Forecast Rationality Tests in the Presence of Instabilities, With Applications to Federal Reserve and Survey Forecasts

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June 8, 2014

Abstract

This paper proposes a framework to implement regression-based tests of predictive ability in unstable environments, including, in particular, forecast unbiasedness and efficiency tests, commonly referred to as tests of forecast rationality. Our framework is general: it can be applied to model-based forecasts obtained either with recursive or rolling window estimation schemes, as well as to forecasts that are model-free. The proposed tests provide more evidence against forecast rationality than previously found in the Federal Reserve's Greenbook forecasts as well as survey-based private forecasts. It confirms, however, that the Federal Reserve has additional information about current and future states of the economy relative to market participants.

Keywords: Forecasting, forecast rationality, regression-based tests of forecasting ability, Greenbook forecasts, survey forecasts, real-time data

J.E.L. Codes: C22, C52, C53.[1]

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1 Introduction

Forecasting is a fundamental tool in economics, as well as statistics, business and other sciences. Judging whether forecasts are good is therefore of great importance, especially since forecasts are used everyday to guide policymakers' and practitioners' decisions. A large literature has provided important insights on how to test whether forecasts are optimal. For example, as the seminal works of Granger and Newbold (1986) and Diebold and Lopez (1996) show, under covariance stationarity and a mean square error loss, forecast errors are mean zero (conditionally and unconditionally) and the h-step-ahead forecast error has zero serial correlation after $(h-1)$ lags. If a forecast is such that its forecast errors satisfy such properties, it is deemed optimal or rational.[2] If a forecast does not satisfy these properties, researchers conclude that the model underlying such forecast can be improved.

However, one of the fundamental assumptions tacitly underlying the existing literature is that of covariance stationarity. Only very recently researchers have become concerned about the consequences of relaxing stationarity assumptions in performing inference regarding predictive ability.[3] For example, Giacomini and Rossi (2010) have developed methods to perform inference on forecast comparisons when the forecasting ability may be affected by instabilities. Besides forecast comparisons, another important issue that forecasters face in practice is to determine whether forecasts are rational or optimal, and that might also be affected by instabilities. In fact, several studies evaluate the robustness of forecast rationality in sub-samples (e.g. Croushore 1998, Patton and Timmermann, 2011, Croushore, 2011). However, while in some cases the choice of the sub-samples may be guided by economic considerations (e.g. sub-samples associated with structural breaks identified by the Great Moderation or Great Recession), in many cases the choice of sub-samples may be ad-hoc. Even when the choice is guided by economic considerations, it may be important to assess

[1] **Acknowledgements:** We are grateful to M. McCracken, A. Patton, C. Vega and participants of seminars at Bank of Canada, CREI, Trinity College Dublin, U. of Barcelona, U. of Montreal, Norges Bank, the 2011 IWH-CIREQ Macroeconometric Workshop, the 2011 JSM, the 2011 MEG, the 2013 Econometric Society Summer Meetings and the 2014 EABCN/Bank of England and ECB workshops for comments. B. Rossi gratefully acknowledges financial support from the European Research Agency's Marie Curie Grant 303434 and the ERC Grant 615608. The views expressed in this paper are those of the authors. No responsibility should be attributed to the Bank of Canada. A previous version was circulated under the title: "Forecast Optimality Tests in the Presence of Instabilities."

[2] Note that we use optimality and rationality interchangeably.

[3] See the discussion in Rossi (2013).

the robustness of the empirical results to other sub-samples, as there might be multiple breaks in the data, or the break date might be uncertain or completely unknown.

This paper proposes forecast rationality tests that are robust to the presence of instabilities. We consider a framework where forecasts are produced either with recursive or rolling estimation schemes, and the size of the estimation window is large relative to the sample size. We propose a "Fluctuation Rationality" test, which is based on testing forecast rationality in rolling windows over the out-of-sample forecast portion of the data. By using rolling windows we avoid averaging out instabilities, and our tests can have greater power to reject forecast rationality than traditional tests when rationality is present only in sub-samples of the data. Our "Fluctuation Rationality" test can be applied to study forecast unbiasedness, efficiency, rationality, encompassing, as well as serial uncorrelation, among other regression-based tests of forecasting ability.[4]

This paper is closely related to Giacomini and Rossi (2010) and West and McCracken (1998). Giacomini and Rossi (2010) propose a "Fluctuation test" to compare forecasting models in the presence of instabilities. While our "Fluctuation Rationality" test is inspired by their work, there are several differences between their framework and ours. Their framework compares models' relative forecasting performance and is focused primarily on a rolling window estimation where the size of the window is fixed (i.e. finite). We are instead interested in measures of absolute predictive ability and tests for forecast rationality. Our framework focuses on an estimation window size that is a fixed fraction of the total sample size, which allows us to take into account parameter estimation error and considerably complicates the analysis. The latter framework is similar to that of West and McCracken (1998). The difference between our tests and West and McCracken's (1998) is that the latter is based on measures of average forecasting ability in the out-of-sample portion of the data, and may lack power in certain directions when there are rationality breakdowns over time. Our tests can be used both when the forecasting model is known (and thus the researcher needs to correct for parameter estimation) and when it is not known (such as in the Greenbook and survey forecasts in our empirical application).

We demonstrate the usefulness of our procedures by evaluating the rationality of the

[4]It should be noted that, as shown in Rossi (2012), our framework can encompass the hypothesis of forecast optimality for more general definitions of optimality. As Patton and Timmermann (2010) suggest, under certain regularity conditions forecast optimality tests can reduce to those considered in this paper for arbitrary loss functions (symmetric or asymmetric) if one adheres to the definition of "generalized forecast error" and/or changes the probability measure.

Federal Reserve's Greenbook forecast of inflation as well as the private sector's forecasts provided by the Survey of Professional Forecasters and Blue Chip Economic Indicators. We revisit the empirical analysis in Romer and Romer (2000), Patton and Timmermann (2011), and Croushore (2012) in a framework that is robust to the presence of instabilities. We then reconsider Romer and Romer's (2000) hypothesis that the Federal Reserve has an information advantage in forecasting inflation beyond what is known to the private forecasters, again using our framework robust to time-variation. In both cases, our empirical results are very different than those in the literature: first, we find more empirical evidence against forecast rationality using our tests than using the traditional tests. In fact, the Fed was consistently under-estimating inflation in the 1970s, due to recurrent and unpredictable oil price shocks, and over-estimating inflation in the 1980s, during Volker's disinflation. Clearly, traditional forecast unbiasedness tests applied over the full sample do not reject forecast unbiasedness because under-predictions, on average, cancel out over-predictions. Similar issues affect tests of forecast rationality in general. Our test, instead, is capable of uncovering the lack of forecast rationality. Our findings are related to Sinclair et al. (2010), who similarly found systematic errors in Fed's forecasts using other techniques. Furthermore, our test uncovers that the informational advantage of the Fed over private sectors' forecasts, while confirmed in the data, has decreased over time.[5]

It is important to consider the trade-offs between our tests and the existing tests for forecast rationality. As our Monte Carlo simulations show, the test is capable of signalling lack of forecast rationality even if it is present in a sub-sample, and therefore has higher power relative to the existing tests in the latter cases; however, because the test is implemented in rolling windows over the out-of-sample period, the number of observations used to implement the test **is** less than the total number of forecasts, thus its power may be lower than that of existing tests in small samples when there are no instabilities in the data.

The paper is structured as follows. The second section discusses the motivation that inspired the development of the techniques proposed in this paper, while the third section presents the econometric methodology. Sections 4 and 5 respectively present the results in the general framework and in special cases that are very relevant for researchers in practice. Section 6 studies the size and power of our "Fluctuation Rationality" test in small samples, while Section 7 discusses the empirical applications. The last section concludes.

[5] This evidence is consistent with that in Gamber and Smith (2009).

2 Motivation

In a very influential paper, Romer and Romer (2000) analyzed the properties of forecasts of US inflation made by the Federal Reserve Board as well as by several private institutions. Their goal was to evaluate whether the forecasts were rational, that is, whether they were unbiased and efficient by using standard Mincer and Zarnowitz's (1969) tests. Based on their empirical analysis, they find no evidence against the rationality of the Federal Reserve Board's staff forecasts. Given that the forecasts are ultimately used by the central bank in guiding monetary policy, it is important that they are unbiased and efficient.

To motivate the methodologies developed in this paper, consider Figure 1. Figure 1 reports one-quarter-ahead forecasts of U.S. inflation made by several institutions. The dash/dotted line reports the forecasts made by the Federal Reserve Board; the dotted line reports the forecasts made by the Blue Chip Economic Indicator (BCEI) and the dashed line reports forecasts made by the Survey of Professional Forecasters (SPF). These forecasts are discussed in detail in Section 7. Note that all forecasts *under-predict* inflation in the 1970s and early 1980s, a time period where the economy was constantly subject to unforecastable oil price shocks. Also, the forecasts constantly *over-predict* inflation in the late 1980s and 1990s, a time period where the monetary authority was constantly fighting inflation. Thus, the forecasts appear not to be unbiased, nor rational: they under-predict the target in the first part of the sample, and over-predict it in the second part of the sample. We will investigate whether that is the case using formal statistical tests that we propose. If the forecasts are not unbiased, then, why did Romer and Romer (2000) conclude that the forecasts were unbiased? They applied their test over the full sample, which comprises periods of over-prediction as well as under-prediction. Thus, on average, the forecasts are unbiased. However, they may not be systematically so.

The goal of this paper is to develop techniques to help researchers detect situation where forecasts are not rational nor, in general, optimal, but the lack of rationality appears only in sub-samples of the data, or presents itself in an unstable fashion. In fact, existing tests, that are based on stationarity assumptions, should not be used in the presence of instabilities: they could lead to the wrong conclusion, as in the empirical example considered here.

3 The Econometric Framework

The main objective of this paper is to test whether the $h-$step ahead, out-of-sample direct forecasts for the variable y_t, which we assume to be a scalar, are optimal ($h > 0$). We assume that the forecasts are based on a model that is characterized by the $(k \times 1)$ parameter vector γ. The forecasts are obtained by dividing the sample of size $(T + h)$ observations into an in-sample portion of size R and an out-of-sample portion of size P, such that $R+P = T+h$. The sequence of P out-of-sample forecast errors depends on the realizations of the forecasted variable and on the in-sample parameter estimates, $\widehat{\gamma}_{t,R}$. According to usual forecasting practices, we assume that these parameters are estimated in either one of two ways: (i) re-estimated at each $t = R, ..., T$ over a window of R observations including data indexed $t - R + 1, ..., t$ (rolling scheme); or (ii) re-estimated at each $t = R, ..., T$ over a window of t observations including data indexed $1, ..., t$ (recursive scheme).

