Likelihood Ratio based Joint Test for the Exogeneity and the Relevance of Instrumental Variables

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Abstract

This paper develops a joint test for the exogeneity and the relevance of instrumental variables using an approach similar to Vuong's (1989) model selection test. The test statistic is derived from the likelihood ratio of two competing models: one with exogenous and possibly relevant instruments and the other with irrelevant and even possibly endogenous instruments. The joint test is asymptotically pivotal under the null hypothesis that the instruments are exogenous and irrelevant, and is consistent against the alternative hypothesis that the instruments are exogenous and relevant. Hence, non-rejection of the joint test should be taken as an evidence suggesting instruments of poor quality. Another salient feature of the test is that its asymptotic null distribution is the same under both the conventional and the weak instruments asymptotic frameworks, which implies it has better size control than the commonly used overidentifying restrictions tests.

Keywords and phrases: Joint test, exogeneity, relevance, instrumental variables, likelihood ratio.

JEL Codes: C12; C30

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1 Introduction

A set of instrumental variables is said to be *relevant* if they are correlated with the endogenous regressors and *exogenous* if uncorrelated with the errors. It is a common practice in empirical researches to check these two conditions since the standard inference results on the structural parameters hold only when these conditions hold. The overidentifying restrictions (OID) test (e.g., Anderson and Rubin, 1949; Sargan, 1958; Basmann, 1960) is widely used for the exogeneity condition, and the first stage F and Wald tests are mostly used for the relevance condition while Hall, Rudebusch and Wilcox (1996), and Stock and Yogo (2005) are more recently developed relevance tests. All aforementioned testing procedures are, however, designed for only one of the two conditions and there is no test considering both conditions simultaneously to the best of our knowledge. More surprisingly, it has not been discussed much in the econometrics literature how to combine and interpret these two types of tests.[1]

Most notably, the null distribution of the OID test is approximated by the Chi-square distribution under the implicit assumption that the instruments are relevant. As Staiger and Stock (1997) point out, however, the Chi-square distribution is a good approximation only if the instruments are strongly correlated with the endogenous regressors. Therefore, without the knowledge of the relevance of the instruments, we cannot be sure about the legitimacy of the Chi-square approximation. One may consider a two-stage testing procedure−testing for the relevance first and continuing to test for the exogeneity if the first stage relevance test rejects no or weak relevance. However, the distribution of the OID test conditional on the rejection of the relevance test can be quite different from the unconditional distribution and we are really not sure, even asymptotically, what the exact error probability is when we use the conventional critical values. See Section 4 for some Monte Carlo experiment results.

The testing procedure developed in this paper considers the relevance and exogeneity conditions at the same time by taking an approach similar to Vuong's (1989) model selection

[1] Recently, Moreira (2003) among others, propose an inferential method that is robust to arbitrarily weak instruments. This weak instruments robust inference, however, requires exogenous instruments and the necessity to check the exogeneity of instruments still remains (e.g., Doko and Dufour, 2008).

test. From a structural equation and its associated reduced form equation, we consider two competing models: one imposing the instruments to be exogenous and the other imposing the instruments to be completely irrelevant. Assuming normality, we show that the likelihood ratio of these two models is equivalent to the difference of the standard first stage Wald statistic and the OID test statistic. We propose a new Q_{IV} test based on this likelihood ratio, and it can be viewed as providing a formal way of interpreting the difference between the commonly used relevance test and the OID test.

More precisely, we set the null hypothesis as the intersection of the two models described above so that the instruments are exogenous and irrelevant. Then, the Q_{IV} statistic is shown to be asymptotically pivotal under the null hypothesis, whereas it diverges to the positive infinity when the instruments are relevant and exogenous. This implies that the probability of rejecting the Q_{IV} test with a large positive value approaches one as the sample size grows if the instruments are indeed exogenous and relevant. Another salient feature of our test statistic is that its asymptotic null distribution is invariant to the asymptotic framework: the limiting distribution is the same either under the conventional or under Staiger and Stock's (1997) weak instruments asymptotic framework. This is a very important property since it implies that our test has better size control than the commonly used OID test.

One caveat is that the Q_{IV} statistic could diverge to the positive infinity even when the instruments are not exogenous to the structural error. A leading case of this instance is when the instruments slightly violate the exogeneity condition while they retain strong correlation with the endogenous regressors, so that the asymptotic bias of the instrumental variables estimator is smaller than that of the ordinary least squares estimator. If a set of instruments indeed reduces the asymptotic bias relative to the OLS estimator, it should be deemed to be of good quality and the Q_{IV} test also concludes to that direction. Nevertheless, it should be noted that rejection of the Q_{IV} test with a large positive value should not be taken as a strong evidence of exactly exogenous and relevant instruments, while non-rejection of the Q_{IV} test should still be taken as an evidence that the set of instruments is of questionable quality.

This paper is organized as follows. Section 2 describes the model and the new test statistic Q_{IV}. Section 3 presents the asymptotic properties of the Q_{IV} test under the null and the alternative hypotheses. Section 4 contains Monte Carlo experiment results. Section 5 concludes with some remarks. All the technical proofs and simulation results are provided in Appendix.

2 Model and Test Statistic

We consider a structural equation and an associated reduced form equation given by

$$y = Y\beta + X\alpha + \varepsilon \tag{1}$$

$$Y = Z\Pi + X\Phi + V, \tag{2}$$

where y is a $T \times 1$ vector, Y is a $T \times n$ matrix of n endogenous variables, X is a $T \times K_1$ matrix of (included) exogenous variables, and Z is a $T \times K_2$ matrix of (excluded) exogenous variables to be used as instruments. The number of instruments, K_2, satisfies $n < K_2 < T$ and it is assumed to be fixed. ε and V are, respectively, a $T \times 1$ vector and a $T \times n$ matrix of random disturbances.

In the standard setup, the set of instrumental variables, Z, is assumed to be uncorrelated with both ε and V, so that Y is correlated with ε only through the correlation between ε and V. We, however, allow for a more general framework using the following structure, under which Z and ε could be also linearly correlated.

Assumption 1 *$\varepsilon = Z\omega + u$ in (1), where u is correlated with V.*

Using Assumption 1, we can rewrite the structural equation (1) as (e.g., Basmann, 1960)

$$y = Y\beta + X\alpha + Z\omega + u. \tag{3}$$

The set of instrumental variables Z is said to be *exogenous* if $\omega = 0$ so that it is orthogonal

to the structural error ε. Z is said to be *relevant* if $\Pi \neq 0$ or more precisely Π is of full column rank, and thus correlations between the instruments and the endogenous regressors are nonzero. Based on (3), we can rewrite the model in a system of equations given by

$$\bar{Y}\begin{pmatrix} 1 & 0 \\ -\beta & I_n \end{pmatrix} = \bar{Z}\begin{pmatrix} \omega & \Pi \\ \alpha & \Phi \end{pmatrix} + \bar{V}, \tag{4}$$

where I_n is the identity matrix with rank n, $\bar{Y} = [y, Y]$, $\bar{V} = [u, V]$ and $\bar{Z} = [Z, X]$. If we define $P_W = W(W'W)^{-1}W'$ and $M_W = I - P_W$ for any given matrix W, we can also concentrate out X from (2) and (3) to have $y^{\perp} = Y^{\perp}\beta + Z^{\perp}\omega + u^{\perp}$ and $Y^{\perp} = Z^{\perp}\Pi + V^{\perp}$, where $A^{\perp} = M_X A$ for any matrix A.

The main interest of this paper is to develop a joint test for the exogeneity and the relevance of a set of instruments by considering the following two conditions at the same time:[2]

$$\omega = 0 \text{ and } \Pi \neq 0. \tag{5}$$

Obviously, the composite hypothesis in (5) cannot be tested in the standard testing framework. We instead take an approach similar to the model selection test of Vuong (1989). To this end, we consider the following two non-nested models:

$$\text{Model } \mu_\omega \quad : \quad \bar{Y}\begin{pmatrix} 1 & 0 \\ -\beta & I_n \end{pmatrix} = \bar{Z}\begin{pmatrix} 0 & \Pi \\ \alpha & \Phi \end{pmatrix} + \bar{V};$$

$$\text{Model } \mu_\Pi \quad : \quad \bar{Y}\begin{pmatrix} 1 & 0 \\ -\beta & I_n \end{pmatrix} = \bar{Z}\begin{pmatrix} \omega & 0 \\ \alpha & \Phi \end{pmatrix} + \bar{V}.$$

The first model μ_ω is (4) with a restriction $\omega = 0$, whereas the second model μ_Π is (4) with a restriction $\Pi = 0$. Note that under the first model μ_ω, the instruments are exogenous though

[2] It may be more useful to test for the exogeneity (i.e., $\omega = 0$) and strength of instruments (i.e., each element of Π is far enough from zero.) However, the weak instruments are formulated under the local-to–zero asymptotics and thus it is hard to test the strength of the instruments using the standard testing setup. See Stock and Yogo (2005) for a possible approach of testing for weak instruments under the exogeneity assumption.

its relevance is not verified. Under the second model μ_Π, the instruments are irrelevant and could be even endogenous depending on the value of ω. The key idea is that the likelihood ratio between the two models, μ_ω and μ_Π, can be used as a model selection test. More formally, we set the null hypothesis as

$$H_0 : \omega = 0 \text{ and } \Pi = 0, \tag{6}$$

which implies that two specifications are equally close to the true data generating model. When the null hypothesis is rejected in favor of the alternative hypothesis

$$H_1 : \omega = 0 \text{ and } \Pi \neq 0 \tag{7}$$

(i.e., the first specification μ_ω is closer to the true model), we may expect the set of instruments to be exogenous and relevant or likely to be so. It should be noted that, though this testing idea looks very similar to Vuong (1989), there is a fundamental difference between these two approaches. For Vuong's (1989) model selection test, each competing model is required to have a unique value of the model parameter vector that minimizes the Kullback-Leibler distance between the given model and the true distribution. This condition is necessary for selecting the model closer to the true distribution with probability approaching one as the sample size grows. In our setup, this condition is not satisfied because some parameters in the model μ_Π are not properly identified. However, this lack of identification in the model μ_Π is not critical because the main purpose of the test in this paper is not selecting a model between μ_ω and μ_Π but to reject the null hypothesis H_0 in favor of the model μ_ω. In other words, it is of little interest to tell the model μ_Π from the null hypothesis H_0.

To derive a test statistic, we assume that $(u_t, V_t')'|Z_t, X_t \sim i.i.d.\mathcal{N}(0, \Sigma)$,[3] where

$$\Sigma = \begin{pmatrix} \sigma_{uu} & \Sigma_{uV} \\ \Sigma_{Vu} & \Sigma_{VV} \end{pmatrix}$$

with $\Sigma_{uV} \neq 0$ and the partition is conformable with $(u_t, V_t')'$. The likelihood function is denoted as $L(\theta)$ with $\theta = (\beta', \omega', vec(\Pi)', vec(\Phi)', vec(\Sigma)')'$. Then, the likelihood ratio between the non-nested models μ_ω and μ_Π can be derived as

$$\begin{aligned} 2LR &= 2 \max_{\theta:\omega=0} \log L(\theta) - 2 \max_{\theta:\Pi=0} \log L(\theta) \\ &= T \left(\log \left| I_n + \frac{1}{T} G_T \right| - \log(1 + \frac{1}{T} \phi(\hat{\beta}_{LIML})) \right) \\ &\simeq tr\,(G_T) - \phi(\hat{\beta}_{LIML}), \end{aligned} \tag{8}$$

where $tr\,(\cdot)$ is the trace operator,

$$G_T = \hat{\Sigma}_{VV}^{-1/2} \left(Y^{\perp\prime} P_{Z^\perp} Y^\perp \right) \hat{\Sigma}_{VV}^{-1/2} \quad \text{and} \quad \phi(\hat{\beta}_{LIML}) = \frac{\hat{\varepsilon}^{\perp\prime} P_{Z^\perp} \hat{\varepsilon}^\perp}{\hat{\varepsilon}^{\perp\prime} M_{Z^\perp} \hat{\varepsilon}^\perp / T}$$

with $\hat{\Sigma}_{VV} = Y^{\perp\prime} M_{Z^\perp} Y^\perp / T$, $\hat{\varepsilon}^\perp = y^\perp - Y^\perp \hat{\beta}_{LIML}$ and $\hat{\beta}_{LIML}$ being the standard LIML estimator. The detailed derivation of (8) is given in Appendix. Notice that the first component of (8), $tr\,(G_T)$, is nothing but the Wald statistic testing for $\Pi = 0$. The commonly used first stage F statistic is equivalent to this statistic when there is only one endogenous regressor; Hall, Rudebusch and Wilcox's (1996), and Stock and Yogo's (2005) statistics are its variants. On the other hand, the second component of (8), $\phi(\hat{\beta}_{LIML})$, is the standard overidentifying restrictions (OID) test statistic. For example, it is Anderson-Rubin (1949) statistic when the true value of β is used instead of $\hat{\beta}_{LIML}$ and is the Basmann's (1960) OID test when the two stage least squares (TSLS) estimator $\hat{\beta}_{TSLS}$ is used.

