \tau_{\bar{u}} \quad \text{if} \quad \frac{\psi}{2}\left[\frac{\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau+\tau_{\bar{u}}}\right]^2 > K'(\tau_{\bar{u}}),$$

$$\tau(i)_u = 0 \quad \text{if} \quad \frac{\psi}{2}\left[\frac{\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau}\right]^2 < K'(0),$$

$$\frac{\psi}{2}\left[\frac{\alpha\gamma_1\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau+\tau(i)_u}\right]^2 - K'(\tau(i)_u) = 0 \quad \text{otherwise.}$$

The latter equation rewrites:

$$\frac{\psi}{2}\left[\frac{\gamma_1(i)}{\tau(i)_u}\right]^2 - K'(\tau(i)_u) = 0 \quad \text{otherwise.} \tag{A.2}$$

These conditions define the best response mapping.
A REE is a fixed point of the above best response mapping, that is a solution of the system:

$$\gamma_0^* = \beta - \alpha\gamma_0^*$$
$$\gamma_1^* = -\left(\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right)\frac{\tau_u^*}{\tau+\tau_u^*}$$

and

$$\tau_u^* = \tau_{\bar{u}} \quad \text{if} \ \frac{\psi}{2}\left[\frac{\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau+\tau_{\bar{u}}}\right]^2 > K'(\tau_{\bar{u}}),$$

$$\tau_u^* = 0 \quad \text{if} \ \frac{\psi}{2}\left[\frac{\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1}{\tau}\right]^2 < K'(0),$$

$$\frac{\psi}{2}\left[\frac{\gamma_1^*}{\tau_u^*}\right]^2 - K'(\tau_u^*) = 0 \quad \text{otherwise.}$$

We now solve for the equilibrium coefficients. The two supply coefficients are characterized as functions of the equilibrium information precision τ_u^*:

$$\gamma_0^* = \frac{\beta}{1+\alpha} > 0 \tag{A.4a}$$

$$\gamma_1^* = -\frac{\tau_u^*}{\tau+\left[1+\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right]\tau_u^*} \leq 0 \tag{A.4b}$$

The optimal information precision τ_u^* is either the solution $\tau_{\bar{u}} > \tilde{\tau}_u > 0$ of the equation (if this solution exists):

$$\frac{\psi}{2}\frac{1}{\left(\tau+\left[1+\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right]\tilde{\tau}_u\right)^2} = K'(\tilde{\tau}_u)$$

or $\tau_u^* = 0$ or $\tau_u^* = \tau_{\bar{u}}$. $\tau_u^* = 0$ if and only if $\frac{\psi}{2}\frac{1}{\tau^2} < K'(0)$, and $\tau_u^* = \tau_{\bar{u}}$ if and only if

$$\frac{\psi}{2}\frac{1}{((1+\alpha)(\tau_{\bar{u}}+\tau))^2} > K'(\tau_{\bar{u}})$$

Together with the above derived Eqs. (A.4a) and (A.4b) the REE is then completely described. □

Proof of Lemma 1. The best response mapping has already been derived while proving Proposition 1. It is given by Eqs. (A.1a), (A.1b) and (A.2).

Assume that the REE satisfies $\tau_{\bar{u}} > \tau_u^* > 0$. The total differentials of these equations evaluated at the REE are given by:

$$d\gamma(i)_0 = -\alpha d\gamma_0$$

$$d\gamma(i)_1 = -\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\frac{\tau_u^*}{\tau+\tau_u^*}d\gamma_1 - \left(\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right)\frac{\tau}{(\tau+\tau_u^*)^2}d\tau(i)_u$$

$$W\,d\gamma(i)_1 - d\tau_u(i) = 0\,,$$

where

$$W \equiv \frac{\psi\frac{\gamma_1^*}{\tau_u^{*2}}}{K''+\psi\frac{\gamma_1^{*2}}{\tau_u^{*3}}} < 0,$$

$$-\left(\alpha\gamma_1^*\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)+1\right) = \gamma_1^*\frac{\tau+\tau_u^*}{\tau_u^*} < 0$$