Let the forecast error associated with the h-step-ahead forecast made at time t be denoted by $v_{t+h}(\widehat{\gamma}_{t,R})$. For example, in a simple linear regression model with h-period lagged $(k \times 1)$ vector of regressors x_t where $E_t y_{t+h} = x_t' \gamma$, we have $v_{t+h}(\widehat{\gamma}_{t,R}) = y_{t+h} - x_t' \widehat{\gamma}_{t,R}$.

We focus on testing for forecast rationality in the framework developed by West and McCracken (1998). Consider the general regression:

$$v_{t+h}(\widehat{\gamma}_{t,R}) = g_t' \cdot \theta + \eta_{t+h}, \ t = R, ..., T, \tag{1}$$

where θ is an $(\ell \times 1)$ parameter vector, $v_{t+h}(\widehat{\gamma}_{t,R})$ is the estimated forecast error, and g_t is an $(\ell \times 1)$ vector of variables known at time t such that $E(g_t g_t') \equiv G$ is an $(\ell \times \ell)$ matrix with full rank. West and McCracken (1998) focus on testing the null hypothesis:

$$H_0 : \theta = \theta_0 \text{ vs. } H_A : \theta \neq \theta_0, \text{ where } \theta_0 = 0. \tag{2}$$

Let $\widehat{\theta}_P$ denote the estimate of θ in regression (1). Consider the following Wald test:

$$\mathcal{W}_P = P \left(\widehat{\theta}_P - \theta_0 \right)' \widehat{V}_{\theta,P}^{-1} \left(\widehat{\theta}_P - \theta_0 \right), \tag{3}$$

where $\widehat{V}_{\theta,P}$ is a consistent estimate of the long run variance of $\sqrt{P}\widehat{\theta}_P$. West and McCracken (1998) show that it is important to correct the estimate of the variance by parameter estimation error in order to estimate the long run variance consistently (cfr. their Theorem 4.1). We report the exact formula for West and McCracken's (1998) case, $\widehat{V}_{\theta,P}$, at the end of Section 4.

The framework in eq. (1) is quite general and includes the following leading cases:

(i) forecast unbiasedness tests, where $g_t = 1$;

(ii) forecast efficiency, where $g_t = y_{t+h|t}$.

(iii) forecast rationality (Mincer and Zarnowitz, 1969), where $g_t = [1 \ y_{t+h|t}]$, $\theta = [\alpha, \beta]'$, and typically a researcher is interested in testing whether α and β are jointly zero; more in general, g_t may also contain any other variable known at time t which was not included in the forecasting model;

(iv) forecast encompassing tests, where g_t is the forecast of the encompassed model;

(v) serial uncorrelation tests, where $g_t = v_t$.

We refer to all these tests under the maintained assumption that $\theta_0 = 0$ as "tests for forecast rationality"; the zero restriction on the parameter under the null hypothesis ensures that the forecast errors are truly unpredictable given the information set available at the time when the forecast is made.

Our main interest is testing forecast optimality in the presence of instabilities. In fact, in the presence of instabilities, tests that focus on the average out-of-sample performance of a model may be misleading, as they may average out instabilities. Instead, we consider the following rolling regression approach. Let $\widehat{\theta}_j$ be the parameter estimate in regression (1) computed at time j over rolling windows of size m.[6] That is, $\widehat{\theta}_j$ is recursively estimated in regression (1) for $j = R+m, ..., T$.[7] Also, let the Wald test in the corresponding regressions be defined as:

$$\mathcal{W}_{j,m} = m\widehat{\theta}_j' \ \widehat{V}_\theta^{-1}\widehat{\theta}_j, \text{ for } j = R+m, ..., T, \tag{4}$$

where, for example, in some special cases (such as forecast unbiasedness or efficiency), $\widehat{V}_\theta$ is a HAC estimator of the asymptotic variance of the parameter estimates in the rolling windows obtained as in West and McCracken (1998), that is, implemented by replacing P in their notation with m. We refer to $\max_{j \in \{R+m,...,T\}} \mathcal{W}_{j,m}$ as the "Fluctuation Rationality" test and we use it to test the null hypothesis:[8]

$$H_0 : \theta_j = \theta_0 \text{ vs. } H_A : \theta_j \neq \theta_0, \ \forall \ j = R+m, ..., T \tag{5}$$

[6]Without loss of generality, we removed the dependence of $\widehat{\theta}_j$ from m (m is the same for all $\widehat{\theta}_j' s$).

[7]E.g., $\widehat{\theta}_{R+m}$ is estimated in equation (1) using $v_{R+h}(\widehat{\gamma}_{R,R})$, ..., $v_{R+m+h}(\widehat{\gamma}_{R+m,R})$; $\widehat{\theta}_{R+m+1}$ is estimated in equation (1) using $v_{R+1+h}(\widehat{\gamma}_{R+1,R})$, ..., $v_{R+m+h+1}(\widehat{\gamma}_{R+m+1,R})$; ... and $\widehat{\theta}_T$ is estimated in equation (1) using $v_{T-m+1+h}(\widehat{\gamma}_{T-m+1,R})$, ..., $v_{T+h}(\widehat{\gamma}_{T,R})$.

[8]In the construction of the test we associate the end of period date of the fixed window m with the parameter estimate $\widehat{\theta}_j$. In fact, that need not necessarily be the case. If one prefers, one can choose to associate the mid-period date of the fixed window m with the parameter estimate, for example.

where $\theta_0 = 0$ and θ_j is the true parameter value.

4 General Results

Let the $(k \times 1)$ true parameter vector be denoted by γ^*, $v_{t+h}(\gamma^*) \equiv v_{t+h}$, $f_{t+h}(\widehat{\gamma}_{t,R}) \equiv g_t v_{t+h}(\widehat{\gamma}_{t,R})$ (an $(\ell \times 1)$ vector), $f_{t+h} \equiv g_t v_{t+h} = f_{t+h}(\gamma^*)$, $f_{t+h,\gamma} \equiv \frac{\partial f_{t+h}(\gamma^*)}{\partial \gamma}$, $F \equiv E\left(\frac{\partial f_{t+h}(\gamma^*)}{\partial \gamma}\right)$ (an $(\ell \times k)$ matrix).

We make the following assumptions:

Assumption 1:

(i) *The estimate* $\widehat{\gamma}_{t,R}$ *satisfies* $\widehat{\gamma}_{t,R} - \gamma^* = B_t H_t$ *where* B_t *is* $(k \times q)$ *matrix and* H_t *is* $(q \times 1)$ *with (a)* $B_t \underset{p}{\rightarrow} B$ *with rank k; (b)* $H_t = t^{-1}\sum_{r=1}^{t} h_r(\gamma^*)$ *for the recursive estimation method or* $H_t = R^{-1}\sum_{r=t-R+1}^{t} h_r(\gamma^*)$ *for the rolling for a* $(q \times 1)$ *orthogonality condition* $h_r(\gamma^*)$*; (c)* $E(h_r(\gamma^*)) = 0$.

(ii) *In some neighborhood N around* γ^**, and with probability 1,* $v_t(\gamma)$ *and* $g_t(\gamma)$ *are measurable and twice continuously differentiable, and* $E(g_t g_t') \equiv G$ *is an* $(\ell \times \ell)$ *of rank* ℓ.

Assumption 2:

(i) $lim_{T\rightarrow\infty} sup_j \; m^{-1/2} \sum_{t=j-m+1}^{j} (f_{t+h,\gamma} - F) B H_t = o_p(1)$;

(ii) $lim_{T\rightarrow\infty} sup_j \; m^{-1/2} F \sum_{t=j-m+1}^{j} (B_t - B) H_t = o_p(1)$;

(iii) $lim_{T\rightarrow\infty} sup_j \; m^{-1/2} \sum_{t=j-m+1}^{j} (f_{t+h,\gamma} - F)(B_t - B) H_t = o_p(1)$;

(iv) $lim_{T\rightarrow\infty} sup_j \left[\left(m^{-1} \sum_{t=j-m+1}^{j} g_t g_t'\right)^{-1} - G^{-1}\right] = o_p(1)$.

(v) *There is a finite constant D such that, for all t,* $sup_{\gamma\in N}|\partial^2 v_t(\gamma)/\partial\gamma\partial\gamma'| < m_t$ *for a measurable* m_t *such that* $E(m_t^4) < D$ *and the same holds when* v_t *is replaced by any element of* g_t.

Assumption 3. $lim_{T\rightarrow\infty} m/T = \mu \in (0,1)$ *as* $m \rightarrow \infty$*;* $lim_{T\rightarrow\infty} R/T = \rho \in (0,1)$ *as* $R \rightarrow \infty$*; let* $j = [\tau T]$, $t = [sT]$*, where [.] denotes the integer function and* $\tau \in (0,1)$, $s \in (0,1)$, $h < \infty$.

Assumption 1(i) allows the in-sample parameter estimates to be obtained by general estimation procedures such as Ordinary Least Squares (OLS), maximum likelihood, and

GMM, for example. Assumption 1(ii) imposes differentiability and full rank conditions. Assumption 3 requires j, R and m to be large relative to the sample size (in particular, relative to the finite horizon, h) and ensures the consistency of the out-of-sample test statistics. The assumption on ρ accommodates rolling and recursive estimation schemes.

The Appendix shows that, under Assumptions 1, 2 and 3, we have:

$$m^{1/2}\widehat{\theta}_j = G^{-1}\left(\frac{T}{m}\right)^{1/2}[I_\ell,\ FB]\left\{\frac{1}{\sqrt{T}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j, \tag{6}$$

where A_j is a remainder term such that $lim_{T\to\infty} sup_j A_j = o_p(1)$ and I_ℓ is an $(\ell \times \ell)$ identity matrix. Let $\sum_{t=R}^{j} H_t = \sum_{t=1}^{j} a_{R,t,j} h_t$, where direct calculations show that:

(i) for the recursive estimation scheme $\left(H_t = t^{-1}\sum_{r=1}^{t} h_r\right)$:

$$a_{R,t,j} = \left(R^{-1} + ... + j^{-1}\right)\cdot 1\,(t \le R) + \left(t^{-1} + ... + j^{-1}\right)\cdot 1\,(R < t \le j)\,; \tag{7}$$

(ii) for the rolling estimation scheme $\left(H_t = R^{-1}\sum_{r=t-R+1}^{t} h_r\right)$:
if $\tau - \rho \ge \rho$:

$$\begin{aligned} a_{R,t,j} = &\left(\frac{1}{R}\right)\{t\cdot 1\,(t \le R) + R\cdot 1\,(R < t \le j-R) \\ &+ (j-t)\cdot 1\,(j-R < t \le j)\}; \end{aligned} \tag{8}$$

whereas if $\tau - \rho < \rho$:

$$\begin{aligned} a_{R,t,j} = &\left(\frac{1}{R}\right)\{t\cdot 1\,(t < j-R) + (j-R)\cdot 1\,(j-R \le t \le R) \\ &+ (j-t)\cdot 1\,(R < t \le j)\}. \end{aligned} \tag{9}$$

In addition, for all estimation schemes let $b_{R,j,t} = 1\,(t > R)$.