From the model selection point of view, a large positive value of the LR statistic in (8) indicates that the model μ_ω has a Kullback-Leibler distance to the true model smaller than

[3] The normality assumption is not needed for our main asymptotic results presented in the next section.

that of the model μ_Π. In addition, as the sample size grows, we can show the LR statistic tends to the positive infinity when the model μ_ω is true with $\Pi \neq 0$, whereas it shifts to the opposite direction if the model μ_Π is true with $\omega \neq 0$. Under the intersection of the two models (i.e., both $\omega = 0$ and $\Pi = 0$), the LR statistic is asymptotically pivotal. (See the remark following Theorem 1.) It is thus natural to consider a testing procedure which concludes that a given set of instruments is closer to being exogenous and relevant (i.e., of good quality) when the LR in (8) takes a large positive value. More formal discussions can be found in the following section.

The new joint test statistic developed in this paper has basically the same structure as the LR statistic in (8). Specifically, the test statistic (on the quality of instrumental variables: Q_{IV}) that we consider is defined as[4]

$$Q_{IV} = \lambda_{\min}\left(G_T^0\right) - \phi(\hat{\beta}(k_T)), \tag{9}$$

where $\lambda_{\min}(\cdot)$ is the minimum eigenvalue of a given matrix and

$$\begin{aligned} G_T^0 &= \tilde{\Sigma}_{VV}^{-1/2}\left(Y^{\perp\prime}P_{Z^\perp}Y^\perp\right)\tilde{\Sigma}_{VV}^{-1/2} \quad \text{with } \tilde{\Sigma}_{VV} = \frac{1}{T}Y^{\perp\prime}Y^\perp, \\ \phi(\hat{\beta}(k_T)) &= \frac{\tilde{\varepsilon}^{\perp\prime}P_{Z^\perp}^\perp\tilde{\varepsilon}^\perp}{\tilde{\varepsilon}^{\perp\prime}M_{Z^\perp}\tilde{\varepsilon}^\perp/T} \quad \text{with } \tilde{\varepsilon}^\perp = y^\perp - Y^\perp\hat{\beta}(k_T). \end{aligned}$$

Here we compute the covariance matrix of the reduced form error V with assuming $\Pi = 0$. G_T^0 and G_T are asymptotically equivalent if $\Pi = 0$, and $\lambda_{\min}(G_T)$ is the test statistic for instrument weakness suggested by Stock and Yogo (2005), which is based on Cragg and Donald's (1993) statistic. $\hat{\beta}(k_T)$ is the standard k-class estimator defined as

$$\hat{\beta}(k_T) = (Y^{\perp\prime}(I_T - k_T M_{Z^\perp})Y^\perp)^{-1}Y^{\perp\prime}(I_T - k_T M_{Z^\perp})y^\perp, \tag{10}$$

in which $k_T = 1$ for the TSLS estimator; $k_T = \hat{k}_T$ for the LIML estimator; and $k_T =$

[4] Apparently, Q_{IV} is a modified version of LR. The justification for such modification is given after Theorem 2 in the next section.

$\hat{k}_T - 1/(T - K_1 - K_2)$ for the Fuller-k estimator with $\hat{k}_T$ being the smallest root satisfying $|\bar{Y}'M_X\bar{Y} - \hat{k}_T\bar{Y}'M_{\bar{Z}}\bar{Y}| = 0$. An interesting point is that the new test statistic Q_{IV} is the difference between a weak instrument test statistic (e.g., Stock and Yogo, 2005; Cragg and Donald, 1993) and the standard OID test statistic. Therefore, our test procedure can also be viewed as providing a formal way of interpreting the difference between the commonly used relevance test and the OID test.

3 Asymptotic Results

We first derive the asymptotic distribution of the Q_{IV} statistic (9) under the null hypothesis (6). We let $\rho = \Sigma_{VV}^{-1/2}\Sigma_{Vu}\sigma_{uu}^{-1/2}$ and $\Omega = S_{ZZ} - S_{ZX}S_{XX}^{-1}S_{XZ}$, where

$$S = E\left(\bar{Z}_t\bar{Z}_t'\right) = \begin{pmatrix} S_{ZZ} & S_{XZ} \\ S_{ZX} & S_{XX} \end{pmatrix}$$

with $\bar{Z}_t'$ being the t-th row of $\bar{Z}$. We make the high level assumptions following Staiger and Stock (1997).

Assumption 2

(a) $T^{-1}\bar{V}'\bar{V} \xrightarrow{p} \Sigma$ *and* $T^{-1}\bar{Z}'\bar{Z} \xrightarrow{p} S$ *as* $T \to \infty$*, where both* Σ *and* S *are positive definite and finite.*

(b) $(X'u, Z'u, X'V, Z'V)/\sqrt{T} \xrightarrow{d} (\Psi_{Xu}, \Psi_{Zu}, \Psi_{XV}, \Psi_{ZV})$ *as* $T \to \infty$*,*

where $(\Psi_{Xu}', \Psi_{Zu}', vec(\Psi_{XV})', vec(\Psi_{ZV})')' \sim \mathcal{N}(0, \Sigma \otimes S)$.

Based on Assumption 2, we also define Gaussian random matrices $z_u = \Omega^{-1/2}(\Psi_{Zu} - S_{ZX}S_{XX}^{-1}\Psi_{Xu})\sigma_{uu}^{-1/2}$ and $z_V = \Omega^{-1/2}(\Psi_{ZV} - S_{ZX}S_{XX}^{-1}\Psi_{XV})\Sigma_{VV}^{-1/2}$ so that $(z_u', vec(z_V)')' \sim \mathcal{N}\left(0, \overline{\Sigma} \otimes I_{K_2}\right)$ with $\overline{\Sigma} = \left(\begin{smallmatrix} 1 & \rho' \\ \rho & I_n \end{smallmatrix}\right)$. The first theorem derives the asymptotic distribution of the Q_{IV} statistic under the null hypothesis (6).

Theorem 1 *We suppose* $\omega = 0$ *and* $\Pi = 0$. *Under Assumptions 1 and 2,* $\kappa_T = T(\hat{k}_T - 1) \xrightarrow{d} \kappa^*$ *as* $T \to \infty$, *where* κ^* *is the smallest root satisfying* $|(\eta, z_V)'(\eta, z_V) - \kappa^* I_{n+1}| = 0$ *and* $\eta = (z_u - z_V\rho)/\sqrt{1-\rho'\rho}$ *so that* $(\eta', vec(z_V)')' \sim \mathcal{N}\left(0, I_{(n+1)K_2}\right)$. *Furthermore,*

$$Q_{IV} \xrightarrow{d} Q_{IV}^0 = \lambda_{\min}(z_V' z_V) - \frac{\eta'(I_{K_2} - z_V(z_V' z_V - \kappa I_n)^{-1} z_V')^2 \eta}{1 + \eta' z_V (z_V' z_V - \kappa I_n)^{-2} z_V' \eta},$$

where $\kappa = 0$ *for the TSLS estimator;* $\kappa = \kappa^*$ *for the LIML estimator; and* $\kappa = \kappa^* - 1$ *for the Fuller-k estimator.*

The limiting null distribution of the Q_{IV} statistic is nuisance parameter free; it depends only on the number of instrumental variables (K_2) and the number of endogenous regressors (n). Tables 1.A to 1.C in Appendix report the relevant quantiles of Q_{IV}^0. It is also evident from Theorem 1 that the LR statistic in (8) is asymptotically pivotal. The following theorem derives the asymptotic behavior of the Q_{IV} statistic under various hypotheses including the alternative hypothesis (7).

Theorem 2 *Under Assumptions 1 and 2, as* $T \to \infty$ *we have the following asymptotic results:*

(i) If $\omega = 0$ *and* $\Pi \neq 0$, *then* $Q_{IV} \xrightarrow{p} \infty$.

(ii) If $\omega \neq 0$ *and* $\Pi = 0$, *then* $\kappa_T = T(\hat{k}_T - 1) = O_p(1)$. *Moreover, let* $\kappa_T \xrightarrow{d} \kappa^*$, *then* $Q_{IV} \xrightarrow{d} Q_{IV}^{\omega}$, *where*

$$Q_{IV}^{\omega} = \lambda_{\min}(z_V' z_V) - \frac{\omega' \Omega^{1/2\prime} \left(I_{K_2} - z_V(z_V' z_V - \kappa I_n)^{-1} z_V'\right)^2 \Omega^{1/2} \omega}{\omega' \Omega^{1/2\prime} z_V (z_V' z_V - \kappa I_n)^{-2} z_V' \Omega^{1/2} \omega}$$

with $\kappa = 0$ *for the TSLS estimator;* $\kappa = \kappa^*$ *for the LIML estimator; and* $\kappa = \kappa^* - 1$ *for the Fuller-k estimator.*

(iii) If $\omega \neq 0$ *and* $\Pi \neq 0$, *then* $\hat{k}_T \xrightarrow{p} k^*$ *with* k^* *being the smallest root satisfying*

$|\Theta - (k^* - 1)\Sigma| = 0$ *with* $\Theta = \left(\begin{smallmatrix} \omega'\Omega\omega & \omega'\Omega\Pi \\ \Pi'\Omega\omega & \Pi'\Omega\Pi \end{smallmatrix}\right)$, *and*

$$\text{plim}_{T\to\infty}\frac{Q_{IV}}{T} = \lambda_{\min}((\Pi'\Omega\Pi + \Sigma_{VV})^{-1}\Pi'\Omega\Pi) - \frac{\omega'\Omega\omega + b\,(k)'\,\Pi'\Omega\Pi b\,(k) - 2\omega'\Omega\Pi b\,(k)}{\sigma_{uu} + b\,(k)'\,\Sigma_{VV} b\,(k) - 2\Sigma_{uV} b\,(k)}, \tag{11}$$

where $b\,(k) = \text{plim}_{T\to\infty}(\hat{\beta}(k_T) - \beta) = (\Pi'\Omega\Pi - k\Sigma_{VV})^{-1}(\Pi'\Omega\omega - k\Sigma_{Vu})$ *with* $k = 0$ *for the TSLS estimator; and* $k = k^* - 1$ *for the LIML and the Fuller-k estimators.*

Theorem 2-(i) indicates that the Q_{IV} statistic diverges to the positive infinity with exogenous and relevant instruments (i.e., under H_1), and thus the probability to reject the null hypothesis approaches one as the sample size grows. This result is the basic building block of our new test Q_{IV}: we reject H_0 ($\omega = 0$ and $\Pi = 0$) in (6) in favor of H_1 ($\omega = 0$ and $\Pi \neq 0$) in (7) if Q_{IV} is large enough. Furthermore, Monte Carlo experiments indicate that the distribution of Q_{IV} shifts to the left when $\omega \neq 0$ and $\Pi = 0$ (i.e., Q^{ω}_{IV} in Theorem 2-(ii)) and the probability to erroneously reject H_0 in favor of H_1 remains under the controlled level, though it does not diverge to the negative infinity as the sample size grows. See Section 4 for the relevant simulation results.