Using matrices, this system can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\gamma_1^*\frac{\tau}{\tau_u^*(\tau+\tau_u^*)} \\ 0 & W & -1 \end{bmatrix}\begin{pmatrix} d\gamma(i)_0 \\ d\gamma(i)_1 \\ d\tau(i)_u \end{pmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\frac{\tau_u^*}{\tau+\tau_u^*} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\tau_u \end{pmatrix}$$

We write this system as $x' = Px$, where $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. We have indeed:

$$P = A^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & D \\ 0 & W & -1 \end{bmatrix}^{-1}\begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha C & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -C\frac{\alpha}{DW+1} & 0 \\ 0 & -CW\frac{\alpha}{DW+1} & 0 \end{bmatrix}$$

Some computations gives:

$$\frac{\alpha C}{DW-1} = -\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\frac{\tau_u^*}{\tau+\tau_u^*}}{-\gamma_1^*\frac{\tau}{\tau_u^*(\tau+\tau_u^*)}\frac{\psi\frac{\gamma_1^*}{\tau_u^{*2}}}{K''+\psi\frac{\gamma_1^{*2}}{\tau_u^{*3}}}+1} = -\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\left(K''\tau_u^{*3}+\psi\gamma_1^{*2}\right)}{\psi\gamma_1^{*2}+(\tau+\tau_u^*)K''\tau_u^{*2}}$$

The respective eigenvalues λ_1 , λ_2 and λ_3 are:

$$\lambda_1 = 0, \quad \lambda_2 = -\alpha < 0, \quad \lambda_3 = -\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\left(K''\tau_u^{*3}+\psi\gamma_1^{*2}\right)}{\psi\gamma_1^{*2}+(\tau+\tau_u^*)K''\tau_u^{*2}} < 0$$

It follows that the spectral radius of P is either α or $|\lambda_3|$. Stability is then characterized by the condition:

$$\max\{\alpha, |\lambda_3|\} < 1.$$

We now make this condition explicit. Notice first that $|\lambda_3| < 1$ rewrites:

$$\begin{aligned} & \alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\left(K''\tau_u^{*3}+\psi\gamma_1^{*2}\right) < \psi\gamma_1^{*2}+(\tau+\tau_u^*)K''\tau_u^{*2} \\ \Leftrightarrow \quad & \alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\left(K''\tau_u^{*}+2K'\right) < 2K'+(\tau+\tau_u^*)K'' \\ \Leftrightarrow \quad & -\left(\tau_u^*\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau\right)\frac{K''}{K'} < 2\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right) \end{aligned}$$

Assume now $\alpha < 1$ (a necessary condition for stability). We have:

(i) If $\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right) < 1$, then the above inequality holds true and $|\lambda_3| < 1$. In this case, the REE is stable.

(ii) If $\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right) > 1$, then the above inequality rewrites:

$$\frac{K''}{K'} > 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau_u^*\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau} > 0.$$

If this condition is met, $|\lambda_3| < 1$ and the REE is stable. Otherwise, $|\lambda_3| > 1$ and the REE is not stable.

□

Proof of Corollary 1. Consider the RHS of Condition (5). We have: $\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1 > 0$, and (given $\tau_u^* \le \tau_{\bar{u}}$):

$$\tau_u^*\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau > \tau_{\bar{u}}\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau > 0$$

It follows that the RHS increases in τ_u^* from $2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau}$ to $2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau_{\bar{u}}\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau}$. Points (i) and (ii) follow.