Let I_q be a $(q \times q)$ identity matrix. It follows from equations (7), (8) and (9) and $b_{R,j,t} = 1\,(t > R)$ that

$$\tfrac{1}{\sqrt{T}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} = \tfrac{1}{\sqrt{T}}\sum_{t=1}^{j}\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix}\begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix}.$$

We further approximate $b_{R,t,j}$ and $a_{R,t,j}$ as follows. Let $b_{R,t,j} = 1\,(t/T \ge R/T) \simeq 1\,(s \ge \rho) \equiv \sigma_f(s)$ and

(i) for recursive: given equation (7), we follow West (1996) to show that

$$a_{R,t,j} \simeq \left(\int_R^j \frac{1}{k} dk\right) \cdot 1\left(t \leq R\right) + \left(\int_t^j \frac{1}{k} dk\right) \cdot 1\left(R < t \leq j\right)$$

Consider $r = k/T$. Accordingly,

$$\begin{aligned} a_{R,t,j} &\simeq \left(\int_\rho^\tau \frac{1}{r} dr\right) \cdot 1\left(s \leq \rho\right) + \left(\int_s^\tau \frac{1}{r} dr\right) \cdot 1\left(\rho < s \leq \tau\right) \\ &= \left[\ln\left(\tau\right) - \ln\left(\rho\right)\right] \cdot 1\left(s \leq \rho\right) + \left[\ln\left(\tau\right) - \ln\left(s\right)\right] \cdot 1\left(\rho < s \leq \tau\right) \equiv \sigma_h\left(s, \tau\right); \end{aligned} \tag{10}$$

(ii) for rolling: when $j - R \geq R$, we can re-write equation (8) as

$$\begin{aligned} a_{R,t,j} = {} & \frac{t}{R} \cdot 1\left(\frac{t}{T} \leq \frac{R}{T}\right) + \frac{R}{R} \cdot 1\left(\frac{R}{T} < \frac{t}{T} \leq \frac{j-R}{T}\right) \\ & + \frac{j-t}{R} \cdot 1\left(\frac{j-R}{T} < \frac{t}{T} \leq \frac{j}{T}\right); \end{aligned} \tag{11}$$

thus, when $\tau - \rho \geq \rho$,

$$\sigma_h(s,\tau) = \frac{s}{\rho} \cdot 1\left(s \leq \rho\right) + 1 \cdot 1\left(\rho < s \leq \tau - \rho\right) + \frac{\tau - s}{\rho} \cdot 1\left(\tau - \rho < s \leq \tau\right). \tag{12}$$

A similar argument shows that when $\tau - \rho < \rho$, equation (9) can be approximated as

$$\begin{aligned} \sigma_h(s,\tau) = {} & \frac{s}{\rho} \cdot 1\left(s \leq \tau - \rho\right) + \frac{\tau - \rho}{\rho} \cdot 1\left(\tau - \rho < s \leq \rho\right) \\ & + \frac{\tau - s}{\rho} \cdot 1\left(\rho < s \leq \tau\right). \end{aligned} \tag{13}$$

The following table summarizes the approximations we use for the weights $a_{R,t,j}$, $b_{R,t,j}$:

Approximation for the weights $a_{R,t,j}$, $b_{R,t,j}$

Weights	Estimation Scheme	Approximation
$b_{R,t,j}$	All	$\sigma_f\left(s\right) \equiv 1\left(s \geq \rho\right)$
$a_{R,t,j}$	Recursive	$\sigma_h\left(s,\tau\right) \equiv \left[\ln\left(\tau\right) - \ln\left(\rho\right)\right] \cdot 1\left(s \leq \rho\right)$
		$+\left[\ln\left(\tau\right) - \ln\left(s\right)\right] \cdot 1\left(\rho < s \leq \tau\right)$
	Rolling scheme:	
	(a) $\tau - \rho \geq \rho$	$\sigma_h\left(s,\tau\right) \equiv \frac{s}{\rho} \cdot 1\left(s \leq \rho\right) + 1 \cdot 1\left(\rho < s \leq \tau - \rho\right)$
		$+\frac{\tau - s}{\rho} \cdot 1\left(\tau - \rho < s \leq \tau\right)$
	(b) $\tau - \rho < \rho$	$\sigma_h\left(s,\tau\right) \equiv \frac{s}{\rho} \cdot 1\left(s \leq \tau - \rho\right) + \frac{\tau-\rho}{\rho} \cdot 1\left(\tau - \rho < s \leq \rho\right)$
		$+\frac{\tau - s}{\rho} \cdot 1\left(\rho < s \leq \tau\right).$

Define $\xi_j = \sum_{t=1}^{j} \begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix}$ and the stochastic integral of interest as in Hansen (1992, p. 491):

$$\int_0^\tau \begin{pmatrix} \sigma_f(s) \cdot I_\ell & 0 \\ 0 & \sigma_h(s,\tau) \cdot I_q \end{pmatrix} d\xi_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{j} \begin{pmatrix} b_{R,t,j} \cdot I_\ell & 0 \\ 0 & a_{R,t,j} \cdot I_q \end{pmatrix} \begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix}.$$

The following assumption is based on Hansen (1992) and allows us to derive the asymptotic distribution of our parameter of interest, $\widehat{\theta}_j$.

Assumption 4. For some $p > \beta > 2$, $(f_{t+h}, h_t')'$ *is zero mean, strong mixing with mixing coefficients* α_m of size $-p\beta/(p-\beta)$ and $\sup_{t\geq 1} || (f_{t+h}, h_t')' ||_p = C < \infty$. *In addition,* $\underset{T\to\infty}{lim} T^{-1} E(\xi_T \xi_T') = S \equiv \begin{pmatrix} S_{ff} & S_{fh} \\ S_{fh}' & S_{hh} \end{pmatrix}$ *is an* $(l+q) \times (l+q)$ *positive definite and finite matrix as* $T \to \infty$.

Proposition 1 (Preliminary Asymptotic Result) *Under Assumptions 1-4 and* $T^{-1/2}\xi_T \to \xi$ *in* $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$*:*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{j} \begin{pmatrix} b_{R,t,j} \cdot I_\ell & 0 \\ 0 & a_{R,t,j} \cdot I_q \end{pmatrix} \begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} \Rightarrow \int_0^\tau \Omega(s,\tau)^{1/2} d\xi(s),$$

where $\xi(s) = S^{1/2}\mathcal{B}_{\ell+q}(s)$, $\mathcal{B}_{\ell+q}(s)$ *is an* $(\ell+q)\times 1$ *vector of independent standard Brownian motions,* $\mathcal{D}$ *denotes the space of cadlag functions, "⇒" denotes weak convergence with respect to the Skorohod metric, and* $\Omega(s,\tau)^{1/2} \equiv \begin{pmatrix} \sigma_f(s) \cdot I_\ell & 0 \\ 0 & \sigma_h(s,\tau) \cdot I_q \end{pmatrix}$.

We use the result in Proposition 1 to derive the asymptotic distribution of the parameter estimate, $\widehat{\theta}_j$, in the next Proposition.

Proposition 2 (Asymptotic Distribution of $\widehat{\theta}_j$) *Under Assumptions 1-4 and* $T^{-1/2}\xi_T \to \xi$ *in* $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$*:*

$$m^{1/2}\,\widehat{\theta}_j \Rightarrow \int_0^\tau \widetilde{\omega}(s,\tau)\, d\mathcal{B}_{\ell+q}(s) - \int_0^{\tau-\mu} \widetilde{\omega}(s,\tau-\mu)\, d\mathcal{B}_{\ell+q}(s) = \mathcal{B}_{\widetilde{\omega}}(\tau) - \mathcal{B}_{\widetilde{\omega}}(\tau-\mu) \quad (14)$$

$$\underset{d}{=} \int_0^\tau \omega(s,\tau)\, d\mathcal{B}_{\ell+q}(s) \equiv \mathcal{B}_\omega(\tau) = \mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\, ds\right), \quad (15)$$

where

$$\widetilde{\omega}(s,\tau) = \mu^{-\frac{1}{2}} G^{-1}[I_\ell, FB]\,\Omega(s,\tau)^{1/2} S^{1/2}, \quad (16)$$

$$\omega(s,\tau) = \mu^{-\frac{1}{2}} G^{-1} [I_\ell, FB] \Big\{ \left[\Omega(s,\tau)^{1/2} - \Omega(s,\tau-\mu)^{1/2} \right] \cdot 1(s \leq \tau-\mu) + \Omega(s,\tau)^{1/2} \cdot 1(\tau-\mu < s \leq \tau) \Big\} S^{1/2}, \tag{17}$$

$\mathcal{B}_{\ell+q}(s)$ is an $(\ell+q)\times 1$ vector of independent standard Brownian motions and $\underset{d}{=}$ denotes equality in distribution.

Note that both $\mathcal{B}_\omega(\tau)$ as well as $\mathcal{B}_{\widetilde{\omega}}(\tau)$ are $(\ell\times 1)$ Gaussian processes with time-varying variances. $\mathcal{B}_\omega(\tau)$ is Gaussian with quadratic variation $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$.[9] Similarly, $\mathcal{B}_{\widetilde{\omega}}(\tau) \equiv \mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right)$ is Gaussian with quadratic variation $\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds$. The following Proposition calculates the quadratic variation of $\mathcal{B}_\omega(\tau)$ and $\mathcal{B}_{\widetilde{\omega}}(\tau)$ for the rolling and the recursive estimation schemes.

Proposition 3 (Calculation of $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$)

$$\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds = \mu^{-1} G^{-1} \Big\{ \left(\int_{\tau-\mu}^\tau \sigma_f^2(s)\,ds\right) S_{ff} + \left(\int_{\tau-\mu}^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds\right) (FBS_{fh} + S_{fh}B'F') + \int_0^\tau \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \leq \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \leq s \leq \tau) \right] ds FBS_{hh}B'F' \Big\} G^{-1},$$

where

(i) $\int_{\tau-\mu}^\tau \sigma_f^2(s)\,ds = \mu$ for both rolling and recursive cases;

(ii) recursive: let $\widetilde{\pi} \equiv \mu/(\tau-\mu)$;

$\int_{\tau-\mu}^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \mu\left[1 - \widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right]$ and

$\int_0^\tau \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \leq \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \leq s < \tau) \right] ds = 2\mu\left[1 - \widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right]$;

(iii) rolling: let $\pi^\dagger \equiv \frac{\mu}{\rho}$;

(a) if $\mu \geq \rho$, then

$\int_{\tau-\mu}^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \mu\left(1 - \frac{1}{2\pi^\dagger}\right)$ and

$\int_0^\tau \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \leq \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \leq s < \tau) \right] ds = \mu\left(1 - \frac{1}{3\pi^\dagger}\right)$;

(b) if $\mu < \rho$, then

$\int_{\tau-\mu}^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \frac{1}{2}\mu\pi^\dagger$ and

$\int_0^\tau \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s \leq \tau-\mu) + \sigma_h^2(s,\tau) \cdot 1(\tau-\mu \leq s < \tau) \right] ds = \mu\pi^\dagger\left(1 - \frac{1}{3}\pi^\dagger\right)$.