One caveat is that the result in Theorem 2-(iii) implies that the Q_{IV} statistic could diverge to the positive infinity provided $\text{plim}_{T\to\infty}Q_{IV}/T > 0$, even when the instruments are correlated with the structural error ($\omega \neq 0$) as long as they are correlated strongly enough with the endogenous regressors ($\Pi \neq 0$). Therefore, it is important to emphasize that rejection of the Q_{IV} test with a large positive value should not be taken as a strong evidence of good instruments. Rather, non-rejection of the Q_{IV} test should be taken as an evidence that the set of instruments is of questionable quality.

However, the sign of $\text{plim}_{T\to\infty}Q_{IV}/T$ is not completely arbitrary. It is roughly linked to the relative magnitude of the biases between $\hat{\beta}(k_T)$ and the OLS estimator $\hat{\beta}_{OLS}$: $\text{plim}_{T\to\infty}Q_{IV}/T$ is more likely positive when the asymptotic bias of $\hat{\beta}(k_T)$ is smaller than that of $\hat{\beta}_{OLS}$ (i.e., the instruments works properly to reduce the endogeneity problem). More precisely, with $k = 0$ (i.e., the TSLS estimator $\hat{\beta}_{TSLS}$ is used for $\hat{\beta}(k_T)$), note that

$\text{plim}_{T\to\infty} Q_{IV}/T > 0$ implies

$$\begin{aligned}
\frac{\lambda_{\min}(\Lambda'\Lambda)}{1+\lambda_{\min}(\Lambda'\Lambda)} &\geq \frac{\omega'\Omega\omega + b(0)'\Pi'\Omega\Pi b(0) - 2\omega'\Omega\Pi b(0)}{\sigma_{uu} + b(0)'\Sigma_{VV} b(0) - 2\Sigma_{uV} b(0)} \\
&= \frac{\omega'\Omega\omega - \omega'\Omega\Pi(\Pi'\Omega\Pi)^{-1}\Pi'\Omega\omega}{\sigma_{uu} - 2\Sigma_{uV}(\Pi'\Omega\Pi)^{-1}\Pi'\Omega\omega + 2\omega\Omega\Pi(\Pi'\Omega\Pi)^{-1}\Sigma_{VV}(\Pi'\Omega\Pi)^{-1}\Pi'\Omega\omega} \\
&= \frac{\xi' M_\Lambda \xi}{1 - 2\rho'(\Lambda'\Lambda)^{-1}\Lambda'\xi + \xi'\Lambda(\Lambda'\Lambda)^{-2}\Lambda'\xi} \qquad (12)
\end{aligned}$$

from (11), where $\Lambda = \Omega^{1/2}\Pi\Sigma_{VV}^{-1/2}$, $\xi = \Omega^{1/2}\omega\sigma_{uu}^{-1/2}$ and $M_\Lambda = I - \Lambda(\Lambda'\Lambda)^{-1}\Lambda'$. In comparison, for $\hat{\beta}_{OLS} - \beta \xrightarrow{p} (\Pi'\Omega\Pi + \Sigma_{VV})^{-1}(\Pi'\Omega\omega + \Sigma_{Vu})$, the relative magnitude of the biases between $\hat{\beta}_{TSLS}$ and $\hat{\beta}_{OLS}$ with respect to $\Pi'\Omega\Pi$ is given by[5]

$$\begin{aligned}
RB &= \text{plim}_{T\to\infty} \frac{(\hat{\beta}_{TSLS} - \beta)'(\Pi'\Omega\Pi)(\hat{\beta}_{TSLS} - \beta)}{(\hat{\beta}_{OLS} - \beta)'(\Pi'\Omega\Pi)(\hat{\beta}_{OLS} - \beta)} \\
&= \frac{\omega'\Omega\Pi(\Pi'\Omega\Pi)^{-1}\Pi'\Omega\omega}{(\omega'\Omega\Pi + \Sigma_{Vu}')(\Pi'\Omega\Pi + \Sigma_{VV})^{-1}(\Pi'\Omega\Pi)(\Pi'\Omega\Pi + \Sigma_{VV})^{-1}(\Pi'\Omega\omega + \Sigma_{Vu})} \\
&= \frac{\xi' P_\Lambda \xi}{\rho' B_\Lambda \rho + 2\rho' B_\Lambda \Lambda'\xi + \xi'\Lambda B_\Lambda \Lambda'\xi} \qquad (13)
\end{aligned}$$

where $P_\Lambda = \Lambda(\Lambda'\Lambda)^{-1}\Lambda'$ and $B_\Lambda = (\Lambda'\Lambda + I)^{-1}(\Lambda'\Lambda)(\Lambda'\Lambda + I)^{-1}$. Now, suppose none of $\xi' P_\Lambda \xi$ and $\xi' M_\Lambda \xi$ is zero but the cross products $\rho' B_\Lambda \Lambda'\xi$ and $\rho'(\Lambda'\Lambda)^{-1}\Lambda'\xi$ are close to zero. Then, both expressions in (12) and (13) have the same structure: they are non-negative and increasing in $\|\xi\|$. It thus suggests that for a given pair of (ρ, Λ), small-enough ξ will satisfy the inequality in (12), which is also likely to satisfy $RB < 1$ as well (i.e., TSLS has a smaller bias than OLS). In this sense, roughly speaking, the sign of $\text{plim}_{T\to\infty} Q_{IV}/T$ is related to the relative bias RB. Therefore, when a set of instruments slightly violates the exogeneity condition but its correlation with the endogenous regressors remains strong enough, so that the asymptotic bias of the instrumental variables estimator is smaller than that of the ordinary least squares estimator, the quality of the instruments should be deemed

[5] It is similar to the relative magnitude of the biases with respect to $Y^{\perp\prime}Y^{\perp}$ (e.g., Stock and Yogo, 2005). Note that $Y^{\perp\prime}Y^{\perp}/T \xrightarrow{p} \Pi'\Omega\Pi + \Sigma_{VV}$ and $\Pi'\Omega\Pi$ only considers the pure signal from the IV.

to be good enough and the Q_{IV} test concludes to that direction.[6]

In addition, as extreme cases, if ξ is in the null space of Λ, $\hat{\beta}_{TSLS}$ has no asymptotic bias but we do not necessarily conclude that the instruments are valid. This implies that the Q_{IV} test is unnecessarily tough. If ξ is in the range space of Λ (or equivalently $\omega = \Pi c$ for some vector c), we always conclude that the instruments are valid even when $\omega \neq 0$ regardless of the relative bias. Recall that the standard OID tests share the same feature so that it has no power against such a violation of the exogeneity condition (e.g., Newey, 1985).

Finally, in order to investigate the local power property of the new test statistic Q_{IV}, we assume the following local-to-zero assumptions similarly as Staiger and Stock (1997).

Assumption 3 *$\omega = d/\sqrt{T}$ and $\Pi = C/\sqrt{T}$ for some $0 < d, C < \infty$.*

One of the novelties of the Q_{IV} statistic is that its limiting distribution under $\omega = 0$ and $\Pi = 0$ is invariant to the asymptotic framework. That is the limiting distribution is the same either under the conventional or under Staiger and Stock's weak instruments framework. This is a very important feature because the error probabilities of our test is controlled much better than the standard OID tests.

Theorem 3 *Under Assumptions 1, 2 and 3, we have $Q_{IV} \xrightarrow{d} Q_{IV}^{loc}$ as $T \to \infty$, where*

$$Q_{IV}^{loc} \equiv \lambda_{\min}((z_V + \Lambda_C)'(z_V + \Lambda_C)) - \frac{(z_u - (z_V + \Lambda_C)\Delta_\xi(\kappa))'(z_u - (z_V + \Lambda_C)\Delta_\xi(\kappa))}{1 - 2\rho'\Delta_\xi(\kappa) + \Delta_\xi(\kappa)'\Delta_\xi(\kappa)},$$

$\Delta_\xi(\kappa) = ((z_V + \Lambda_C)'(z_V + \Lambda_C) - \kappa I_n)^{-1}\left[(z_V + \Lambda_C)'(z_u + \xi_d) - \kappa\rho\right]$, $\Lambda_C = \Omega^{1/2} C \Sigma_{VV}^{-1/2}$ and $\xi_d = \Omega^{1/2} d \sigma_{uu}^{-1/2}$. $\kappa = 0$ for the TSLS estimator; $\kappa = \kappa^*$ for the LIML estima-

[6] The LR statistic in (8) also satisfies the same asymptotic behavior as the Q_{IV} statistic. Particularly when $\omega \neq 0$ and $\Pi \neq 0$, we can show that

$$\text{plim}_{T\to\infty} \frac{LR}{T} = tr(\Sigma_{VV}^{-1}\Pi'\Omega\Pi) - \frac{\omega'\Omega\omega + b_{LIML}'\Pi'\Omega\Pi b_{LIML} - 2\omega\Omega\Pi b_{LIML}}{\sigma_{uu} + b_{LIML}'\Sigma_{VV} b_{LIML} - 2\Sigma_{uV} b_{LIML}},$$

where $b_{LIML} = \text{plim}_{T\to\infty}(\hat{\beta}_{LIML} - \beta)$. Note that $tr(\Sigma_{VV}^{-1}\Pi'\Omega\Pi)$ is the sum of all the eigenvalues of $\Lambda'\Lambda$ and therefore, it is not only larger than $\lambda_{\min}(\Lambda'\Lambda)/(1 + \lambda_{\min}(\Lambda'\Lambda))$ but also could be unbounded above. Consequently, it is more difficult to find any relationship between the sign of $\text{plim}_{T\to\infty} LR/T$ and the relative bias RB. In this point of view, the Q_{IV} statistic is more preferable but it is also true that the non-rejection of the LR test is a stronger indication that the set of instruments is of questionable quality.

tor; and $\kappa = \kappa^* - 1$ *for the Fuller-k estimator with* κ^* *being the smallest root satisfying* $\left|(z_u + \xi_d, z_V + \Lambda_C)'(z_u + \xi_d, z_V + \Lambda_C) - \kappa^* \overline{\Sigma}\right| = 0$.

Obviously, Q_{IV}^{loc} depends on a nuisance parameter ρ unless $\omega = 0$ and $\Pi = 0$. Furthermore, if there are multiple endogenous variables ($n \geq 2$), Q_{IV}^{loc} depend on all the eigenvalues of $\Lambda_C' \Lambda_C$, as Stock and Yogo (2005) point out. Therefore, this asymptotic distribution cannot be directly used for inferences. See Section 4 where we report the local power of the Q_{IV} test obtained from simulating Q_{IV}^{loc}.

4 Monte Carlo Simulation

4.1 Null rejection probability of the standard OID test

In this subsection, we demonstrate via Monte Carlo experiments the difficulties arising when the relevance test and exogeneity test are used in the conventional manner. First, we show the dependence of the limiting null distribution of the standard OID test on the correlation between the instruments and endogenous variables. In particular, the size of the OID test can depart from the nominal level by a large margin when the correlation between the instruments and endogenous variables is weak. One may consider a two-stage testing procedure−testing for the relevance first and continuing to test for the exogeneity if the first stage relevance test rejects no or weak relevance. In this case, we show that the dependence on the instrumental strength gets intensified causing even larger size distortion of the OID test.

Let $\phi(\hat{\beta}(k_T))$ and $\lambda_{\min}(G_T)$ be the standard OID test and Stock and Yogo's (2005) weak instruments test, respectively, as defined in (8), where $\hat{\beta}(k_T)$ is the k-class estimator as in (10). From Theorem 3, the limit expressions for these statistics are

$$\phi(\hat{\beta}(k_T)) \xrightarrow{d} \phi_\infty \equiv \frac{(z_u - (z_V + \Lambda_C)\Delta_\xi(\kappa))'(z_u - (z_V + \Lambda_C)\Delta_\xi(\kappa))}{1 - 2\rho'\Delta_\xi(\kappa) + \Delta_\xi(\kappa)'\Delta_\xi(\kappa)} \tag{14}$$

$$\lambda_{\min}(G_T) \xrightarrow{d} g_\infty \equiv \lambda_{\min}((z_V + \Lambda_C)'(z_V + \Lambda_C)) \tag{15}$$

with $\xi_d = 0$. We simulate the limiting quantities ϕ_∞ and g_∞ in order to avoid any other finite sample complications.