To prove Point (iii), rewrite Condition (5) as

$$vr > 2\frac{\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)-1}{\tau_u^*\left(1-\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)+\tau}, \tag{A.6}$$

with $\tau_u^* = \frac{\sqrt{\frac{\psi}{2K'(\tau_u^*)}}-\tau}{1+\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)} > 0$. τ_u^* decreases from 0 to $\tau_{\bar{u}}$ when $K'(\tau_u^*)$ decreases from $\psi/2\left(\left(1+\alpha\left(1+\frac{\tau}{\tau_{\bar{u}}}\right)\right)\tau_{\bar{u}}+\tau\right)^2$ to $\psi/2\tau^2$. It follows that stability obtains iff $K'(\tau_u^*)$ is larger than a certain threshold (such that the two handsides of the above inequality are equal). □

Proof of Corollary 2. To check the comments, differentiate the polynomial equation $H(\gamma_1^*) = \tau_u$ defining γ_1^*:

$$\begin{aligned}&\left[3(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right] d\gamma_1^* \\ &= -2\gamma_1^*\left[(\gamma_1^*)^2\alpha\tau_\varepsilon\right] d\alpha - \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\right] d\tau_\varepsilon - \gamma_1^* d\tau + (1-\gamma_1^*)\, d\tau_u,\end{aligned}$$

implying that

$$\begin{aligned}\frac{d\tau_p}{d\alpha} &= 2\alpha\gamma_1^{*2}\tau_\varepsilon + 2\alpha^2\gamma_1^*\tau_\varepsilon \frac{-2\gamma_1^*\left[(\gamma_1^*)^2\alpha\tau_\varepsilon\right]}{\left[3(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right]} \\ &= 2\alpha\gamma_1^{*2}\tau_\varepsilon \frac{(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u}{\left[3(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right]} > 0 \\ \frac{d\tau_p}{d\tau_\varepsilon} &= \alpha^2\gamma_1^{*2} + 2\alpha^2\gamma_1^*\tau_\varepsilon \frac{-\gamma_1^*\left[(\gamma_1^*)^2\alpha^2\right]}{\left[3(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right]} \\ &= \alpha^2\gamma_1^{*2} \frac{(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u}{\left[3(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right]} > 0\end{aligned}$$

To prove the corollary, define $X = \sqrt{\frac{\tau_u}{\alpha^2\tau_\varepsilon}}$. Existence of a SREE is characterized by $\gamma_1^* < X$. $\gamma_1^* < X$ is equivalent to $H(X) > \tau_u$ (because $H' > 0$). The condition $H(X) > \tau_u$ rewrites:

$$\begin{aligned}\tau + 2\tau_u &> \sqrt{\alpha^2\tau_\varepsilon\tau_u}, \\ \tau^2 + \left(4\tau - \alpha^2\tau_\varepsilon\right)\tau_u + 4\tau_u^2 &> 0\end{aligned}$$

If $\alpha^2\tau_\varepsilon < 8\tau$, then the condition holds true. If $\alpha^2\tau_\varepsilon > 8\tau$, then the condition becomes:

$$\tau_u < \frac{\alpha^2\tau_\varepsilon - 4\tau - \sqrt{\alpha^2\tau_\varepsilon\left(\alpha^2\tau_\varepsilon - 8\tau\right)}}{8} \text{ or } \tau_u > \frac{\alpha^2\tau_\varepsilon - 4\tau + \sqrt{\alpha^2\tau_\varepsilon\left(\alpha^2\tau_\varepsilon - 8\tau\right)}}{8}.$$

□

Proof of Proposition 5. The coefficient γ_1^* is the unique solution of the polynomial

$$H(\gamma_1^*) \equiv \gamma_1^*\left[(\gamma_1^*)^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right] = \tau_u,$$

and the additional first order condition of the maximisation of profit is:

$$\frac{\psi}{2}\left(\frac{\gamma_1^*}{\tau_u}\right)^2 = K'(\tau_u)$$

Combining the 2 equations shows that

$$\gamma_1^* = \sqrt{\frac{2K'(\tau_u)}{\psi}}\tau_u$$

and

$$\sqrt{\frac{2K'(\tau_u)}{\psi}}\left[\frac{2K'(\tau_u)}{\psi}\tau_u^2\alpha^2\tau_\varepsilon + \tau + \tau_u\right] = 1,$$