[9] We eliminated the vector dimension in the notation for $\mathcal{B}_\omega(\cdot)$, $\mathcal{B}_{\widetilde{\omega}}(\cdot)$ as they are always dimension $(\ell\times 1)$.

Proposition 4 (Calculation of $\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds$)

$$\int_0^\tau \tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)' ds = \mu^{-1}G^{-1}\left\{\left(\left(\int_0^\tau \sigma_f^2(s)\,ds\right)S_{ff} + \left(\int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds\right)\times \right.\right.$$
$$\left. \times\left(FBS_{fh} + S_{fh}B'F'\right) + \left(\int_0^\tau \sigma_h^2(s,\tau)\,ds\right)FBS_{hh}B'F'\right\}G^{-1},$$

where

(i) $\int_0^\tau \sigma_f^2(s)\,ds = (\tau-\rho)$ *for both rolling and recursive cases;*

(ii) recursive:

$\int_{\tau-\mu}^\tau \sigma_h(s,\tau)\,\sigma_f(s)\,ds = (\tau-\rho)\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)$ *and*

$\int_0^\tau \sigma_h^2(s,\tau)\,ds = 2(\tau-\rho)\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right);$

(iii) rolling:

(a) if $\tau-\rho\geq\rho$*, then*

$\int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \left(\tau-\frac{3}{2}\rho\right)$ *and* $\int_0^\tau \sigma_h^2(s)\,ds = \left(\tau-\frac{4}{3}\rho\right)$*;*

(b) if $\tau-\rho<\rho$*, then*

$\int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds = \frac{1}{2\rho}(\tau-\rho)^2$ *and* $\int_0^\tau \sigma_h^2(s)\,ds = \frac{1}{3\rho^2}(\tau-\rho)^2(4\rho-\tau)$.

The next Proposition discusses the asymptotic distribution of the $\mathcal{W}_{j,m}$ test statistic presented in equation (4).

Theorem 5 (Main Proposition) *Under Assumption 1-4,*

$$\mathcal{W}_{j,m} = m\widehat{\theta}_j'\,V_{\theta,\tau}^{-1}\widehat{\theta}_j$$
$$\Rightarrow \left[\mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)' ds\right)\right]' V_{\theta,\tau}^{-1}\left[\mathcal{B}_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)' ds\right)\right], \quad (18)$$

where

$$V_{\theta,\tau} = Avar\left(m^{1/2}\widehat{\theta}_j\right) \quad (19)$$
$$= G^{-1}[I_\ell,\ FB]\,Avar\left(\frac{1}{\sqrt{m}}\sum_{t=j-m+1}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right)[I_\ell,\ FB]\,G^{-1}$$
$$= \int_0^\tau \omega(s,\tau)\,\omega(s,\tau)' ds, \quad (20)$$

$j=[\tau T]$, $m=[\mu T]$ *and* $\mathcal{B}_\ell(\cdot)$ *is a standard* ℓ*-dimensional Brownian motion. Let* θ_j *be the true parameter value. We reject the null hypothesis:*

$$H_0:\theta_j=\theta_0\ ,\ \theta_0=0 \text{ for all } j=R+m,...,T \quad (21)$$

if $max_{j\in\{R+m,...,T\}}\ \mathcal{W}_{j,m} > \kappa_{\alpha,\ell}$*, where* $\kappa_{\alpha,\ell}$ *are the critical values at the* $100\alpha\%$ *significance level that can be simulated for given values of* μ*,* ℓ*,* G*,* F*,* B *and* S*.*

$V_{\theta,\tau}$ can be estimated using Proposition 3 and replacing the population values of $S_{ff,}$ S_{fh}, S_{hh} with a consistent estimate. For example, one could use Newey and West's (1987) covariance estimator of long run variance of $\left\{(f_{t+h}, h_t')'\right\}_{t=R}^{T}$.

The asymptotic distribution of the test statistic $W_{j,m}$ is non-standard and depends on nuisance parameters. We obtain its critical values, $\kappa_{\alpha,\ell}$, via Monte Carlo simulations by the following steps:

1. Simulate $B_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right)$ by using the approximation $\sqrt{T}\sum_{t=1}^{j}\widetilde{\omega}\left(\frac{t}{T},\frac{j}{T}\right)\vartheta_\ell$, where ϑ_ℓ is an $(\ell\times 1)$ vector of independent standard Normal random variables;

2. Simulate $B_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right)$ similarly by $\sqrt{T}\sum_{t=1}^{j-m}\widetilde{\omega}\left(\frac{t}{T},\frac{j-m}{T}\right)\vartheta_\ell$;[10]

3. Then, we obtain

$$B_\ell\left(\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds\right) = B_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right) - B_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\,\widetilde{\omega}(s,\tau)'\,ds\right);$$

4. Finally, conditional on the estimated value of $V_{\theta,\tau}$ obtained by equation (19) and Proposition 4, we simulate the $W_{j,m}$ test statistic by using equation (18);

5. The critical values at significance level α can be obtained by the $(1-\alpha)\,th$ quantile of the simulated distribution of $W_{j,m}$.

   Note that when S is full rank, $S^{1/2}$ can be calculated as the Cholesky factor of S; when S is rank deficient, one can use a singular value decomposition in Rao (Section 8a.4) to approximate $\tilde{\omega}(t/T, j/T)$.

The case of a linear regression model in equation (3) is the same as that considered in West and McCracken (1998). Note the similarity between the results in Proposition (3) and West and McCracken's (1998) variance. The latter define the variance in exactly the same way, except that in their case $m = P$, $\pi = lim_{T\to\infty}(P/R)$ and $\tau = 1$. Consequently, by letting $\mu = \pi(1+\pi)^{-1}$, $\rho = (1+\pi)^{-1}$ and $(1-\rho) = \pi(1+\pi)^{-1}$ in Propositions 3 and 4 we recover their results:[11]

[10] It is important that the random variable ϑ_ℓ used to simulate the two Brownian motions is the same.

[11] Recall that West and McCracken's (1998) test statistic is obtained by rescaling by $P^{1/2}$ rather than $T^{1/2}$ as shown in equation (3).

(i) recursive scheme:

$$V_{\theta,P} = G^{-1}\left\{S_{ff} + \left(1-\pi^{-1}\ln(1+\pi)\right)\left[FBS'_{fh} + B'F'S_{fh}\right] + 2\left(1-\pi^{-1}\ln((1+\pi))\right)FBS_{hh}B'F'\right\}G^{-1};$$

(ii) rolling scheme:

$$V_{\theta,P} = \begin{cases} G^{-1}\left\{S_{ff} + \left(1-\frac{1}{2\pi}\right)\left[FBS'_{fh} + B'F'S_{fh}\right] - \left(1-\frac{1}{3\pi}\right)BFS_{hh}F'B'\right\}G^{-1}; & \pi \geq 1 \\ G^{-1}\left\{S_{ff} + \frac{\pi}{2}\left[FBS'_{fh} + B'F'S_{fh}\right] + \left(\pi - \frac{\pi^2}{3}\right)BFS_{hh}F'B'\right\}G^{-1}; & \pi < 1. \end{cases}$$

The difference between West and McCracken (1998) and our approach is that we aim at testing forecast optimality at each point in the out-of-sample period, based on rolling window averages, while they focus on optimality on average over the whole out-of-sample portion of the data. In the case of West and McCracken (1998), the tests take into account parameter estimation error by simple adjustment factors in the variance, which result in tests with asymptotic distributions that are nuisance parameter free. In our case, instead, we need to adjust the asymptotic distribution to take into account the parameter estimation error, which induces a time-varying variance; under very general conditions, this implies that the critical values depend on the data generating process and need to be simulated specifically for the individual application at hand.

5 Forecast Unbiasedness and Efficiency Tests, and Survey Forecasts

The general results presented in the previous section simplify considerably in two cases important for practitioners. A first important special case is when parameter estimation error is irrelevant ($F = 0$). This may be often of interest in practice when the model that generated the forecasts is not available and, thus, the correction for parameter estimation error is not applicable. Relevant examples are survey forecasts and judgemental forecasts produced by central banks; for instance, the Greenbook forecasts by the Federal Reserve Board or private sector forecasts such as those produced by the Blue Chip Economic Indicators.[12]

[12] In addition, if a researcher were to consider a null hypothesis on the forecast errors evaluated at the estimated models' parameter, the asymptotic distribution of the test statistic would similarly be nuisance parameter free. For a discussion and implementation, see Rossi (2012, 2013).

Proposition 6 (Special Case I: Irrelevant Parameter Estimation Error) *Under the condition $F = 0$, parameter estimation error becomes irrelevant and $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$ becomes $G^{-1}S_{ff}G^{-1}$ and $\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds = \frac{(\tau-\rho)}{\mu}G^{-1}S_{ff}G^{-1}$ for all estimation schemes.*

A second special case involves testing for forecast unbiasedness and efficiency using t-tests under general conditions, as well as several other tests under more specific assumptions. As discussed in West and McCracken (1998) Corollary 5, in such cases a special condition holds, which considerably simplifies the asymptotic distributions of our test statistic. As in West and McCracken (1998), the results in the special case below also hold for encompassing and serial correlation tests when the errors are conditionally homoskedastic.

Proposition 7 (Special Case II: Forecast Unbiasedness and Efficiency Tests) *Under the condition:*

$$S_{ff} = -\frac{1}{2}(FBS_{hf} + S_{fh}B'F') = FBS_{hh}B'F', \tag{22}$$

$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds$ in Proposition 3 becomes:
(i) recursive case: $\frac{\tau-\rho}{\mu}G^{-1}S_{ff}G^{-1}$;
(ii) rolling case:
(a) $\frac{1}{\mu}\frac{2\rho}{3}G^{-1}S_{ff}G^{-1}$, if $\tau-\rho \geq \rho$; and
(b) $\frac{(\tau-\rho)}{\mu}\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}$, if $\tau-\rho < \rho$.
Furthermore, $\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$ in Proposition 3 becomes $\lambda G^{-1}S_{ff}G^{-1}$, where:
(i') recursive case: $\lambda = 1$;
(ii') rolling case: let $\pi^\dagger \equiv \frac{\mu}{\rho}$; then,
(a) $\lambda = \frac{2}{3\pi^\dagger}$, if $\mu \geq \rho$; and
(b) $\lambda = \left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)$, if $\mu < \rho$.

Note that, when either Proposition 6 or 7 holds, $V_{\theta,\tau} = \int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$ does not depend on τ; thus, the variance is not time-varying. The next proposition shows that, in these special cases, the distribution of the test statistic simplifies and its critical values can be tabulated.