Tables 2.A and 2.B. in Appendix report the rejection probabilities of the OID test based on several different testing procedures, where Tables 2.A is based on the TSLS estimator and Tables 2.B is based on the Fuller-k estimator. The case of one endogenous regressor ($n = 1$) and 3, 9 instrumental variables ($K_2 = 3, 9$) are presented but the results remain qualitatively unchanged for other values of n and K_2. For each value of K_2, there are three columns: "rej," "n-rej," and "uncond." Each of these three columns corresponds to the rejection probabilities of ϕ_∞ conditional on the rejection of g_∞ (i.e., $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ rejects)), conditional on the non-rejection of g_∞ (i.e., $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ not reject)) and unconditionally (i.e., $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05})$), respectively. The numbers in the parenthesis next to "rej" are $\mathbb{P}(g_\infty$ rejects) and those next to "n-rej" are $\mathbb{P}(g_\infty$ not rejects). Note that the first-stage weak IV tests using (15) are based on the 10% TSLS/Fuller-k bias (see Stock and Yogo, 2005, for the precise definition) at the 5% significance level. The second-stage OID tests using (14) are based on the Chi-square distribution. Three values of $\Lambda_C'\Lambda_C$ are simulated:[7] 0.5, 0.8, and 1.2 times of $\lambda^*_{\min}$, where $\lambda^*_{\min}$ is the boundary value for the weak instruments set based on the 10% TSLS/Fuller-k bias. Hence, the cases of 0.5 and 0.8 correspond to the weak instruments, while 1.2 to the strong instruments. For each value of $\Lambda_C'\Lambda_C$, $\rho = 0.1$, 0.3, 0.5, 0.8 and 1.0 are considered.

The first observation in Table 2.A is that the size distortion of ϕ_∞ increases (e.g., see the "uncond." columns) as the instruments get weaker, which shows the danger of applying the standard OID test without knowing the strength of the instruments. Note that the actual sizes vary from less than 4% to more than 20% when the instruments are weakly correlated with the endogenous variables. The second observation is that the sequential procedure creates more size distortion in that the rejection probabilities of ϕ_∞ conditional on the rejection of g_∞ is always greater than their unconditional counterparts. Also, the size of ϕ_∞ is very liberal near $\rho = 1$, while it is mildly conservative near $\rho = 0$. Table 2.B reports

[7] $\Lambda_C'\Lambda_C$ is the weak instrument limit of the concentration matrix: $\Sigma^{1/2\prime}\Pi Z^{\perp\prime}Z^{\perp}\Pi\Sigma^{1/2} \xrightarrow{p} \Lambda_C'\Lambda_C$.

the results obtained using the Fuller-k estimator instead of the TSLS estimator. Overall, Table 2.B exhibits a great deal of similarity to Table 2.A and the general conclusions from Table 2.A remain valid. One notable difference is that the largest size distortion is not associated with $\rho = 1$ but with $\rho = 0.1$.

4.2 Size and power properties of the Q_{IV} test

We also conduct Monte Carlo experiments for the finite sample size and power of the Q_{IV} test. The model we simulate is based on (2) and (3) without X:

$$y = Y\beta + Z\omega + u \quad \text{and} \quad Y = Z\Pi + V.$$

We consider the number of endogenous variables $n = 1$, 2, 3 and the number of instruments $K_2 = n+1$, $n+3$, $n+5$. The errors $(u_t, V_t')'$ are specified as

$$u_t = e_t + \Sigma_{Vu}' E_t \quad \text{and} \quad V_t = E_t,$$

where $(e_t, E_t')' \sim i.i.d.\mathcal{N}(0, I_{n+1})$ and Σ_{Vu} is a vector of ones multiplied by $0.5/\sqrt{n}$. Z_t is from a multivariate normal with unit mean and identity variance covariance matrix. Also, we let $\beta = 0$ since the Q_{IV} test is exactly invariant to the value of β. The number of replications is 5,000. The sample size is $T = 50$, 100, 200 and 300.

For the finite sample size simulation, we assume $\omega = 0$ and $\Pi = 0$. The results are reported in Tables 3.A and 3.B. For any sample size T, the Q_{IV} test shows that actual sizes are very close to the nominal 5% whether it is based on the TSLS or Fuller-k estimator.

For the finite sample power simulation, we set $\omega = 0$ and $\Pi = 0.25R_{n,K_2}$, where the columns of R_{n,K_2} are a set of orthonormalized vectors which are randomly selected from a uniform distribution for each value of n and K_2. Tables 4.A and 4.B report the results. They show that the probability rejecting the null hypothesis (i.e., $H_0 : \omega = 0$ and $\Pi = 0$) with a large positive value of Q_{IV} quickly approaches one as the sample size grows. The last power experiments assume $\Pi = 0$ while ω is a vector of zeros except for the last element,

which is equal to 0.5. This particular shape of ω reflects that only one or two instruments violate the exogeneity condition, which is very likely if the researcher is careful enough. Tables 5.A and 5.B show that the rejection probabilities are much smaller than the nominal 5% for all n and k_2. The last row in Table 5.A corresponds to the limiting case which are obtained from simulating Q_{IV}^{ω} given in Theorem 2. Note that the power does not collapse to zero because Q_{IV} does not diverge to the negative infinity.

Finally, we simulate Q_{IV}^{loc} to see the local power of the Q_{IV} test. Q_{IV}^{loc} depends on ρ and we consider three cases: ρ is proportional to a vector of ones with $||\rho|| = 0.2$, 0.5 and 0.8. We present only the results of the Fuller-k estimator since the TSLS and LIML estimators give very similar results. Also, the results are quite stable across different pairs of (n, K_2) and we report three cases $(n, K_2) = (1, 3)$, $(2, 5)$, and $(3, 7)$. The number of replications is 20,000.

Figure 1.A shows the results when $\omega = 0$ but $\Pi = C/\sqrt{T}$ as C gets away from zero. More precisely, we let $\xi_d = 0$ and $\Lambda_C = cR_{n,K_2}$, where c varies from 0 to 5. In all cases, the power increases toward one as $|c|$ increases. Figure 1.B, on the other hand, shows the results when $\Pi = 0$ but $\omega = d/\sqrt{T}$ as d gets away from zero. We let $\Lambda_C = 0$ and $\xi_d = (0, \ldots, 0, \xi_{K_2})'$, where ξ_{K_2} varies from 0 to 5. In all cases, the power decreases toward zero as $|\xi_{K_2}|$ increases.

5 Conclusion

A joint test for the relevance and the exogeneity conditions is proposed using an approach similar to Vuong's (1989) model selection test. In particular, the test statistic is derived from two competing models: one imposing the instruments to be exogenous and the other imposing the instruments to be completely irrelevant. The likelihood ratio of these two models is shown to be equivalent to the difference of the standard first stage Wald statistic and the OID test statistic.

The proposed Q_{IV} test is a slight modification of the likelihood ratio. The null hypoth-

esis is set to be the intersection of the two models described above so that the instruments are exogenous and irrelevant. Then, the Q_{IV} statistic is shown to be asymptotically pivotal under the null hypothesis, whereas it is consistent against the alternative hypothesis that the instruments are relevant and exogenous. This result implies that non-rejection of the Q_{IV} test should be taken as an evidence that the set of instruments is of questionable quality. Furthermore, the asymptotic null distribution of the Q_{IV} test is invariant either under the conventional or under Staiger and Stock's (1997) weak instruments asymptotic framework. This is an important property since it implies that our test has better size control than the commonly used overidentifying restrictions test. Lastly, it should be noted that the Q_{IV} statistic could diverge to the positive infinity even if the instruments are not strictly exogenous to the structural error. Therefore, rejection of the Q_{IV} test should not be taken as a strong evidence of exact exogeneity, while the rejection still tells the instruments to be of good quality in the sense that the 2SLS estimator based on these instruments is likely to have smaller asymptotic bias relative to the OLS estimator.

Appendix

A.1 Mathematical Proofs

Likelihood Ratio (8) Derivation First, we impose $\omega = 0$ and write the log-likelihood as

$$\begin{aligned}\log L(\theta) &= -\frac{T}{2}\log|\sigma_{uu}| - \frac{1}{2\sigma_{uu}}\varepsilon^{\perp\prime}\varepsilon^{\perp} \\ &\quad -\frac{T}{2}\log\left|\Sigma_{V|u}\right| - \frac{1}{2}tr\left[\left(Y^{\perp} - Z^{\perp}\Pi - \varepsilon^{\perp}\delta'\right)'\Sigma_{V|u}^{-1}\left(Y^{\perp} - Z^{\perp}\Pi - \varepsilon^{\perp}\delta'\right)\right],\end{aligned}$$

where $\varepsilon^{\perp} = y^{\perp} - Y^{\perp}\beta$, $\delta = \Sigma_{Vu}/\sigma_{uu}$ and $\Sigma_{V|u} = \Sigma_{VV} - \Sigma_{Vu}\Sigma_{uV}/\sigma_{uu} = \Sigma_{VV}^{1/2}\left(I_n - \rho\rho'\right)\Sigma_{VV}^{1/2}$. We denote the estimates obtained imposing $\omega = 0$ with a subscript 0. From the first order conditions, we obtain $\hat{\sigma}_{uu,0} = \hat{\varepsilon}^{\perp\prime}\hat{\varepsilon}^{\perp}/T$, where $\hat{\varepsilon}^{\perp} = y^{\perp} - Y^{\perp}\hat{\beta}_{LIML}$, and

$$\begin{aligned}\hat{\delta}_0 &= \left(Y^{\perp} - Z^{\perp}\hat{\Pi}_0\right)'\hat{\varepsilon}^{\perp}(\hat{\varepsilon}^{\perp\prime}\hat{\varepsilon}^{\perp})^{-1} = Y^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp}(\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp})^{-1} \\ \hat{\Pi}_0 &= (Z^{\perp\prime}Z^{\perp})^{-1}Z^{\perp\prime}(Y^{\perp} - \hat{\varepsilon}^{\perp}\hat{\delta}_0') = (Z^{\perp\prime}Z^{\perp})^{-1}Z^{\perp\prime}(Y^{\perp} - \hat{\varepsilon}^{\perp}\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}Y^{\perp}(\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp})^{-1}) \\ \hat{\Sigma}_{V|u,0} &= \frac{1}{T}\left(Y^{\perp} - Z^{\perp}\hat{\Pi}_0 - \hat{\varepsilon}^{\perp}\hat{\delta}_0'\right)'\left(Y^{\perp} - Z^{\perp}\hat{\Pi}_0 - \hat{\varepsilon}^{\perp}\hat{\delta}_0'\right) \\ &= \frac{1}{T}Y^{\perp}M_{Z^{\perp}}Y^{\perp} - \frac{1}{T}Y^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp}(\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp})^{-1}\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}Y^{\perp}.\end{aligned}$$

Note that

$$\left|\hat{\Sigma}_{V|u,0}\right| = \frac{1}{T^n}\left|\hat{\Gamma}'\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\hat{\Gamma}\right|\frac{1}{\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp}} = \frac{1}{T^n}\left|\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\right|\frac{1}{\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp}},$$

where $\hat{\Gamma} = \begin{pmatrix} 1 & 0 \\ -\hat{\beta}_{LIML} & I_n \end{pmatrix}$. We then have (up to a constant addition)

$$\begin{aligned}\max_{\theta:\omega=0}\log L(\theta) &= -\frac{T}{2}\log\left(|\hat{\sigma}_{uu,0}|\left|\hat{\Sigma}_{V|u,0}\right|\right) \\ &= -\frac{T}{2}\log\left(\frac{\hat{\varepsilon}^{\perp\prime}\hat{\varepsilon}^{\perp}}{\hat{\varepsilon}^{\perp\prime}M_{Z^{\perp}}\hat{\varepsilon}^{\perp}}\frac{1}{T^{n+1}}\left|\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\right|\right) \\ &= -\frac{T}{2}\log(1 + \frac{1}{T}\phi(\hat{\beta}_{LIML})) - \frac{T}{2}\log\left|\frac{1}{T}\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\right|\end{aligned}$$