The LHS of the latter equation is an increasing function of τ_u (increasing from $\sqrt{\frac{2K'(0)}{\psi}}\tau$ to $+\infty$ when τ_u increases from 0 to $+\infty$). This implies that there is a unique τ_u solving this equation. □

Proof of Lemma 2. We first make precise the assumptions needed to define aggregate demand when information precisions are heterogenous among agents. Consider that the demand of every firm j is

$$x(j) = \psi[(1-\gamma(j)_2)p - \gamma(j)_0 - \gamma(j)_1(\theta + u(j))],$$

with $\tau_u(j)$ the precision of $u(j)$ ($\theta + u(j) = s(j)$ is the private signal). Assume that $\int_0^1 \gamma(j)_0 dj$, $\int_0^1 \gamma(j)_1 dj$ and $\int_0^1 \gamma(j)_2 dj$ exist. Aggregate demand is then defined as:

$$\int_0^1 x(j)dj = \psi[(1-\gamma_2)p - \gamma_0 - \gamma_1\theta - \int_0^1 \gamma(j)_1 u(j)dj].$$

Assume that $\int_0^1 \gamma(j)_1 u(j)dj = 0$ almost surely (this is a law of large numbers: every $\gamma(i)_1 u(i)$ is a centered gaussian variable. It follows that aggregate demand is

$$\int_0^1 x(j)dj = \psi[(1-\gamma_2)p - \gamma_0 - \gamma_1\theta],$$

and aggregate behavior is well characterized by $(\gamma_0, \gamma_1, \gamma_2)$.

Now, deriving the best response of a firm i to $(\gamma_0, \gamma_1, \gamma_2)$ is purely routine. Profit $\pi(i)$ of firm i is given by:

$$\pi(i) = [p-\theta]x(i) - \frac{1}{2}\frac{1}{\psi}[x(i)]^2 - K(\tau(i)_u)$$

Clearly, the profit maximizing output is $x(i) = \psi(p - E[\theta|p, s(i)])$. Given that $p = \beta - \frac{1}{\phi}\int_0^1 x(j)dj + \varepsilon$, we have

$$p = \frac{\beta + \alpha[\gamma_0 + \gamma_1\theta] + \varepsilon}{1 + \alpha(1-\gamma_2)}$$

and we have

$$E[\theta|p, s(i)] = \frac{\alpha^2\gamma_1^2\tau_\varepsilon \hat{p} + \tau(i)_u s(i)}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon},$$

where $\hat{p} = \theta + \frac{\varepsilon}{\alpha\gamma_1}$(the use of $\hat{p}$ makes computations simpler: consider $E[\theta|p, s(i)] = E[\theta|\hat{p}, s(i)]$). We have then:

$$(1-\gamma(i)_2)p - \gamma(i)_0 - \gamma(i)_1 s(i) = p - \frac{\alpha^2\gamma_1^2\tau_\varepsilon\frac{(1+\alpha(1-\gamma_2))p - (\beta+\alpha\gamma_0)}{\alpha\gamma_1} + \tau(i)_u s(i)}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}$$

implying:

$$\gamma(i)_0 = -\frac{\alpha\gamma_1\tau_\varepsilon(\beta + \alpha\gamma_0)}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}$$
$$\gamma(i)_1 = \frac{\tau(i)_u}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}$$
$$\gamma(i)_2 = \frac{\alpha\gamma_1\tau_\varepsilon(1 + \alpha(1-\gamma_2))}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}$$

To compute the optimal precision, consider the expected profit:

$$E[\pi(i)] = E\left((p-\theta)x(i) - \frac{1}{2}\frac{1}{\psi}[x(i)]^2\right) - K(\tau(i)_u)$$