Theorem 8 (Main Proposition in Special Cases) *(a) Under Assumption 1-4 and Condition (22), we have:*

$$\mathcal{W}_{j,m} \Rightarrow \mathcal{W}_{\tau,\mu}, \tag{23}$$

where $\mathcal{W}_{\tau,\mu}$ *is: (i) Recursive estimation:*

$$\mathcal{W}_{\tau,\mu} = \left[\mathcal{B}_\ell\left(\tau-\rho\right) - \mathcal{B}_\ell\left(\tau-\rho-\mu\right)\right]' \left[\mathcal{B}_\ell\left(\tau-\rho\right) - \mathcal{B}_\ell\left(\tau-\rho-\mu\right)\right], \tag{24}$$

(ii) Rolling estimation:

$$\begin{aligned}\mathcal{W}_{j,m} \Rightarrow\ & 1\left(\mu+\rho \leq \tau < 2\rho\right)\left\{\mathcal{B}_\ell\left(\frac{\tau-\rho}{\mu}\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right) - \mathcal{B}_\ell\left(\frac{\tau-\mu-\rho}{\mu}\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\} \\ & + 1\left(2\rho < \tau \leq 2\rho+\mu\right)\left\{\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right) - \mathcal{B}_\ell\left(\frac{\tau-\mu-\rho}{\mu}\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\},\end{aligned} \tag{25}$$

where $V_\theta = \left(\frac{2}{3\pi^\dagger}\right)\cdot 1\left(\mu \geq \rho\right) + \left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)\cdot 1\left(\mu < \rho\right) = \lambda$, λ *defined in Proposition 7.*
(b) Furthermore, under Assumptions 1-4 and condition $F = 0$*, eq. (23) holds with*

$$\mathcal{W}_{\tau,\mu} = \left[\mathcal{B}_\ell\left(\tau-\rho\right) - \mathcal{B}_\ell\left(\tau-\rho-\mu\right)\right]' \left[\mathcal{B}_\ell\left(\tau-\rho\right) - \mathcal{B}_\ell\left(\tau-\rho-\mu\right)\right]. \tag{26}$$

We reject the null hypothesis:

$$H_0 : \theta_j = \theta_0,\ \theta_0 = 0 \text{ for all } j = R+m, ..., T \tag{27}$$

if $\max_{j\in\{R+m,\ldots,T\}} W_{j,m} > \kappa_{\alpha,\ell}$*, where* $\kappa_{\alpha,\ell}$ *are the critical values at the* $100\alpha\%$ *significance level and are reported in Table 1a for eq. (24) and (26) for various values of* $\mu = [m/T]$ *and number of restrictions,* ℓ*; and in Table 1b for eq. (25) for various combinations of* μ, ρ, ℓ*.*[13]

INSERT TABLES 1a, 1b AND 1c HERE

Under the special cases considered in Propositions 6 or 7, which are the ones more commonly used in the literature, the critical values do not depend on the data generating process and can be tabulated. More specifically, Theorem 8 shows that this is the case when either: (i) $F = 0$; or (ii) when testing forecast unbiasedness and rationality via t-tests (that is, when concerned about mean prediction errors and efficiency); or (iii) when testing encompassing and serial correlation with conditionally homoskedastic errors. In these cases, our method results in an adjustment procedure similar to that of West and McCracken (1998), where the test statistics could be calculated similarly, by substituting P in their

[13] The critical values can be obtained by Monte Carlo simulation. The critical values at significance level $100\alpha\%$ are such that $\Pr\left\{\sup_{\tau\in\{\rho+\mu,\ldots,1\}} \mathcal{W}_{\tau,\mu} > \kappa_{\alpha,\ell}\right\} = \alpha$.

notation with m of our notation, provided inference is conducted using the critical values provided in this paper.

Note that for $\widetilde{j} \equiv j - R$, $\widetilde{\tau} \equiv \tau - \rho$, $sup_{\tau} \left[\mathcal{B}_{\ell}(\tau - \rho) - \mathcal{B}_{\ell}(\tau - \rho - \mu)\right]' \left[\mathcal{B}_{\ell}(\tau - \rho) - \mathcal{B}_{\ell}(\tau - \rho - \mu)\right]$ $= sup_{\widetilde{\tau}} \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \mu)\right]' \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \mu)\right]$. Thus, the critical values that we provide in Table 1a do not depend on ρ. Note also that in the case of model-free forecasts, the only sample available to researchers is P: they do not have an available R; therefore we define $\widetilde{\mu}$ such that $m = [\widetilde{\mu} P]$ (that is, we define m as a fraction of the number of observations P, as opposed to being a fraction of the total sample size T) and provide critical values for the test statistic $sup_{\widetilde{\tau}} \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \widetilde{\mu})\right]' \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \widetilde{\mu})\right]$, for $\widetilde{\tau} = 1, ..., P$. Clearly there is a relationship between μ and $\widetilde{\mu}$, and we could have provided only one table of critical values for μ. However, that would not easily map into critical values for $\widetilde{\mu}$ that would be commonly used in empirical applications. Thus, we provide a separate table of critical values (Table 1c) for $sup_{\widetilde{\tau}} \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \widetilde{\mu})\right]' \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \widetilde{\mu})\right]$, as discussed in the following Corollary.

Corollary 9 (Main Proposition for Survey and Model-Free Forecasts) *Under Assumptions 1-4 and condition $F = 0$, the alternative test statistic*

$$\mathcal{W}_{\widetilde{j},m} = m \widehat{\theta}'_{\widetilde{j}} \, \widehat{V}_{\theta}^{-1} \widehat{\theta}_{\widetilde{j}}, \; for \; \widetilde{j} = m, ..., P,$$

implemented over the sequence of P forecasts is such that:

$$\sup_{\widetilde{j} \in \{m,...,P\}} \mathcal{W}_{\widetilde{j},m} \Longrightarrow \sup_{\widetilde{\tau} \in \{\widetilde{\mu},...,1\}} \mathcal{W}_{\widetilde{\tau},\widetilde{\mu}}, \tag{28}$$

where $\widetilde{\tau} \equiv \tau - \rho$, $\tilde{\mu} \equiv [m/P]$ and

$$\mathcal{W}_{\widetilde{\tau},\widetilde{\mu}} = \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \tilde{\mu})\right]' \left[\mathcal{B}_{\ell}(\widetilde{\tau}) - \mathcal{B}_{\ell}(\widetilde{\tau} - \tilde{\mu})\right]. \tag{29}$$

We reject the null hypothesis:

$$H_0 : \theta_j = \theta_0, \; \theta_0 = 0 \; for \; all \; \widetilde{j} = m, ..., P \tag{30}$$

if $max_{\widetilde{j} \in \{m,...,P\}} \mathcal{W}_{\widetilde{j},m} > \kappa_{\alpha,\ell}$, where $\kappa_{\alpha,\ell}$ are the critical values at the $100\alpha\%$ significance level and are reported in Table 1c for various values of $\tilde{\mu} = [m/P]$.

6 Monte Carlo Analysis

We study the small sample performance of the methods that we propose in a series of Monte Carlo experiments inspired by West and McCracken (1998). Let the Data Generating Process (DGP) be: $y_t = \gamma y_{t-1} + \varepsilon_t$, where $\gamma = 0.5$, $\varepsilon_t \sim iid(0,1)$, and y_0 is drawn from its unconditional distribution, a normal with zero mean and variance $(1-\gamma^2)^{-1}$. In each sample, $t = 1, ..., T+1$, we split the data into $T+1 = P+R$, and we utilize either a rolling or a recursive scheme to generate P one-step ahead out-of-sample forecasts, $y^f_{R+1}, ..., y^f_{T+1}$. The forecasting model for the recursive estimation scheme implies $E_t y^f_{t+1} = \gamma y_t$, while for the rolling estimation scheme it implies $E_t y^f_{t+1} = \gamma_0 + \gamma_1 y_t$. The forecast errors are denoted by $\varepsilon^f_{t+1} \equiv y_{t+1} - y^f_{t+1}$, for $t = R, ..., T$.[14]

First, we consider the size properties of our test and compare it to the "Traditional tests" typically used in the literature. Consider the following regression model:

$$E_t \varepsilon^f_{t+1} = \theta_0 + \theta_1 Z_t. \tag{31}$$

The "Traditional tests" include testing:

(i) Zero mean prediction error (or forecast unbiasedness). We test whether the mean of the sequence of forecast errors is zero. The test is implemented by a two-sided t-test for $\theta_0 = 0$ in regression (31), where there are no regressors other than the constant. Let $\widehat{\theta}_0 = P^{-1} \sum_{t=R}^{T} \varepsilon^f_{t+1}$ and $\widehat{\sigma}^2_{\theta_0} = P^{-1} \sum_{t=R}^{T} \left(\varepsilon^f_{t+1}\right)^2$. We consider a t-test with West and McCracken's (1998) variance correction: $t^c_\alpha = P^{1/2} \widehat{\theta}_0 \widehat{\sigma}^{-1} \lambda^{-1/2}_{WM}$, where $\lambda_{WM} = 1$ for the recursive scheme and, for the rolling scheme, $\lambda_{WM} = 1 - \pi^2/3$ when $\pi \leq 1$ and $\lambda_{WM} = 2/(3\pi)$ when $\pi > 1$.

(ii) Forecast efficiency. The test is implemented by a two-sided t-test for $\theta_1 = 0$ in the regression (31), where $Z_t = y^f_{t+1}$, and a constant is included in the regression. Let $\widehat{\theta}_1$ be the OLS estimate of the slope coefficient in the regression (31), and $\widehat{\sigma}^2_{\theta_1}$ be its estimated standard error. We consider a t-test that utilizes West and McCracken's (1998) correction: $t^c_{\theta_1} = \widehat{\beta} \widehat{\sigma}^{-1}_{\theta_1} \lambda^{-1/2}_{WM}$, for the same values of λ_{WM} as in (i).

In addition, we consider our proposed "Fluctuation Rationality" test, equation (4), implemented in rolling regressions over the out-of-sample period with a rolling window size $m = 50$.

[14] The advantage of using the same DGP as West and McCracken (1998) is that we can directly compare our results to theirs. In addition, in order for Condition (22) to hold, one would need to include a constant in the estimation equation. For a reference, see West and McCracken (1998), proof of Theorem 7.1.

The estimate of the asymptotic variance is obtained by using a simple homoskedastic covariance estimate of S, as in West and McCracken (1998). The number of Monte Carlo replications is 1,000.

Table 2 reports results for the recursive and the rolling estimation schemes, respectively. Panel A reports results for testing forecast unbiasedness and panel B for forecast efficiency. The tables shows that the empirical rejection frequencies of our proposed tests (reported in the column labeled "Fluctuation Test") as well as those of the traditional tests (reported in the column labeled "Traditional Test") are close to the nominal value except in very small sample sizes. The size distortions in small samples are mild for the recursive scheme for both tests and for the rolling scheme for the mean prediction error test, and a little bit larger (10%) for the rolling case in the efficiency test. In general, for the small samples, i.e. $R <= 100$, the recursive estimation scheme results in a better sized tests than the rolling estimation scheme.