Now, we impose $\Pi = 0$ and write

$$\begin{aligned}
\log L(\theta) &= -\frac{T}{2}\log\left|\sigma_{u|V}\right| - \frac{1}{2\sigma_{u|V}}\left(y^{\perp} - Z^{\perp}\omega - Y^{\perp}\varphi\right)'\left(y^{\perp} - Z^{\perp}\omega - Y^{\perp}\varphi\right) \\
&\quad -\frac{T}{2}\log\left|\Sigma_{VV}\right| - \frac{1}{2}tr\left[\Sigma_{VV}^{-1}Y^{\perp\prime}Y^{\perp}\right],
\end{aligned}$$

where $\varphi = \gamma + \beta$, $\sigma_{u|V} = \sigma_{uu} - \Sigma_{uV}\Sigma_{VV}^{-1}\Sigma_{Vu}$ and $\gamma = \Sigma_{VV}^{-1}\Sigma_{Vu}$. We use a subscript 1 to all estimates obtained with $\Pi = 0$ restriction. Similarly as the first case, we have

$$\begin{aligned}
\hat{\Sigma}_{VV,1} &= \frac{1}{T}Y^{\perp\prime}Y^{\perp} \\
\hat{\omega}_1 &= (Z^{\perp\prime}Z^{\perp})^{-1}Z^{\perp\prime}\left(y^{\perp} - Y^{\perp}\hat{\varphi}_1\right) = (Z^{\perp\prime}Z^{\perp})^{-1}Z^{\perp\prime}\left(y^{\perp} - Y^{\perp}(Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp})^{-1}Y^{\perp\prime}M_{Z^{\perp}}y^{\perp}\right) \\
\hat{\varphi}_1 &= (Y^{\perp\prime}Y^{\perp})^{-1}Y^{\perp\prime}\left(y^{\perp} - Z^{\perp}\hat{\omega}_1\right) = (Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp})^{-1}Y^{\perp\prime}M_{Z^{\perp}}y^{\perp} \\
\hat{\sigma}_{u|V,1} &= \frac{1}{T}\left(y^{\perp} - Z^{\perp}\hat{\omega}_1 - Y^{\perp}\hat{\varphi}_1\right)'\left(y^{\perp} - Z^{\perp}\hat{\omega}_1 - Y^{\perp}\hat{\varphi}_1\right) \\
&= \frac{1}{T}y^{\perp\prime}M_{Z^{\perp}}y^{\perp} - \frac{1}{T}y^{\perp\prime}M_{Z^{\perp}}Y^{\perp}(Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp})^{-1}Y^{\perp\prime}M_{Z^{\perp}}y^{\perp} = \frac{1}{T}\frac{\left|\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\right|}{\left|Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp}\right|}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\hat{\Sigma}_{VV,1}\right| &= \frac{1}{T^n}\left|Y^{\perp\prime}Y^{\perp}\right| = \frac{1}{T^n}\left|Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp} + Y^{\perp\prime}P_{Z^{\perp}}Y^{\perp}\right| \\
&= \frac{1}{T^n}\left|(Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp})^{1/2}(I_n + \frac{1}{T}G_T)(Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp})^{1/2}\right| \\
&= \frac{1}{T^n}\left|I_n + \frac{1}{T}G_T\right|\left|Y^{\perp\prime}M_{Z^{\perp}}Y^{\perp}\right|,
\end{aligned}$$

which yields (up to a constant addition)

$$\begin{aligned}
\max_{\theta:\Pi=0}\log L(\theta) &= -\frac{T}{2}\log\left(\left|\hat{\sigma}_{u|V,1}\right|\left|\hat{\Sigma}_{VV,1}\right|\right) \\
&= -\frac{T}{2}\log\left(\left|\frac{1}{T}\bar{Y}^{\perp\prime}M_{Z^{\perp}}\bar{Y}^{\perp}\right|\right) - \frac{T}{2}\log\left(\left|I_n + \frac{1}{T}G_T\right|\right).
\end{aligned}$$

Therefore, the LR statistic can be derived as

$$\begin{aligned} 2LR &= 2\max_{\theta:\omega=0}\log L(\theta) - 2\max_{\theta:\Pi=0}\log L(\theta) \\ &= T\left(\log\left|I_n + \frac{1}{T}G_T\right| - \log(1+\frac{1}{T}\phi(\hat{\beta}_{LIML}))\right) \\ &\simeq tr\,(G_T) - \phi(\hat{\beta}_{LIML}), \end{aligned}$$

where the last approximation is valid since, by construction, $\|G_T\|$ is small under $\Pi = 0$ and so is $\phi(\hat{\beta}_{LIML})$ under $\omega = 0$. ■

Proof of Theorem 1 This is a special case of Theorem 3 with $C = 0$ and $d = 0$. Note that $z_u = (1-\rho'\rho)^{1/2}\eta + z_V\rho$ and

$$\begin{aligned} \Delta_0(\kappa) &= (z_V'z_V - \kappa I)^{-1}(z_V'z_u - \kappa\rho) \\ &= \rho + (1-\rho'\rho)^{1/2}(z_V'z_V - \kappa I)^{-1}z_V'\eta, \end{aligned}$$

which implies

$$\begin{aligned} z_u - z_V\Delta_0(\kappa) &= (1-\rho'\rho)^{1/2}(I - z_V(z_V'z_V - \kappa I)^{-1}z_V')\eta \\ \Delta_0(\kappa)'\Delta_0(\kappa) &= \rho'\rho + (1-\rho'\rho)\eta' z_V(z_V'z_V - \kappa I)^{-2}z_V'\eta + 2(1-\rho'\rho)^{1/2}\rho'(z_V'z_V - \kappa I)^{-1}z_V'\eta \\ \rho'\Delta_0(\kappa) &= \rho'\rho + (1-\rho'\rho)^{1/2}\rho'(z_V'z_V - \kappa I)^{-1}z_V'\eta. \end{aligned}$$

Therefore,

$$Q_{IV} \xrightarrow{d} \lambda_{\min}(z_V'z_V) - \frac{(z_u - z_V\Delta_0(\kappa))'(z_u - z_V\Delta_0(\kappa))}{1 - 2\rho'\Delta_0(\kappa) + \Delta_0(\kappa)'\Delta_0(\kappa)},$$

in which the second component is given by

$$\frac{(z_u - z_V\Delta_0(\kappa))'(z_u - z_V\Delta_0(\kappa))}{1 - 2\rho'\Delta_0(\kappa) + \Delta_0(\kappa)'\Delta_0(\kappa)} = \frac{(1-\rho'\rho)\eta'(I - z_V(z_V'z_V - \kappa I)^{-1}z_V')^2\eta}{1 - \rho'\rho + (1-\rho'\rho)\eta' z_V(z_V'z_V - \kappa I)^{-2}z_V'\eta},$$

where $\kappa = 0$ for the TSLS estimator; $\kappa = \kappa^*$ for the LIML estimator; and $\kappa = \kappa^* - 1$ for the Fuller-k estimator with κ^* being the smallest root satisfying $|(z_u, z_V)'(z_u, z_V) - \kappa^*\bar{\Sigma}| = 0$. Note that $|(z_u, z_V)'(z_u, z_V) - \kappa^*\bar{\Sigma}| = |D'(\eta, z_V)'(\eta, z_V)D - \kappa^* D'D| = |(\eta, z_V)'(\eta, z_V) - \kappa^* I_{n+1}| = 0$, where $D = \left(\begin{smallmatrix} (1-\rho'\rho)^{1/2} & 0 \\ \rho & I_n \end{smallmatrix}\right)$. ■

Proof of Theorem 2 Part (i) is trivial and omitted. For part (ii), the limit of the first component $\lambda_{\min}(G_T^0)$ is the same as in Theorem 1. For the second component, since the roots of $\left|\bar{Y}^{\perp\prime}\bar{Y}^{\perp} - k_T\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}\right| = 0$ are the same as those of $\left|J'\bar{Y}^{\perp\prime}\bar{Y}^{\perp}J - k_T J'\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}J\right| =$

0 with $J = \begin{pmatrix} 1 & 0 \\ -\beta & I_n \end{pmatrix}$, we consider

$$\begin{aligned} 0 &= |J'\bar{Y}^{\perp\prime}\bar{Y}^{\perp}J - \hat{k}_T J'\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}J| \\ &= |J'\bar{Y}^{\perp\prime}P_{Z^\perp}\bar{Y}^{\perp}J - T(\hat{k}_T - 1)\frac{1}{T}J'\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}J| \\ &= |\hat{\Sigma}^{-1/2\prime}J'\bar{Y}^{\perp\prime}P_{Z^\perp}\bar{Y}^{\perp}J\hat{\Sigma}^{-1/2} - \kappa_T I_{n+1}|, \end{aligned}$$

where κ_T is the smallest root and $\hat{\Sigma} = T^{-1}J'\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}J$. Under $\omega \neq 0$ and $\Pi = 0$, we have $\hat{\Sigma} = T^{-1}J'\bar{Y}^{\perp\prime}M_{Z^\perp}\bar{Y}^{\perp}J \xrightarrow{p} \Sigma$ and we can write

$$P_{Z^\perp}\bar{Y}^{\perp}J = P_{Z^\perp}[y^\perp, Y^\perp]J = [Z^\perp\omega, 0] + [P_{Z^\perp}u^\perp, P_{Z^\perp}V^\perp] \equiv A_1 + A_2.$$

Note that $\lambda_{\min}(\hat{\Sigma}^{-1/2\prime}A_1'A_1\hat{\Sigma}^{-1/2}) = 0$ for all T. We denote by ν_{A_1} the eigenvector of $\hat{\Sigma}^{-1/2\prime}A_1'A_1\hat{\Sigma}^{-1/2}$ corresponding to the zero eigenvalue. Also, $\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2} = O_p(1)$, which implies that the largest eigenvalue of $\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2}$ is $O_p(1)$ as well. We let ν_{A_2} be the eigenvector of $\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2}$ corresponding to the largest eigenvalue. Let ν be the eigenvector of $\hat{\Sigma}^{-1/2\prime}J'\bar{Y}^{\perp\prime}P_{Z^\perp}\bar{Y}^{\perp}J\hat{\Sigma}^{-1/2}$ associated with κ_T. Then, for a nonzero vector x,

$$\begin{aligned} \kappa_T &= \min_{\|x\|=1} x'\hat{\Sigma}^{-1/2\prime}J'\bar{Y}^{\perp\prime}P_{Z^\perp}\bar{Y}^{\perp}J\hat{\Sigma}^{-1/2}x \\ &= \nu'\hat{\Sigma}^{-1/2\prime}J'\bar{Y}^{\perp\prime}P_{Z^\perp}\bar{Y}^{\perp}J\hat{\Sigma}^{-1/2}\nu \\ &= \nu'\hat{\Sigma}^{-1/2\prime}\left(A_1'A_1 + A_2'A_2 + A_1'A_2 + A_2'A_1\right)\hat{\Sigma}^{-1/2}\nu \\ &\leq \nu_{A_1}'\hat{\Sigma}^{-1/2\prime}\left(A_1'A_1 + A_2'A_2 + A_1'A_2 + A_2'A_1\right)\hat{\Sigma}^{-1/2}\nu_{A_1} \\ &= 0 + \nu_{A_1}'\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2}\nu_{A_1} + 2\nu_{A_1}'\hat{\Sigma}^{-1/2\prime}A_1'A_2\hat{\Sigma}^{-1/2}\nu_{A_1} \\ &\leq \nu_{A_2}'\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2}\nu_{A_2} + 2\left(\nu_{A_1}'\hat{\Sigma}^{-1/2\prime}A_1'A_1\hat{\Sigma}^{-1/2}\nu_{A_1}\right)^{1/2}\left(\nu_{A_1}'\hat{\Sigma}^{-1/2\prime}A_2'A_2\hat{\Sigma}^{-1/2}\nu_{A_1}\right)^{1/2} \\ &= O_p(1) + 0. \end{aligned}$$