The partial derivative with respect to $\tau_u(i)$ is then:

$$\frac{\partial E[\pi(i)]}{\partial\tau(i)_u} = \frac{\partial}{\partial\tau(i)_u}E\left((p-\theta)x(i) - \frac{1}{2}\frac{1}{\psi}[x(i)]^2\right) - K'(\tau(i)_u)$$

Straigthforwardly, $E((p-\theta)x(i))$ does not depend on $\tau(i)_u$. Thus, some computations shows that

$$\frac{\partial E[\pi(i)]}{\partial \tau(i)_u} = -\frac{\psi}{2}\gamma(i)_1^2 \frac{\partial E\left(s(i)^2\right)}{\partial \tau(i)_u} - K'(\tau(i)_u)$$
$$= \frac{\psi}{2}\left[\frac{\gamma(i)_1}{\tau(i)_u}\right]^2 - K'(\tau(i)_u)$$

The first order condition is then:

$$\frac{\partial E[\pi(i)]}{\partial \tau(i)_u} \leq 0 \text{ and } \tau(i)_u \frac{\partial E[\pi(i)]}{\partial \tau(i)_u} = 0 \text{ (given the constraint } \tau(i)_u \geq 0)$$

$\frac{\partial E[\pi(i)]}{\partial \tau(i)_u} \leq 0$ rewrites

$$\frac{\psi}{2}\frac{1}{[\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon]^2} \leq K'(\tau(i)_u).$$

The LHS is decreansing and the RHS is increasing. This implies that $\tau(i)_u = 0$ if $\frac{\psi}{2}\frac{1}{[\tau+\alpha^2\gamma_1^2\tau_\varepsilon]^2} < K'(0)$ and $\tau(i)_u$ is the unique solution of

$$\frac{\psi}{2}\frac{1}{[\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon]^2} = K'(\tau(i)_u)$$

otherwise. These conditions define the best response mapping. □

Proof of Lemma 3. The relevant dynamical system is given by Eqs. (7) – (11). The total differentials of these equations evaluated at the REE are given by:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\gamma_0^*}{Z} \\ 0 & 1 & 0 & -\frac{1-\gamma_1^*}{Z} \\ 0 & 0 & 1 & \frac{\gamma_2^*}{Z} \\ 0 & W & 0 & -1 \end{bmatrix} \begin{pmatrix} d\gamma'(i)_0 \\ d\gamma'(i)_1 \\ d\gamma'(i)_2 \\ d\tau'(i)_u \end{pmatrix} =$$
$$\begin{bmatrix} -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} & -\frac{[\beta\tau_\varepsilon + \alpha\gamma_0^* + 2\alpha\gamma_1^*\gamma_0^*]\alpha\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & -\frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{Z} & 0 & 0 \\ 0 & -\frac{\alpha\tau_\varepsilon[(1+\alpha(1-\gamma_2^*)) - 2\alpha\gamma_1^*\gamma_2^*]}{Z} & -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} d\gamma_0 \\ d\gamma_1 \\ d\gamma_2 \\ d\tau_u \end{pmatrix}$$

where $Z = \tau + \tau_u^* + \alpha^2\gamma_1^{*2}\tau_\varepsilon$ and $W \equiv \frac{\psi\frac{\gamma_1}{\tau_u^2}}{K'' + \psi\frac{\gamma_1{}^2}{\tau_u^3}}$. We write this system as $Ax' = Bx$. It follows that the Jacobian of the best response dynamics at the REE is the matrix $P = A^{-1}B$. Since it turns out that P is a triangular matrix, its eigenvalues are equal the elements on its main diagonal. The respective eigenvalues $\lambda_1 \ldots \lambda_4$ are:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = -\frac{\alpha^2\gamma_1^{*2}\tau_\varepsilon}{\tau_u^*}, \quad \lambda_4 = -\frac{2\alpha^2\gamma_1^{*2}\tau_\varepsilon}{Z - (1-\gamma_1^*)W}$$

The condition for stability of this dynamical system and, thus, the condition for existence of a locally SREE is that all eigenvalues are less than one in absolute value.