INSERT TABLE 2 HERE

In order to evaluate the power of our test in the presence of time variation, we consider experiments based on three DGPs. All DGPs are based on the model: $y_t = \gamma y_{t-1} + \varepsilon_t + b_t$, where $\gamma = 0.5, \varepsilon_t \sim iid(0,1), y_0$ is drawn from the unconditional distribution of y_t. In DGP A (labeled as "A. Non-stationary"), $b_t = b \cdot 1(1 < t \leq 345) - b \cdot 1(345 < t < T)$. In DGP B (labeled as "B. Non-stationary"), $b_t = 0 \cdot b \cdot 1(1 < t \leq 345) + 2 \cdot b \cdot 1(345 < t < T)$. DGP C (labeled as "C. Stationary") considers $b_t = b$, for all t. Furthermore, $b = \{0, 0.1, ..., 1\}$ for the power exercises in the case of mean prediction error and $b = \{0, 0.5, ..., 5\}$ for the power exercises in the case of efficiency. Predictions are based on an AR(1) model. DGP A is used to assess the power of the"Fluctuation Rationality" test. DGP B is a non-stationary DGP where traditional tests can also have some power. DGP C is a stationary model that we use to study the power loss from using our test that is robust to the presence of instabilities. The power loss occurs since our test uses fewer observations than the traditional tests, i.e. $m < P$. In all cases parameters are estimated with a recursive scheme, and $T = 400$, $R = 300$ and $m = 50$.

The power comparisons are reported in Table 3. The table shows that, in DGP A, the traditional tests do not have power to reject the null hypothesis, and in fact their rejection frequencies approach zero under the alternative hypothesis, whereas our proposed tests do have substantial power (see Panel A). In DGP B, the traditional test has some power, although our test has a higher power to detect lack of rationality (see Panel B). Finally, Panel C illustrates the loss of power in our test relative to the traditional test when there are no instabilities in the data. Clearly, there is a trade-off between the proposed tests and the traditional ones: if one is certain that the forecast environment is stable, the traditional tests would have more power to detect lack of rationality in small samples; however, when the forecast environment is unstable, the traditional tests may have no power at all, even asymptotically, in certain situations, while our proposed test would have power.

INSERT TABLE 3 HERE

7 Are the Federal Reserve and Private Sector's Forecasts Rational?

The quality of private sector's forecasts relative to the internal forecasts of the Federal Reserve has been frequently considered in the literature. As anticipated in Section 2, in important contribution, Romer and Romer (2000) showed that the Federal Reserve has more information relative to the private sector when forecasting inflation. Hence, it would be optimal for a third party with access to both forecasts to put all the weight on the forecasts provided by the Federal Reserve and zero weight on the ones provided by the commercial forecasters.

We revisit the existing empirical evidence from two points of view. First, we consider the rationality of private sector's as well as the Federal Reserve's Greenbook forecasts, as in Romer and Romer (2000), Faust and Wright (2010), Patton and Timmermann (2011) and Croushore (2012), among others. These papers have found that forecast rationality tests for the various, competing inflation forecasts are sensitive to the sub-sample period used for forecast evaluation. The novelty of our approach is to study whether forecast rationality holds by using our "Fluctuation Rationality" test robust to instabilities. One of the advantages of our approach is that it does not require researchers to know or impose a sub-sample date a-priori. Second, we evaluate whether Romer and Romer's (2000) finding that Federal Reserve forecasts are superior to private sectors' forecasts continues to hold when we allow for instabilities.

We consider the Federal Reserve's inflation forecasts provided in the Greenbook and compare them with two commercial forecasts: the Blue Chip Economic Indicators (BCEI) and the Survey of Professional Forecasters (SPF). In what follows, we describe the data from each of the sources.

Greenbook forecasts are made by the staff of the Federal Reserve Board of Governors prior to each Federal Open Market Committee (FOMC) meeting. The Greenbook provides quarterly forecasts (from contemporaneous up to nine quarters) for a variety of economic indicators and for several forecast horizons under a maintained assumption about monetary policy; the forecast horizons can vary depending on when the forecasts were made. We consider only forecasts up to five quarters to ensure a sample large enough for inference. We focus on inflation forecasts provided by the Greenbook, which are measured by (annualized) quarter-over-quarter GNP deflator growth rates from 1965 to 1991 and by (annualized) quarter-over-quarter GDP deflator growth rates afterwards. Greenbook forecasts are available only with a five-year lag. Thus, our current sample includes data up to 2005:IV. The data are provided by the Federal Reserve Bank of Philadelphia, which matches the timing of the Greenbook forecasts with that of the SPF. The database includes forecasts from four of the annual FOMC meetings whose the date is closest to the middle of the quarter.[15] In order to make the two data sets comparable, we omit the first 3 years of observations and start the series in 1968:IV.

The Survey of Professional Forecasters (SPF) provides forecasts for inflation as well as a variety of economic fundamentals at the quarterly frequency. These include nowcasts (forecasts of the current quarter) as well as forecasts up to four-quarter-ahead. We use the forecasts of (annualized) quarterly GNP/GDP deflator growth rates whose timing is consistent with that of the Greenbook forecasts. The survey is conducted roughly at the end of every second month in the quarter, and it includes 34 professionals' forecasts. The series start in 1968:IV. We use the median forecast and terminate our series at 2005:IV to obtain a data set spanning the same period of time as that of the Greenbook.

The Blue Chip Economic Indicators (BCEI) provides monthly forecasts of quarterly economic series starting from 1980. It is a survey-based forecast database where about 50 U.S. business economists participate each month. Though this is a monthly series, in order to match the Greenbook and SPF forecasts we take only four forecasts per year corresponding to the mid-quarter, i.e. February, May, August, and November, from 1980 to 2005.[16]

To evaluate the Greenbook, SPF and Blue Chip forecasts, we use realized values of

[15] Greenbook forecasts can be obtained from the Philadelphia Fed web-site at http://www.philadelphiafed.org/research-and-data/real-time-center/greenbook-data/, while the SPF forecasts are provided by the same source at http://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/.

[16] Although the BCEI forecasts are available from August 1976, the forecasts for the initial four years are for annual changes in key economic variables as opposed to quarterly, thus we omit the earlier period.

(annualized) quarter-over-quarter growth rates of the GNP/GDP deflator constructed from the quarterly vintages in the real-time data set discussed by Croushore and Stark (2001). Our forecast evaluation approach is consistent with that in Romer and Romer (2000), who use the second revision as the benchmark for forecast evaluation, i.e. the specific quarter data available at the last month of the consecutive quarter. Given the real-time nature of the data set, the way we construct the (annualized) quarter-over-quarter GNP/GDP deflator based inflation rate is as follows. For example, the deflator for 1968:IV uses the 1969:I vintage and applies the following transformation: $400ln(PGDP68 : IV/PGDP68 : III)$. We do so for all the vintages up to 2007:IV, then take the diagonal elements of the resulting matrix. This way we obtain a real-time, annualized measure of the quarter-over-quarter inflation rate of the previous quarter, which we use to evaluate the nowcast or the corresponding h-quarter-ahead forecast over time.[17]

Figure 2 compares the Greenbook forecasts with those of the SPF and BCEI. Each panel in Figure 2 corresponds to a forecast horizon, where the horizon h ranges from 0 to 5. $h = 0$ corresponds to the nowcast of inflation. The figure also plots the realized values for inflation at each horizons (reported by the solid line, labeled "actual"). In the figure, not all forecasts have the same starting point. In addition, there are several missing values at several horizons across the different sources, and even for the same source depending on when the forecast has been made. However, overall, the forecasts appear to be correlated with each other: the correlation coefficients between the Greenbook and the private sector's forecasts range from 0.94 to 0.96 across various horizons. Table 4 reports the mean squared forecast errors (MSFE) for the forecast plotted in Figure 2. It appears that the SPF forecasts are inferior to those of the Greenbook at all horizons whereas the BCEI forecasts appear to be superior. However, as we show further, there is substantial evidence of instabilities, and the difference is most likely associated by the different sample period that BCEI covers.

INSERT FIGURE 2 AND TABLE 4 HERE

To evaluate the forecast performance, we consider the following regression:

$$\pi_{t+h} = \alpha + \delta \hat{\pi}_{t+h,t} + \epsilon_{t+h}, \tag{32}$$

where π_{t+h} is the realized inflation rate, $\hat{\pi}_{t+h,t} = E_t \pi_{t+h}$ is the inflation expectation for $t + h$ based on the information available at the time t, h is the forecast horizon and ϵ_{t+h} is a

[17] In the real-time data set provided by the Philadelphia Fed, the observation for 1995:IV is missing in the vintage of 1996:I. We use the value available in the vintage of 1996:II as a substitute value.

forecast error. We consider $h = 0, 1, ..., 6$ for the Greenbook forecast, $h = 0, 1, ..., 5$ for the BCEI, and $h = 0, 1, ..., 4$ for the SPF. For the Greenbook, the choice of h is constrained by the need to have a sample size large enough for inference. The choice of h for the BCEI and SPF is dictated by data availability. In order to test forecast optimality in the framework discussed in Section 3, where the test involves zero restrictions on the parameters, we rewrite equation (32) as follows:

$$\pi_{t+h} - \hat{\pi}_{t+h,t} = \alpha + \beta\hat{\pi}_{t+h,t} + \epsilon_{t+h}, \tag{33}$$

where $\beta = \delta - 1$. Table 5 presents results for both traditional forecast rationality tests as well as the "Fluctuation Rationality" test that we propose. The former relies on the maintained assumption that the parameters of the regression are time invariant and it is implemented using a simple Wald-type test in equation (33), where the parameters are estimated by OLS; we use a HAC variance estimate (Newey and West, 1987) with a bandwidth equal to $P^{1/4}$. Our proposed test instead assesses whether the parameters equal the values implied by optimal forecasts at any given point in time, and it is robust to instabilities. The test is implemented as in eq. (4), where $\widehat{\theta}_j$ are the OLS estimates of α and β from equation (33) in rolling regressions with a window size $m = 60$.

The column labeled "Fluctuation" in Table 5 reports the test statistic $\max_{\widetilde{j}\in\{m,...,P\}} W_{\widetilde{j},m}$ in Proposition 9 and the column labeled "Traditional" reports the test statistic W_P in eq. (3); both are reported for several horizons h, listed in the first column. Asterisks denote significance at the 5% significance level. Table 5 suggests that traditional forecast rationality tests fail to reject the null hypothesis of forecast rationality at the 5% significance level for the Greenbook and SPF forecasts, whereas they reject forecast rationality for the BCEI forecasts. However, as we show later, this difference in the results highly depends on the evaluation period, as the sample for the BCEI forecasts starts much later. It is in fact during a period of time when the Greenbook and SPF forecasts fail the rationality test as well. In contrast, the Fluctuation Rationality test rejects the null hypothesis of rationality for all forecasts.