Now, for $\hat{\beta}(k_T) = \beta + (V^{\perp\prime}(I_T - k_T M_{Z^\perp})V^\perp)^{-1}V^{\perp\prime}(I_T - k_T M_{Z^\perp})(Z^\perp\omega + u^\perp)$, we have

$$T^{-1/2}(\hat{\beta}(k_T) - \beta) \xrightarrow{d} \Sigma_{VV}^{-1/2}(z_V'z_V - \kappa I_n)^{-1}z_V'\Omega^{1/2}\omega$$

because

$$\begin{aligned} V^{\perp\prime}(I_T - k_T M_{Z^\perp})V^\perp &= V^{\perp\prime}V^\perp - k_T V^{\perp\prime}M_{Z^\perp}V^\perp \\ &= V^{\perp\prime}P_{Z^\perp}V^\perp - (k_T - 1)V^{\perp\prime}M_{Z^\perp}V^\perp \xrightarrow{d} \Sigma_{VV}^{1/2\prime}z_V'z_V\Sigma_{VV}^{1/2} - \kappa\Sigma_{VV} \end{aligned}$$

and

$$
\begin{aligned}
& T^{-1/2}V^{\perp\prime}(I-k_T M_{Z^\perp})(Z^\perp\omega+u^\perp) \\
= \; & T^{-1/2}V^{\perp\prime}(Z^\perp\omega+u^\perp) - T^{-1/2}k_T V^{\perp\prime}M_{Z^\perp}(Z^\perp\omega+u^\perp) \\
= \; & T^{-1/2}V^{\perp\prime}P_{Z^\perp}(Z^\perp\omega+u^\perp) - T(k_T-1)T^{-3/2}V^{\perp\prime}M_{Z^\perp}(Z^\perp\omega+u^\perp) \xrightarrow{d} \Sigma_{VV}^{1/2\prime}z_V'\Omega^{1/2}\omega,
\end{aligned}
$$

where $\kappa=0$ for the TSLS estimator, $\kappa=\kappa^*$ for the LIML estimator and $\kappa=\kappa^*-1$ for the Fuller-k estimator if we let $\kappa_T \xrightarrow{d} \kappa^*$. In addition, $\tilde{\varepsilon}^\perp = y^\perp - Y^\perp\hat{\beta}(k_T) = Z^\perp\omega + u^\perp - V^\perp(\hat{\beta}(k_T)-\beta)$ since $Y^\perp = V^\perp$. We thus have

$$
\begin{aligned}
T^{-1}\tilde{\varepsilon}^{\perp\prime}P_{Z^\perp}\tilde{\varepsilon}^\perp \;=\; & T^{-1}\omega'Z^{\perp\prime}Z^\perp\omega + T^{-1}(\hat{\beta}(k_T)-\beta)'V^{\perp\prime}P_{Z^\perp}V^\perp(\hat{\beta}(k_T)-\beta) \\
& -2T^{-1}\omega'Z^{\perp\prime}V^\perp(\hat{\beta}(k_T)-\beta) + o_p(1) \\
& \xrightarrow{d} \omega'\Omega\omega + \omega\Omega^{1/2\prime}z_V(z_V'z_V-\kappa I)^{-1}z_V'z_V(z_V'z_V-\kappa I)^{-1}z_V'\Omega^{1/2}\omega \\
& -2\omega\Omega^{1/2\prime}z_V(z_V'z_V-\kappa I)^{-1}z_V'\Omega^{1/2}\omega
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-2}\tilde{\varepsilon}^{\perp\prime}M_{Z^\perp}\tilde{\varepsilon}^\perp \;=\; & T^{-2}\tilde{\varepsilon}^{\perp\prime}\tilde{\varepsilon}^\perp + o_p(1) \\
= \; & T^{-2}(\hat{\beta}(k_T)-\beta)'V^{\perp\prime}V^\perp(\hat{\beta}(k_T)-\beta) + o_p(1) \\
& \xrightarrow{d} \omega'\Omega^{1/2\prime}z_V'(z_V'z_V-\kappa I)^{-2}z_V'\Omega^{1/2}\omega.
\end{aligned}
$$

Therefore,

$$
\phi(\hat{\beta}(k_T)) = \frac{\tilde{\varepsilon}^{\perp\prime}P_{Z^\perp}^\perp\tilde{\varepsilon}^\perp}{\tilde{\varepsilon}^{\perp\prime}M_{Z^\perp}\tilde{\varepsilon}^\perp/T} \xrightarrow{d} \frac{\omega'\Omega^{1/2\prime}\left(I-z_V(z_V'z_V-\kappa I)^{-1}z_V'\right)^2\Omega^{1/2}\omega}{\omega'\Omega^{1/2\prime}z_V'(z_V'z_V-\kappa I)^{-2}z_V'\Omega^{1/2}\omega}.
$$

For part (iii), note that Assumption 2 implies that $T^{-1}V^{\perp\prime}V^\perp \xrightarrow{p} \Sigma_{VV}$, $T^{-1}V^{\perp\prime}u^\perp \xrightarrow{p} \Sigma_{Vu}$, $T^{-1}u^{\perp\prime}u^\perp \xrightarrow{p} \sigma_{uu}$, $T^{-1}Z^{\perp\prime}Z^\perp \xrightarrow{p} \Omega$, $T^{-1}Z^{\perp\prime}Y^\perp \xrightarrow{p} \Omega\Pi$, $T^{-1}Z^{\perp\prime}y^\perp \xrightarrow{p} \Omega\omega$, and $\tilde{\Sigma}_{VV} \xrightarrow{p} \Pi'\Omega\Pi + \Sigma_{VV}$. Hence, $\lambda_{\min}(G_T^0/T) \xrightarrow{p} \lambda_{\min}((\Pi'\Omega\Pi+\Sigma_{VV})^{-1}\Pi'\Omega\Pi)$. Recall that $\hat{\beta}(k_T) = (Y^{\perp\prime}(I_T-k_TM_{Z^\perp})Y^\perp)^{-1}Y^{\perp\prime}(I_T-k_TM_{Z^\perp})y^\perp$, where $k_T=1$ for the TSLS estimator; $k_T=\hat{k}_T$ for the LIML estimator; and $k_T=\hat{k}_T-1/(T-K_1-K_2)$ for the Fuller-k estimator with $\hat{k}_T$ being the smallest root of $|\bar{Y}'M_X\bar{Y}-\hat{k}_T\bar{Y}'M_{\bar{Z}}\bar{Y}|=0$. For any T, the roots of $|\bar{Y}'M_X\bar{Y}-\hat{k}_T\bar{Y}'M_{\bar{Z}}\bar{Y}|=0$ are the same as those of $|T^{-1}\bar{Y}'M_X\bar{Y}-\hat{k}_TT^{-1}\bar{Y}'M_{\bar{Z}}\bar{Y}|=0$. Thus, $\text{plim}_{T\to\infty}\hat{k}_T = k^*$ where k^* is the smallest root of $|\text{plim}_{T\to\infty}T^{-1}\bar{Y}'M_X\bar{Y} - k^*\text{plim}_{T\to\infty}T^{-1}\bar{Y}'M_{\bar{Z}}\bar{Y}| = |\Theta+\Sigma-k^*\Sigma| = |\Theta-(k^*-1)\Sigma| = 0$ with Θ as defined in

the theorem, and $\text{plim}_{T\to\infty} k_T = k^*$ for both the LIML and Fuller-k estimators. Hence,

$$\begin{aligned}\hat{\beta}(k_T) &= (Y^{\perp\prime}P_{Z^\perp}Y^\perp - (k_T-1)Y^{\perp\prime}M_{Z^\perp}Y^\perp)^{-1}(Y^{\perp\prime}P_{Z^\perp}y^\perp - (k_T-1)Y^{\perp\prime}M_{Z^\perp}y^\perp) \\ &\xrightarrow{p} \beta + (\Pi'\Omega\Pi - (k^*-1)\Sigma_{VV})^{-1}(\Pi'\Omega\omega - (k^*-1)\Sigma_{Vu}).\end{aligned}$$

Now, $\hat{\varepsilon}^\perp = y^\perp - Y^\perp\hat{\beta}(k_T) = Z^\perp\omega + u^\perp - Y^\perp(\hat{\beta}(k_T) - \beta)$ and

$$\frac{\phi(\hat{\beta}(k_T))}{T} = \frac{\hat{\varepsilon}^{\perp\prime}P_{Z^\perp}\hat{\varepsilon}^\perp/T}{\hat{\varepsilon}^{\perp\prime}M_{Z^\perp}\hat{\varepsilon}^\perp/T} \xrightarrow{p} \frac{\omega'\Omega\omega + b(k)'\Pi'\Omega\Pi b(k) - 2\omega'\Omega\Pi b(k)}{\sigma_{uu} + b(k)'\Sigma_{VV}b(k) - 2\Sigma_{uV}b(k)}. \quad \blacksquare$$

Proof of Theorem 3 Obvious from Theorems 1 and 3 in Staiger and Stock (1997) and thus omitted. ■

A.2 Simulation Results

TABLE 1.A. Critical Values for the Q_{IV} test, TSLS

	$n=1$				$n=2$				$n=3$			
	.90	.95	.975	.99	.90	.95	.975	.99	.90	.95	.975	.99
$K_2=2$	3.97	5.35	6.74	8.60								
3	4.93	6.46	7.99	10.06	1.88	2.57	3.24	4.16				
4	5.70	7.37	9.03	11.21	2.43	3.24	4.01	5.03	1.21	1.65	2.11	2.72
5	6.45	8.28	10.09	12.40	2.90	3.82	4.72	5.83	1.62	2.19	2.74	3.45
6	7.03	8.99	10.89	13.19	3.31	4.34	5.30	6.57	1.94	2.59	3.22	3.99
7	7.60	9.69	11.64	14.14	3.68	4.85	5.93	7.33	2.22	2.98	3.70	4.59
8	8.13	10.39	12.45	15.05	4.01	5.28	6.51	7.92	2.48	3.33	4.15	5.14
9	8.59	10.94	13.12	15.83	4.35	5.70	6.98	8.57	2.71	3.67	4.55	5.63
10	9.02	11.42	13.66	16.30	4.56	6.00	7.37	9.07	2.91	3.94	4.89	5.97
11	9.49	11.99	14.33	17.26	4.83	6.41	7.80	9.56	3.10	4.23	5.28	6.50
12	9.89	12.49	14.89	17.80	5.11	6.78	8.26	10.04	3.28	4.51	5.61	6.91
13	10.28	13.02	15.50	18.59	5.34	7.07	8.63	10.48	3.40	4.72	5.90	7.32
14	10.65	13.50	16.09	19.21	5.55	7.38	9.06	11.00	3.57	4.95	6.18	7.69
15	11.06	13.96	16.58	19.74	5.76	7.68	9.37	11.46	3.63	5.12	6.39	7.87
16	11.38	14.36	17.14	20.27	5.97	7.98	9.78	11.87	3.76	5.33	6.73	8.33
17	11.71	14.81	17.61	20.88	6.15	8.17	10.05	12.35	3.88	5.50	6.95	8.63
18	12.04	15.23	18.11	21.61	6.29	8.48	10.38	12.66	4.00	5.72	7.20	9.01
19	12.36	15.55	18.47	22.05	6.53	8.76	10.72	13.03	4.08	5.88	7.45	9.26
20	12.71	16.04	18.99	22.54	6.69	9.01	11.01	13.41	4.18	6.09	7.79	9.62
21	12.98	16.39	19.50	23.24	6.78	9.20	11.29	13.78	4.20	6.13	7.90	9.84
22	13.16	16.60	19.80	23.53	7.05	9.49	11.65	14.19	4.40	6.44	8.16	10.24
23	13.50	17.02	20.29	23.94	7.18	9.67	11.89	14.54	4.38	6.48	8.23	10.35
24	13.84	17.46	20.67	24.82	7.28	9.91	12.14	14.84	4.42	6.60	8.48	10.71
25	14.17	17.86	21.08	25.11	7.41	10.04	12.37	15.17	4.62	6.82	8.74	11.02

Note: n is the number of endogenous regressors and K_2 is the number of instruments.