As $H(\gamma_1^*) = \tau_u^*$, $H' > 0$, $H(0) = 0$ and $H(1) = \tau_u^*$, it follows that $1 > \gamma_1^* > 0$. Furthermore $\gamma_1^* > 0$ implies $0 \leq W \leq \frac{\tau_u^*}{\gamma_1^*}$ and $H(\gamma_1^*) = \tau_u^*$ implies $Z = \frac{\tau_u^*}{\gamma_1^*}$. Hence, one has:

$$Z - (1-\gamma_1^*)W \geq \frac{\tau_u^*}{\gamma_1^*} - (1-\gamma_1^*)\frac{\tau_u^*}{\gamma_1^*} = \tau_u^* > 0.$$

Hence, the condition for stability is $\lambda_3 > -1$ and $\lambda_4 > -1$, that is:

$$\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \min\left(\tau_u^*, \frac{Z - (1-\gamma_1^*)W}{2}\right). \tag{A.7}$$

□

Proof of Proposition 6. To prove Point (i), notice that, given that $Z - (1-\gamma_1^*)W \geq \tau_u^*$, the above condition (A.7) implies that

$$\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \frac{\tau_u^*}{2},$$

is a sufficient condition for existence of a locally SREE.

To prove Point (ii), notice that: if we now assume that marginal costs of information acquisition are constant and equal to $\bar{\kappa}$, such that $K'' = 0$, we have $W = \frac{\tau_u}{\gamma_1}$ and the condition (A.7) rewrites:

$$\alpha^2 \gamma_1^{*2} \tau_\varepsilon < \frac{\tau_u^*}{2}.$$

Point (iii) is obvious. □

Proof of Lemma 4. The map $\mathcal{T}$ is defined as (see Lemma 2):

$$\gamma(i)_0 = -\frac{\alpha\gamma_1\tau_\varepsilon(\beta + \alpha\gamma_0)}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}, \tag{A.8}$$

$$\gamma(i)_1 = \frac{\tau(i)_u}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}, \tag{A.9}$$

$$\gamma(i)_2 = \frac{\alpha\gamma_1\tau_\varepsilon(1 + \alpha(1-\gamma_2))}{\tau + \tau(i)_u + \alpha^2\gamma_1^2\tau_\varepsilon}, \tag{A.10}$$

where $\tau(i)_u = 0$ if $\tau + \alpha^2\gamma_1^2\tau_\varepsilon > Q$ and $\tau_u(i) = Q - \tau - \alpha^2\gamma_1^2\tau_\varepsilon$ otherwise. It follows that

$$\tau_u(i) = \max\{0, Q - \tau - \alpha^2\gamma_1^2\tau_\varepsilon\}. \tag{A.11}$$

From equation (A.9), we have that $\tau(i)_u = Q\gamma(i)_1$ in the case $\tau + \alpha^2\gamma_1^2\tau_\varepsilon < Q$. Given that $\gamma(i)_1 = \tau(i)_u = 0$ in the other case $\tau + \alpha^2\gamma_1^2\tau_\varepsilon > Q$, it follows that we always have $\tau(i)_u = Q\gamma(i)_1$. Hence, every element $(\gamma_0, \gamma_1, \gamma_2, \tau_u)$ in $\mathcal{T}(\mathbb{R}^3 \times \mathbb{R}_+)$ satisfies $\gamma_1 = \tau_u/Q$.