INSERT TABLE 5 HERE

Figures 3-5 plots $\mathcal{W}_{\widetilde{j},m}$ together with the critical values for the $\max_{\widetilde{j}\in\{m,...,P\}} \mathcal{W}_{\widetilde{j},m}$ test statistic at the 5% significance level. The timing on the horizontal axis provides useful information about the timing of the forecast rationality breakdown. Figure 3 focuses on the Greenbook's forecasts. The figure shows three substantial breakdowns: the first two are associated with the beginning and the end of 1990s. It appears that the forecasts deteriorate over the 1990s and rationality tends to recover by the 2000s. However, for almost all forecast horizons with the exception of five quarters ahead, forecast rationality breaks down again in 2005. Overall, it appears that the empirical evidence in favor of forecast rationality supported by the traditional forecast rationality tests, reported in Table 6, is driven mainly by the good performance of the Greenbook forecasts at the beginning of our sample.

INSERT FIGURES 3, 4 AND 5 HERE

Figure 4 plots the Fluctuation Rationality test for the BCEI forecasts and Figure 5 reports the same test for the SPF forecasts. Figure 5 suggests that the empirical evidence on forecast rationality for SPF forecasts is qualitatively similar to that of the Greenbook. However, the recovery of forecast rationality during the first half of 2000s is less pronounced for SPF than for the Greenbook. By comparing Figure 5 with Figure 4, we note that they behave similarly in the overlapping part of the evaluation period. This suggests that the traditional forecast rationality test results for the BCEI reported in Table 6 are different from the other forecasts solely due to the different sample period. The BCEI forecasts are overall qualitatively similar to the SPF forecasts, with a notable exception: the non-existence of the breakdown of forecast rationality in the BCEI forecasts in 2005. In general, the empirical evidence in Figures 3-5 does not support forecast rationality for any of the forecasts at any horizons.

Our second objective is to assess whether the Federal Reserve has an information advantage over private sector's forecasts. To do so, we consider the following regression:

$$\pi_{t+h} - \hat{\pi}^i_{t+h,t} = \delta + \beta_g \hat{\pi}^G_{t+h,t} + \beta_i \hat{\pi}^i_{t+h,t} + \nu_{t+h}, \tag{34}$$

where $\hat{\pi}^g_{t+h,t}$ is the Greenbook forecast and $\hat{\pi}^i_{t+h,t}$, $i = SPF, BCEI$ denote the SPF and BCEI forecasts, respectively. The Federal Reserve forecasts are useful beyond that of the private sector in predicting inflation if and only if $\beta_g \neq 0$. We test this hypothesis both with the traditional tests as well as with our robust Fluctuation-type test. The latter test is implemented as in eq. (4), where $\widehat{\theta}_j$ are the OLS estimates of δ and β_g from equation (34) in rolling regressions with a window size $m = 60$.

The results are reported in Table 6. The table reports the traditional test statistics (column labeled "Traditional") and the Fluctuation-type test statistic (column labeled "Fluctuation"); asterisks denote significance at the 5% level. According to the table, both the traditional tests and the Fluctuation-type test suggest statistically significant evidence that the Federal Reserve has additional information relative to the private sector's forecasts. Figure 6 sheds additional light on this conclusion. The figure plots the Fluctuation-type test statistics over time and shows that the information advantage of the Federal Reserve has deteriorated after 2003. In fact, the rejections of the hypothesis of no information advantage of the Federal Reserve based on the Fluctuation test appear mostly at the beginning of the sample. The result holds also for both commercial forecasts, that is the BCEI and the SPF.

INSERT TABLE 6 AND FIGURE 6 HERE

Figure 7 plots the coefficients on Federal Reserve's Greenbook forecasts, β_g in equation (34), estimated in the rolling regressions. The figure suggests that the coefficient averages around unity. However, the coefficient seems to have been decreasing over time over all horizons. For example, the bottom two panels depict the explanatory power of the Greenbook forecasts over that of the SPF's forecasts, and show that it clearly decreased over time. The picture also shows a mild revival of the information advantage around 1995-2001. The top two panels in the figure depict the explanatory power of the Greenbook over that of the BCEI forecasts; they reinforce the evidence in favor of the presence of additional explanatory power of the Greenbook forecasts around 1995, which starts diminishing around 2001.

INSERT FIGURE 7 HERE

8 Conclusion

This paper proposes new forecast rationality tests that can be used in unstable environments. The tests we propose can be applied to test forecast unbiasedness, efficiency, encompassing, serial uncorrelation and, in general, regression-based tests of forecasting ability. Our test statistics have non-standard limiting distributions and depend on nuisance parameters; in special cases that are very relevant in practice, the critical values can be tabulated, thus making the test easily implementable. Our paper also analyzes the size properties of the test that we propose in small samples, as well as the power of our tests relative to traditional tests in the presence of instabilities. We show that traditional tests may fail to reject forecast optimality in the presence of instabilities whereas our test performs well in that regard.

The methods we propose are robust to data mining because the critical values are based on the supremum of the statistics across all samples. The test is not robust to data mining due to the choice of the out-of-sample window size; to resolve the latter issue, the reader is referred to Inoue and Rossi (2012) and references therein. Furthermore, we should note that our test is designed to signal lack of rationality in sub-samples of the data: it might still be that our test signals lack of optimality in sub-samples but forecasts are, on average, rational, as in the empirical analysis. It also possible that a researcher may want to allow for a learning period or for some violations of optimality at specific points in time; the test can be adapted to these situations by examining a plot of the test statistic $W_{j,m}$ over time.

The empirical analysis compares various private sector forecasts to those of the Federal Reserve Greenbook. We reject the forecast rationality of all these forecasts at some point in time. However, even after allowing for time-variation, we find significant evidence in favor of the Fed's additional information advantage over the private sector when predicting future inflation.

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Appendix

Proof of equation (6). Note that

$$\widehat{\theta}_j = \left(m^{-1} \sum_{t=j-m+1}^{j} g_t g_t' \right)^{-1} \left(m^{-1} \sum_{t=j-m+1}^{j} g_t v_{t+h} \left(\widehat{\gamma}_{t,R}\right) \right) \quad (35)$$
$$= \left(m^{-1} \sum_{t=j-m+1}^{j} g_t g_t' \right)^{-1} \left(m^{-1} \sum_{t=j-m+1}^{j} f_{t+h} \left(\widehat{\gamma}_{t,R}\right) \right).$$

From a mean value expansion of $v_{t+h}\left(\widehat{\gamma}_{t,R}\right)$ around γ^* we have: $v_{t+h}\left(\widehat{\gamma}_{t,R}\right) = v_{t+h} + \frac{\partial v_{t+h}}{\partial \gamma}\left(\widehat{\gamma}_{t,R} - \gamma^*\right) + \widetilde{w}_{t+h}$, where $\widetilde{w}_{t+h}$ is the remainder. Thus, $f_{t+h}\left(\widehat{\gamma}_{t,R}\right) = f_{t+h} + f_{t+h,\gamma} \cdot \left(\widehat{\gamma}_{t,R} - \gamma^*\right) + w_{t+h}$, where $w_{t+h} \equiv g_t \widetilde{w}_{t+h}$.[18] Furthermore, by Assumption 1,

$$m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h}\left(\widehat{\gamma}_{t,R}\right) = m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h} + m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h,\gamma} \cdot B_t H_t \quad (36)$$
$$+ m^{-1/2} \sum_{t=j-m+1}^{j} w_{t+h},$$

As in the proof of equation (4.1) in West (1996), note that

$$m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h,\gamma} \cdot B_t H_t = m^{-1/2} F B \sum_{t=j-m+1}^{j} H_t + \widetilde{A}_j, \text{ where} \quad (37)$$

$$\widetilde{A}_j \equiv m^{-1/2} \sum_{t=j-m+1}^{j} \left(f_{t+h,\gamma} - F\right) B H_t + m^{-1/2} F \sum_{t=j-m+1}^{j} \left(B_t - B\right) H_t$$
$$+ m^{-1/2} \sum_{t=j-m+1}^{j} \left(f_{t+h,\gamma} - F\right)\left(B_t - B\right) H_t.$$

Assumption 2 implies that the last three terms in $\widetilde{A}_j$ are $o_p(1)$.

Therefore, by equations (35), (36) and (37), we have:

$$m^{1/2}\widehat{\theta}_j = \left(m^{-1} \sum_{t=j-m+1}^{j} g_t g_t' \right)^{-1} \left(m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h}\left(\widehat{\gamma}_{t,R}\right) \right)$$
$$= \left(m^{-1} \sum_{t=j-m+1}^{j} g_t g_t' \right)^{-1} \left(m^{-1/2} \sum_{t=j-m+1}^{j} f_{t+h} + m^{-1/2} F B \sum_{t=j-m+1}^{j} H_t + \widetilde{A}_j + m^{-1/2} \sum_{t=j-m+1}^{j} w_{t+h} \right)$$
$$= G^{-1}\left[I_\ell, FB\right] \left\{ m^{-1/2} \sum_{t=j-m+1}^{j} \begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \right\} + A_j,$$

[18] $\frac{\delta f_{t+h}(\gamma)}{\delta \gamma} = \frac{\delta g_t}{\delta \gamma} v_{t+h}(\gamma) + g_t \frac{\delta v_{t+h}(\gamma)}{\delta \gamma}$. The forecast error has mean zero given information at time t and thus it is asumptotically irrelevant (as in West and McCracken, 1998).

where the last line follows from Assumption 1(ii), Assumption 2 and

$$A_j \equiv \left(m^{-1}\sum_{t=j-m+1}^{j} g_t g_t'\right)^{-1}\left[\widetilde{A}_j + m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h}\right].$$

Thus,

$$m^{1/2}\widehat{\theta}_j = G^{-1}\left[I_\ell, FB\right]\left(\frac{T}{m}\right)^{1/2}\left\{\frac{1}{T^{1/2}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{T^{1/2}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right\} + A_j.$$

By Assumption 2(v) and arguments similar to West (1996, proof of equation 4.1), $m^{-1/2}\sum_{t=j-m+1}^{j} w_{t+h} = o_p(1)$. Therefore, Assumptions 1 and 2 ensure that $\lim_{T\to\infty}\sup_j A_j = o_p(1)$. ■

Proof of Proposition 1. By Hansen (1992), under Assumptions 1-4 and $T^{-1/2}\xi_T \to \xi$ in $\mathcal{D}_{\mathbb{R}^{(\ell+q)}}[0,1]$ then