Table 1.B. Critical Values for the Q_{IV} test, LIML

	$n = 1$				$n = 2$				$n = 3$			
	.90	.95	.975	.99	.90	.95	.975	.99	.90	.95	.975	.99
$K_2 = 2$	3.99	5.37	6.76	8.60								
3	4.96	6.49	8.01	10.08	1.90	2.58	3.25	4.17				
4	5.76	7.42	9.05	11.23	2.48	3.28	4.05	5.05	1.23	1.67	2.13	2.73
5	6.52	8.33	10.12	12.42	2.99	3.89	4.76	5.87	1.68	2.23	2.78	3.47
6	7.13	9.06	10.94	13.24	3.43	4.44	5.37	6.64	2.03	2.66	3.28	4.04
7	7.73	9.78	11.71	14.20	3.87	4.98	6.05	7.40	2.37	3.10	3.79	4.68
8	8.28	10.49	12.53	15.11	4.25	5.45	6.64	8.04	2.70	3.49	4.28	5.23
9	8.77	11.07	13.22	15.89	4.64	5.90	7.13	8.70	3.00	3.88	4.71	5.75
10	9.25	11.57	13.75	16.40	4.91	6.28	7.57	9.21	3.28	4.21	5.09	6.11
11	9.73	12.16	14.46	17.34	5.26	6.72	8.02	9.74	3.55	4.57	5.54	6.71
12	10.17	12.66	15.03	17.91	5.61	7.13	8.52	10.25	3.82	4.88	5.90	7.15
13	10.57	13.23	15.68	18.70	5.92	7.47	8.94	10.73	4.04	5.19	6.25	7.56
14	10.97	13.71	16.26	19.35	6.17	7.84	9.41	11.27	4.29	5.48	6.59	8.02
15	11.41	14.21	16.78	19.88	6.47	8.19	9.76	11.76	4.47	5.71	6.87	8.25
16	11.77	14.63	17.34	20.42	6.75	8.53	10.20	12.18	4.70	6.03	7.28	8.74
17	12.14	15.10	17.85	21.06	6.99	8.80	10.51	12.68	4.92	6.28	7.54	9.09
18	12.50	15.56	18.35	21.79	7.24	9.12	10.93	13.05	5.12	6.55	7.88	9.53
19	12.85	15.92	18.74	22.27	7.49	9.48	11.28	13.46	5.31	6.80	8.17	9.79
20	13.23	16.37	19.25	22.81	7.73	9.75	11.63	13.89	5.53	7.12	8.56	10.23
21	13.52	16.77	19.76	23.47	7.93	10.03	11.95	14.27	5.66	7.26	8.76	10.53
22	13.75	17.04	20.12	23.77	8.26	10.39	12.36	14.75	5.93	7.58	9.11	10.96
23	14.10	17.43	20.59	24.17	8.44	10.60	12.63	15.08	6.06	7.73	9.27	11.11
24	14.47	17.92	21.00	25.07	8.63	10.88	12.96	15.45	6.23	7.95	9.55	11.50
25	14.80	18.31	21.50	25.36	8.83	11.11	13.19	15.85	6.49	8.25	9.88	11.87

Note: n is the number of endogenous regressors and K_2 is the number of instruments.

Table 1.C. Critical Values for the Q_{IV} test, Fuller-k

	$n=1$				$n=2$				$n=3$			
	.90	.95	.975	.99	.90	.95	.975	.99	.90	.95	.975	.99
$K_2=2$	3.96	5.35	6.74	8.60								
3	4.94	6.47	8.00	10.07	1.80	2.52	3.20	4.12				
4	5.73	7.40	9.04	11.23	2.40	3.22	4.00	5.01	1.06	1.55	2.03	2.65
5	6.50	8.32	10.12	12.41	2.92	3.84	4.73	5.84	1.54	2.13	2.70	3.42
6	7.11	9.05	10.93	13.24	3.37	4.39	5.34	6.61	1.91	2.57	3.21	3.99
7	7.71	9.77	11.70	14.19	3.81	4.94	6.02	7.38	2.25	3.02	3.73	4.63
8	8.26	10.48	12.52	15.10	4.19	5.42	6.61	8.02	2.60	3.42	4.22	5.18
9	8.76	11.06	13.21	15.89	4.59	5.87	7.11	8.68	2.91	3.81	4.66	5.72
10	9.23	11.56	13.75	16.39	4.87	6.25	7.54	9.20	3.19	4.15	5.04	6.08
11	9.72	12.15	14.45	17.34	5.21	6.69	8.00	9.72	3.47	4.51	5.49	6.67
12	10.15	12.65	15.02	17.91	5.57	7.10	8.50	10.23	3.73	4.83	5.86	7.12
13	10.55	13.22	15.67	18.69	5.88	7.44	8.92	10.72	3.97	5.13	6.21	7.53
14	10.96	13.71	16.25	19.35	6.12	7.81	9.40	11.26	4.22	5.43	6.56	7.99
15	11.40	14.20	16.77	19.88	6.43	8.16	9.74	11.74	4.41	5.66	6.83	8.22
16	11.76	14.63	17.34	20.41	6.71	8.51	10.18	12.16	4.64	5.99	7.26	8.72
17	12.13	15.09	17.84	21.05	6.95	8.77	10.49	12.67	4.86	6.25	7.51	9.07
18	12.49	15.55	18.35	21.79	7.21	9.10	10.91	13.04	5.06	6.51	7.85	9.51
19	12.84	15.91	18.73	22.27	7.46	9.45	11.26	13.45	5.24	6.75	8.14	9.76
20	13.22	16.37	19.24	22.81	7.70	9.73	11.61	13.87	5.47	7.08	8.53	10.20
21	13.51	16.76	19.76	23.47	7.89	10.01	11.93	14.25	5.60	7.22	8.73	10.50
22	13.74	17.03	20.12	23.77	8.23	10.37	12.35	14.75	5.88	7.54	9.08	10.94
23	14.09	17.43	20.59	24.17	8.41	10.58	12.61	15.07	6.01	7.69	9.23	11.09
24	14.46	17.92	21.00	25.07	8.60	10.87	12.94	15.44	6.18	7.92	9.53	11.48
25	14.79	18.31	21.49	25.36	8.80	11.09	13.18	15.84	6.44	8.22	9.85	11.84

Note: n is the number of endogenous regressors and K_2 is the number of instruments.

Table 2.A. Rejection Probabilities of the Overidentifying Restrictions Test, TSLS

$n = 1$		$K_2 = 3$			$K_2 = 9$		
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.01)	n-rej (0.99)	uncond.	rej (0.00)	n-rej (0.99)	uncond.
0.5	0.1	0.0550	0.0364	0.0365	0	0.0450	0.0450
	0.3	0.0661	0.0410	0.0411	0	0.0506	0.0506
	0.5	0.0752	0.0530	0.0531	0.1667	0.0660	0.0660
	0.8	0.1229	0.1142	0.1143	0.1667	0.1179	0.1179
	1.0	0.3321	0.2008	0.2016	0.3333	0.1880	0.1880
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.03)	n-rej (0.97)	uncond.	rej (0.01)	n-rej (0.99)	uncond.
0.8	0.1	0.0419	0.0395	0.0395	0.0442	0.0465	0.0465
	0.3	0.0498	0.0437	0.0438	0.0549	0.0508	0.0508
	0.5	0.0613	0.0545	0.0547	0.0869	0.0617	0.0619
	0.8	0.0972	0.0986	0.0985	0.1631	0.0943	0.0948
	1.0	0.2328	0.1472	0.1494	0.3125	0.1335	0.1347
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.10)	n-rej (0.90)	uncond.	rej (0.19)	n-rej (0.81)	uncond.
1.2	0.1	0.0473	0.0413	0.0419	0.0486	0.0472	0.0475
	0.3	0.0516	0.0449	0.0455	0.0542	0.0500	0.0508
	0.5	0.0600	0.0536	0.0542	0.0663	0.0570	0.0587
	0.8	0.0996	0.0825	0.0841	0.1046	0.0756	0.0810
	1.0	0.1808	0.1092	0.1161	0.1609	0.0933	0.1058

Note: n is the number of endogenous regressors; K_2 is the number of instruments; ρ represents the degree of endogeneity; $\Lambda_C'\Lambda_C$ corresponds to the weak limit of the concentration matrix; and $\lambda^*_{\min}$ is the boundary value of the minimum eigenvalue for the weak instruments set based on the 10% TSLS bias. The numbers in parenthesis next to rej. are $\mathbb{P}(g_\infty$ rejects) and those next to n-rej. are $\mathbb{P}(g_\infty$ not rejects). The column rej. shows $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ rejects), n-rej $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ not reject), and uncond. $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05})$.

Table 2.B. Rejection Probabilities of the Overidentifying Restrictions Test, Fuller-k

$n = 1$		$K_2 = 3$			$K_2 = 9$		
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.01)	n-rej (0.99)	uncond.	rej (0.01)	n-rej (0.09)	uncond.
0.5	0.1	0.1027	0.1745	0.1739	0.1887	0.2830	0.2823
	0.3	0.1655	0.1817	0.1815	0.3032	0.2887	0.2888
	0.5	0.2240	0.1786	0.1789	0.4218	0.2769	0.2779
	0.8	0.2924	0.1438	0.1449	0.5782	0.1878	0.1907
	1.0	0.2753	0.1062	0.1074	0.5040	0.0749	0.0780
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.03)	n-rej (0.97)	uncond.	rej (0.03)	n-rej (0.97)	uncond.
0.8	0.1	0.0906	0.1454	0.1440	0.1667	0.2440	0.2420
	0.3	0.1309	0.1404	0.1402	0.2427	0.2316	0.2319
	0.5	0.1666	0.1266	0.1277	0.3304	0.2026	0.2060
	0.8	0.2115	0.0867	0.0900	0.4147	0.1067	0.1147
	1.0	0.1939	0.0668	0.0702	0.3380	0.0502	0.0578
$\Lambda_C'\Lambda_C/\lambda^*_{\min}$	ρ	rej (0.09)	n-rej (0.91)	uncond.	rej (0.09)	n-rej (0.91)	uncond.
1.2	0.1	0.0933	0.1216	0.1192	0.1520	0.2071	0.2023
	0.3	0.1183	0.1082	0.1091	0.2044	0.1803	0.1824
	0.5	0.1420	0.0901	0.0945	0.2495	0.1411	0.1505
	0.8	0.1572	0.0581	0.0665	0.2670	0.0618	0.0796
	1.0	0.1466	0.0476	0.0560	0.2172	0.0363	0.0520

Note: n is the number of endogenous regressors; K_2 is the number of instruments; ρ represents the degree of endogeneity; $\Lambda_C'\Lambda_C$ corresponds to the weak limit of the concentration matrix; and $\lambda^*_{\min}$ is the boundary value of the minimum eigenvalue for the weak instruments set based on the 10% TSLS bias. The numbers in parenthesis next to rej. are $\mathbb{P}(g_\infty$ rejects$)$ and those next to n-rej. are $\mathbb{P}(g_\infty$ not rejects$)$. The column rej. shows $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ rejects$)$, n-rej $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05}|g_\infty$ not reject$)$, and uncond. $\mathbb{P}(\phi_\infty > \chi^2_{K_2-n,0.05})$.

TABLE 3.A. Finite Sample Sizes of the Q_{IV} test, TSLS

	$n=1$			$n=2$			$n=3$		
K_2	2	4	6	3	5	7	4	6	8
$T=50$	.0428	.0374	.0324	.0504	.0560	.0478	.0560	.0552	.0476
100	.0524	.0526	.0402	.0518	.0494	.0478	.0550	.0516	.0500
200	.0442	.0498	.0458	.0542	.0544	.0538	.0560	.0504	.0530
300	.0458	.0488	.0458	.0500	.0432	.0474	.0568	.0520	.0514

TABLE 3.B. Finite Sample Sizes of the Q_{IV} test, Fuller-k

	$n=1$			$n=2$			$n=3$		
K_2	2	4	6	3	5	7	4	6	8
$T=50$	.0430	.0374	.0324	.0492	.0554	.0478	.0532	.0536	.0482
100	.0522	.0532	.0396	.0510	.0488	.0468	.0552	.0526	.0500
200	.0438	.0502	.0468	.0528	.0552	.0528	.0550	.0502	.0530
300	.0460	.0488	.0454	.0490	.0436	.0476	.0578	.0516	.0516

TABLE 4.A. Finite Sample Powers of the Q_{IV} test, $\omega = 0$ and $\Pi \neq 0$, TSLS

	$n=1$			$n=2$			$n=3$		
K_2	2	4	6	3	5	7	4	6	8
$T=50$	.7102	.3568	.1642	.4856	.2152	.2780	.3790	.1978	.1758
100	.9624	.4042	.4724	.7594	.5534	.4808	.6376	.5252	.4130
200	.8978	.8006	.8326	.9474	.8888	.8040	.9172	.8116	.7262
300	1.000	.9690	.9966	.9872	.9602	.9472	.9838	.9426	.8948

Note: T is the sample size, n is the number of endogenous regressors and K_2 is the number of instruments.