Hence, for every $(\gamma_0, \gamma_1, \gamma_2, \tau_u)$ in $\mathcal{T}(\mathbb{R}^3 \times \mathbb{R}_+)$, equation (A.11) rewrites $\tau(i)_u = T(\tau_u)$. Finally, notice that the projection on the τ_u-axis of $\mathcal{T}(\mathbb{R}^3 \times \mathbb{R}_+)$ is $[0, Q-\tau]$. It is then straigthforward that, for every τ_u in $\mathbb{R}_+$, the optimal $\tau(i)_u$ is $T(\tau_u)$.

It is clear that the dynamics of τ_u is fully characterized by equation (A.11) and is autonomous from the 3 other variables $(\gamma_0, \gamma_1, \gamma_2)$. It follows that the projection of $\mathcal{T}(\mathbb{R}^3 \times \mathbb{R}_+)$ on the τ_u-axis is exactly $T^\infty(\mathbb{R}_+)$. □

Proof of Proposition 7. The proof proceeds in two steps. The first step is to derive some properties of the mapping $T^2(\tau_u)$ in order to find conditions for the existence of a 2–cycle in the best response mapping. The second step then draws the relevant conclusions.

(1) Consider the function $f(\tau_u) = Q - \tau - \frac{\alpha^2\tau_\varepsilon}{Q^2}\tau_u{}^2$, which appears in Eq. (12) and let $f^2(\tau_u)$ denote its second iterate, i.e., $f^2(\tau_u) \equiv f(f(\tau_u))$. It is straightforward to show that (a) $f^2(\tau_u)$ is monotone and increasing and that (b) $f^2(\tau_u)$ has exactly one inflection point for $\tau_u > 0$:

a) With respect to the derivative with respect to τ_u, $f^{2'}(\tau_u)$, we get:

$$f^{2'}(\tau_u) = f'(\tau_u)\, f'(f(\tau_u)) \geq 0$$

because $f'(\tau_u) \leq 0$.

b) The second derivative with respect to τ_u, $f^{2''}(\tau_u)$, is given by:

$$f^{2''}(\tau_u) = 4\left(\frac{\alpha^2\,\tau_\varepsilon}{Q^2}\right)^2 \left[\tau_u\, f'(\tau_u) + f(\tau_u)\right]$$

From this it follows that $f^{2''}(\tau_u) = 0$, if $f(\tau_u) = \tau_u\, f'(\tau_u)$ which is equivalent to:

$$(Q-\tau) = \tau_u^2\,\frac{\alpha^2\,\tau_\varepsilon}{Q^2}$$

With $Q-\tau > 0$, this equation possesses two real roots, such that there are two points of inflection where $f^{2''}(\tau_u) = 0$ and at most one such point in $S = [0, Q-\tau]$. Since $T(\tau_u) = \max\{0, f(\tau_u)\}$, it then follows, that $T^2(\tau_u)$ is monotone increasing on S with at most one point of inflection.

(2) Consider now the case where no SREE exists. This implies $|T'(\tau_u^*)| > 1$ and therefore $T^{2'}(\tau_u^*) > 1$. From the above derived properties of $T^2(\tau_u)$ it then follows that a 2–cycle with $0 < \underline{\tau}_u < \bar{\tau}_u < Q-\tau$ and $T^2(\underline{\tau}_u) = T(\bar{\tau}_u) = \underline{\tau}_u$ exists if and only if $T^2(0) = T(Q-\tau) > 0$. Now, $T(Q-\tau)$ is given by:

$$T(Q-\tau) = \max\left\{0,\, [Q-\tau]\left(1 - \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,[Q-\tau]\right)\right\}$$

As an REE with $\tau_u^* > 0$ requires $Q-\tau > 0$, $T^2(0) > 0$ if and only if:

$$\tau_\varepsilon < \frac{\alpha^2\,[Q-\tau]}{Q^2} \tag{A.12}$$

If this condition is satisfied, a stable 2–cycle is a solution of the mapping $\tau_u' = T(\tau_u)$, and the best response dynamics converge to this 2–cycle. Thus, $S^* = [\underline{\tau}_u, \bar{\tau}_u]$ in this case. Otherwise, no such cycle exists and because τ_u^* is unstable, we have $S^* = S$.

(3) Consider finally the case where $|T'(\tau_u^*)| < 1$ such that a SREE exists. In this case we have $T^{2'}(\tau_u^*) < 1$. Moreover, from $\tau_u^* = T(\tau_u^*)$ we get that:

$$\tau_u^* = \frac{Q^2}{2\,\alpha^2\,\tau_\varepsilon}\left\{\sqrt{4\,(Q-\tau)\frac{\alpha^2\,\tau_\varepsilon}{Q^2} + 1} - 1\right\}$$

With this, our condition for existence of a SREE becomes:

$$\alpha^2\,\gamma_1^{*2}\,\tau_\varepsilon = \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^{*2} < \frac{1}{2}\,\tau_u^* \quad\Leftrightarrow\quad \frac{\alpha^2\,\tau_\varepsilon}{Q^2}\,\tau_u^* < \frac{1}{2}$$

$$\Leftrightarrow\quad (Q-\tau)\,\frac{\alpha^2\,\tau_\varepsilon}{Q^2} < \frac{3}{4}$$

As this implies $T(0) > 0$ (cf. eq. (A.12)), a 2–cycle cannot exist in this case. Hence τ_u^* is the unique stable fixed point of the mapping $\tau_u' = T(\tau_u)$ and $S^* = \tau_u^*$.

□

Proof of Proposition 8. The slopes of the best responses (7) and (9) for a given value of γ_1 are given by:

$$\frac{\partial \gamma_0'}{\partial \gamma_0} = \frac{\partial \gamma_2'}{\partial \gamma_2} = -\frac{\alpha^2 \gamma_1 \tau_\varepsilon}{\tau + \tau_u' + \alpha^2 \gamma_1^2 \tau_\varepsilon} \equiv \Gamma$$

It must be shown that this slope is smaller than one in absolute value for the maximum value, the weight γ_1 can attain, if and only if $T^2(0) > 0$.

Let $\bar{\tau}_u$ denote the precision for which $T(\bar{\tau}_u) = 0$:

$$(Q - \tau)\, \frac{Q^2}{\alpha^2 \tau_\varepsilon} = \bar{\tau}_u^2$$

This precision implies that $\gamma_1 = \bar{\tau}_u / Q$, which is the maximum value γ_1 can attain, as well as $\tau_u' = 0$. In this case, the slope is given by:

$$\begin{aligned}
\Gamma(\bar{\tau}_u) &= -Q\, \frac{\alpha^2 \tau_\varepsilon \bar{\tau}_u}{Q^2 \tau \alpha^2 \tau_\varepsilon \bar{\tau}_u^2} = -Q\, \frac{\alpha^2 \tau_\varepsilon \bar{\tau}_u}{Q^2 \tau \alpha^2 \tau_\varepsilon \left[(Q - \tau)\, \frac{Q^2}{\alpha^2 \tau_\varepsilon} \right]} \\
&= -\frac{1}{Q^2}\, \alpha^2 \bar{\tau}_u \tau_\varepsilon = -\frac{1}{Q^2 \bar{\tau}_u}\, \alpha^2 \bar{\tau}_u^2 \tau_\varepsilon = -\frac{1}{Q^2 \bar{\tau}_u}\, \alpha^2 \tau_\varepsilon (Q - \tau)\, \frac{Q^2}{\alpha^2 \tau_\varepsilon} \\
&= -\frac{Q - \tau}{\bar{\tau}_u}
\end{aligned} \tag{A.13}$$

From (A.13) it follows that $|\Gamma| < 1$ if and only if $Q - \tau < \bar{\tau}_u$, and because $T(\tau_u)$ is monotone decreasing, this requires $T^2(0) = T(T(0)) = T(Q - \tau) > 0$. □