TABLE 4.B. Finite Sample Powers of the Q_{IV} test, $\omega = 0$ and $\Pi \neq 0$, Fuller-k

	$n = 1$			$n = 2$			$n = 3$		
K_2	2	4	6	3	5	7	4	6	8
$T = 50$	.7112	.3602	.1668	.4884	.2200	.2904	.3798	.1998	.1850
100	.9642	.4106	.4794	.7626	.5662	.5018	.6434	.5416	.4350
200	.8996	.8104	.8450	.9502	.8996	.8202	.9238	.8320	.7628
300	1.000	.9732	.9980	.9890	.9676	.9532	.9860	.9586	.9156

TABLE 5.A. Finite Sample Powers of the Q_{IV} test, $\omega \neq 0$ and $\Pi = 0$, TSLS

	$n = 1$			$n = 2$			$n = 3$		
K_2	2	4	6	3	5	7	4	6	8
$T = 50$	.0186	.0070	.0036	.0350	.0178	.0102	.0450	.0270	.0152
100	.0186	.0092	.0020	.0274	.0126	.0044	.0384	.0184	.0118
200	.0174	.0044	.0036	.0294	.0134	.0038	.0362	.0180	.0076
300	.0192	.0040	.0016	.0304	.0114	.0054	.0404	.0140	.0110
∞	.0168	.0034	.0000	.0290	.0099	.0037	.0344	.0118	.0058

TABLE 5.B. Finite Sample Powers of the Q_{IV} test, $\omega \neq 0$ and $\Pi = 0$, Fuller-k

	$n = 1$			$n = 2$			$n = 3$		
K_2	2	4	6	3	5	7	4	6	8
$T = 50$	.0186	.0080	.0046	.0310	.0192	.0124	.0352	.0278	.0178
100	.0198	.0114	.0032	.0254	.0132	.0080	.0310	.0178	.0130
200	.0184	.0060	.0048	.0244	.0154	.0072	.0258	.0200	.0090
300	.0194	.0060	.0032	.0266	.0122	.0076	.0296	.0152	.0150

Note: T is the sample size, n is the number of endogenous regressors and K_2 is the number of instruments.

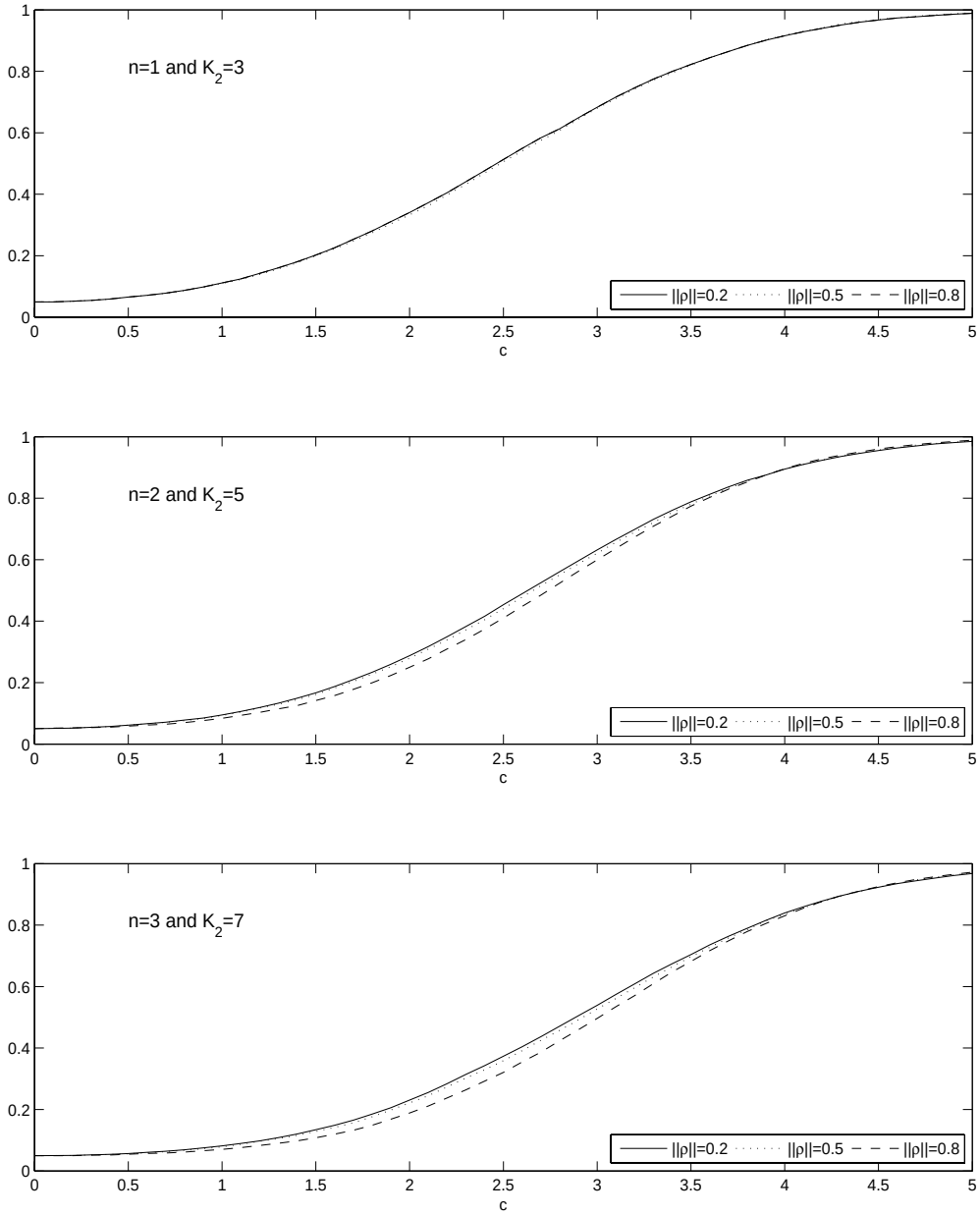

Figure 1.A. Local Power of the Q_{IV} test, $\Lambda_C \neq 0$ and $\xi_d = 0$.

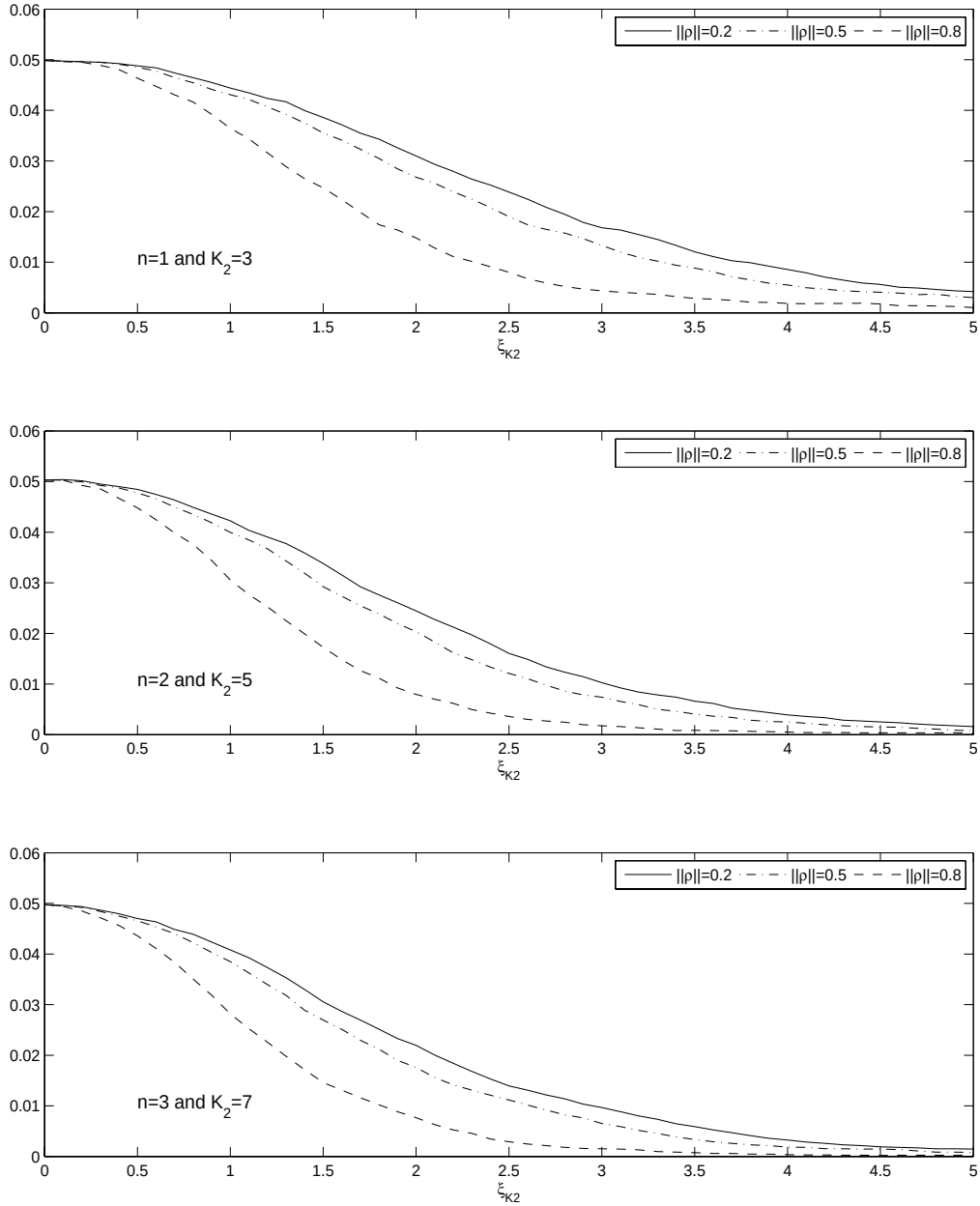

Figure 1.B. Local Power of the Q_{IV} test, $\Lambda_C = 0$ and $\xi_d \neq 0$.

References

[1] Anderson, T. W. and H. Rubin (1949). Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations, *Annals of Mathematical Statistics*, 20, 46-63.

[2] Basmann, R. L. (1960). On Finite Sample Distributions of Generalized Classical Linear Identifiability Test Statistics, *Journal of the American Statistical Association*, 55, 650-659.

[3] Cragg, J. G. and S. G. Donald (1993). Testing Identifiability and Specification in Instrumental Variable Models, *Econometric Theory*, 9, 222-240.

[4] Doko, F. and J-M. Dufour (2008). Instrumental Endogeneity and Identification-robust Tests: Some Analytical Results, *Journal of Statistical Planning and Inference*, 138, 2649-2661.

[5] Hall, A. R., G. D. Rudebusch and D. W. Wilcox (1996). Judging Instrument Relevance in Instrumental Variables Estimation, *International Economic Review*, 37, 283-298.

[6] Moreira, M. J. (2003). A Conditional Likelihood Ratio Test for Structural Models, *Econometrica*, 71, 1027-1048.

[7] Newey, W. K. (1985). Maximum likelihood specification testing and conditional moment tests, *Econometrica*, 53, 1047–1070.

[8] Sargan, J. D. (1958). The estimation of economic relationships using instrumental variables, *Econometrica*, 26, 393–415.

[9] Staiger, D. and J. H. Stock (1997). Instrumental Variables Regression with Weak Instruments, *Econometrica*, 65, 557-586.

[10] Stock, J. H. and M. Yogo (2005). Testing for Weak Instruments in Linear IV Regression, in Identification and Inference for Econometric Models, *Essays in Honor of Thomas Rothenberg*, ed. by D. W. K. Andrews and J. H. Stock, 80-107.

[11] Vuong, Q. H. (1989). Likelihood Ratio Tests for Model Selection and Non-nested Hypotheses, *Econometrica*, 57, 307-333