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{j}\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix}\begin{pmatrix} f_{t+h} \\ h_t \end{pmatrix} - C_T^*(\tau) \Rightarrow \int_0^\tau \begin{pmatrix} \sigma_f(s)\cdot I_\ell & 0 \\ 0 & \sigma_h(s,\tau)\cdot I_q \end{pmatrix} d\xi(s),$$

where $\xi(s) = S^{1/2}\mathcal{B}_{l+q}(s)$, $z_t = \sum_{k=1}^{\infty} E_t\left(\begin{bmatrix} f_{t+h+k} & h_{t+k} \end{bmatrix}'\right)$ and

$$C_T^*(\tau) = \left\{T^{-1/2}\left[\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{j,t,m}\cdot I_q \end{pmatrix} - \begin{pmatrix} b_{R,t-1,j}\cdot I_\ell & 0 \\ 0 & a_{R,t-1,j}\cdot I_q \end{pmatrix}\right] z_t \right.$$
$$\left. - T^{-1/2}\begin{pmatrix} b_{R,t,j}\cdot I_\ell & 0 \\ 0 & a_{R,t,j}\cdot I_q \end{pmatrix} z_{t+1}\right\}.$$

The proof follows from the fact that $\sup_\tau C_T^*(\tau) = o_p(1)$, using the same reasoning as in Cavaliere (2005, Proof of Theorem 4), and the fact that the variances $\sigma_f(s), \sigma_h(s,\tau)$ are square integrable and bounded. ■

Proof of Proposition 2. It follows directly from Proposition 1 and Assumption 2 that

$T^{-1/2}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} \Rightarrow \int_0^\tau \Omega\left(s,\tau\right)^{1/2} S^{1/2} d\mathcal{B}_{l+q}\left(s\right).$ Thus,

$$
\begin{aligned}
m^{1/2}\,\widehat{\theta}_j &= G^{-1}\left(\frac{T}{m}\right)^{1/2}\left[I_\ell, FB\right]\left(\frac{1}{\sqrt{T}}\sum_{t=R}^{j}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix} - \frac{1}{\sqrt{T}}\sum_{t=R}^{j-m}\begin{pmatrix} f_{t+h} \\ H_t \end{pmatrix}\right) + A_j \\
&\Rightarrow \mu^{-1/2}G^{-1}\left[I_\ell, FB\right]\left(\int_0^\tau \Omega\left(s,\tau\right)^{1/2} S^{1/2} d\mathcal{B}_{l+q} - \int_0^{\tau-\mu}\Omega\left(s,\tau-\mu\right)^{1/2} S^{1/2} d\mathcal{B}_{l+q}\right) \qquad (38) \\
&= \mu^{-1/2}G^{-1}\left[I_\ell, FB\right]\left(\int_0^{\tau-\mu}\left[\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)^{1/2}\right] S^{1/2} d\mathcal{B}_{l+q} + \int_{\tau-\mu}^{\tau}\Omega\left(s,\tau\right)^{1/2} S^{1/2} d\mathcal{B}_{l+q}\right) \\
&= \mu^{-1/2}G^{-1}\left[I_\ell, FB\right]\int_0^\tau \begin{pmatrix} \left[\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)^{1/2}\right]\cdot 1\left(s \le \tau-\mu\right) \\ +\Omega\left(s,\tau\right)^{1/2}\cdot 1\left(\tau-\mu < s \le \tau\right) \end{pmatrix} S^{1/2} d\mathcal{B}_{l+q} \\
&= \textstyle\int_0^\tau \omega\left(s,\tau\right) d\mathcal{B}_l\left(s,\tau\right) = \mathcal{B}_l\left(\int_0^\tau \omega\left(s,\tau\right)\omega\left(s,\tau\right)' ds\right),
\end{aligned}
$$

where $\omega\left(s,\tau\right), \widetilde{\omega}\left(s,\tau\right)$ are defined in Proposition 2. The second line follows from Assumptions 2 and 3 as well as Proposition 1; the last equality follows from Lemma 2 in Cavaliere (2005). ■

Proof of Proposition 3. Note that:

$$
\omega\left(s,\tau\right) = \mu^{-\frac{1}{2}}G^{-1}\left[I_\ell, FB\right]\begin{bmatrix} \left(\Omega\left(s,\tau\right)^{1/2} - \Omega\left(s,\tau-\mu\right)\right)\cdot 1\left(s \le \tau-\mu\right) \\ +\Omega\left(s,\tau\right)^{1/2}\cdot 1\left(\tau-\mu < s \le \tau\right) \end{bmatrix} S^{1/2} = \mu^{-\frac{1}{2}}G^{-1}\left[I_\ell, FB\right]\times
$$

$$
\times\begin{bmatrix} \sigma_f\left(s\right)\cdot 1\left(\tau-\mu \le s < \tau\right)\cdot I_\ell & 0 \\ 0 & \begin{matrix}\left(\sigma_h\left(s,\tau\right) - \sigma_h\left(s,\tau-\mu\right)\right)\cdot 1\left(s \le \tau-\mu\right) \\ +\sigma_h\left(s,\tau\right)\cdot 1\left(\tau-\mu \le s \le \tau\right)\cdot I_q\end{matrix} \end{bmatrix} S^{1/2}
$$

$$\int_0^\tau \omega(s,\tau)\,\omega(s,\tau)'\,ds$$
$$= \mu^{-1}G^{-1}\int_0^\tau \sigma_f^2(s)\cdot 1(\tau-\mu\le s<\tau)\,S_{ff}ds G^{-1}$$
$$+\,\mu^{-1}G^{-1}\int_0^\tau \sigma_h(s,\tau)\,\sigma_f(s)\cdot 1(\tau-\mu\le s\le\tau)\left(FBS_{fh}'+S_{fh}B'F'\right)dsG^{-1}$$
$$+\,\mu^{-1}G^{-1}\int_0^\tau [(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu))^2\cdot 1(s\le\tau-\mu)$$
$$+\,\sigma_h^2(s,\tau)\cdot 1(\tau-\mu\le s\le\tau)]dsFBS_{hh}B'F'G^{-1}$$
$$= \mu^{-1}G^{-1}[\left(\int_{\tau-\mu}^\tau \sigma_f^2(s)\,ds\right)S_{ff}+\left(\int_{\tau-\mu}^\tau \sigma_h(s,\tau)\,\sigma_f(s)\,ds\right)\left(FBS_{fh}'+S_{fh}B'F'\right)$$
$$+\left(\int_0^\tau \left[[\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)]^2\cdot 1(s\le\tau-\mu)+\sigma_h^2(s,\tau)\cdot 1(\tau-\mu\le s\le\tau)\right]ds\right)\times$$
$$\times\, FBS_{hh}B'F']G^{-1}.$$

Note that (i) $\int_{\tau-\mu}^\tau \sigma_f^2(s)\,ds=\int_{\tau-\mu}^\tau (1(s\ge\rho))^2\,ds=\int_{\tau-\mu}^\tau ds=\mu$;

(ii) Recursive case:[19]

$$\begin{aligned}\int_{\tau-\mu}^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds &= \int_{\tau-\mu}^\tau 1(s\ge\rho)\cdot([\ln(\tau)-\ln(\rho)]\cdot 1(s<\rho)+[\ln(\tau)-\ln(s)]\cdot 1(s\ge\rho))ds\\ &= \int_{\tau-\mu}^\tau [\ln(\tau)-\ln(s)]\,ds=\int_{\tau-\mu}^\tau \ln(\tau)\,ds-\int_{\tau-\mu}^\tau \ln(s)\,ds\\ &= \ln(\tau)(\tau-\tau+\mu)-(\ln(\tau)\tau-\tau)+(\ln(\tau-\mu)(\tau-\mu)-(\tau-\mu))\\ &= -\ln(\tau)(\tau-\mu)+\tau+\ln(\tau-\mu)(\tau-\mu)-\tau+\mu\\ &= \mu-(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right)=\mu\left[1-\widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right]\end{aligned}$$

where $\widetilde{\pi}\equiv\mu/(\tau-\mu)$. Furthermore,

$$\int_0^{\tau-\mu}(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu))^2\,ds=\int_0^{\tau-\mu}[\ln(\tau)-\ln(\tau-\mu)]^2\,ds=(\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2.$$

[19]Note $\int ln(x)dx = ln(x)x - x + c$; $\int ln(x)^2dx = xln(x)^2 - 2xln(x) + 2x + c$

$$
\begin{aligned}
\int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\,ds &= \int_{\tau-\mu}^{\tau} \left([\ln(\tau)-\ln(\rho)]\cdot 1(s<\rho) + [\ln(\tau)-\ln(s)]\cdot 1(s\geq\rho)\right)^2 ds \\
&= \int_{\tau-\mu}^{\tau} [\ln(\tau)-\ln(s)]^2\,ds = \int_{\tau-\mu}^{\tau} \left(\ln(\tau)^2 - 2\ln(\tau)\ln(s) + \ln(s)^2\right) ds \\
&= \ln(\tau)^2\mu - 2\ln(\tau)\left[\ln(\tau)\tau - \tau - \ln(\tau-\mu)(\tau-\mu) + \tau - \mu\right] + \\
&+ \ln(\tau)^2\tau - 2\tau\ln(\tau) + 2\tau - \ln(\tau-\mu)^2(\tau-\mu) + 2(\tau-\mu)\ln(\tau-\mu) - 2(\tau-\mu) \\
&= \ln(\tau)^2\mu - \ln(\tau)^2\tau + 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) \\
&+ 2\ln(\tau)\mu - 2\tau\ln(\tau) - \ln(\tau-\mu)^2(\tau-\mu) + 2(\tau-\mu)\ln(\tau-\mu) + 2\mu \\
&= 2\mu + \ln(\tau)^2(\mu-\tau) - \ln(\tau-\mu)^2(\tau-\mu) + 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) \\
&+ 2(\tau-\mu)\ln(\tau-\mu) + 2\ln(\tau)(\mu-\tau) \\
&= 2\mu + \ln(\tau)^2(\mu-\tau) - \ln(\tau-\mu)^2(\tau-\mu) + \\
&+ 2\ln(\tau)\ln(\tau-\mu)(\tau-\mu) + 2(\tau-\mu)\ln\left(\frac{\tau-\mu}{\tau}\right) \\
&= 2\mu - 2(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) - (\tau-\mu)\left[\ln(\tau)-\ln(\tau-\mu)\right]^2 \\
&= 2\mu - 2(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) - (\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\tau} \left[(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu))^2 \cdot 1(s\leq\tau-\mu) + \sigma_h^2(s,\tau)\cdot 1(\tau-\mu\leq s<\tau)\right] ds \\
&= \int_0^{\tau-\mu} (\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu))^2\,ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\,ds \\
&= (\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2 + 2\mu - 2(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) - (\tau-\mu)\left[\ln\left(\frac{\tau}{\tau-\mu}\right)\right]^2 \\
&= 2\mu - 2(\tau-\mu)\ln\left(\frac{\tau}{\tau-\mu}\right) = 2\mu\left[1-\widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right]
\end{aligned}
$$

(iii) Rolling case:

In the rolling estimation scheme there are two possible cases. Case (a) occurs when $\tau-\rho\geq\rho$, while (b) when $\tau-\rho<\rho$. We consider the calculation of the respective integrals in these two cases. We show that the covariance is the same in both cases, no matter whether $\mu\geq\rho$ or $\mu<\rho$.

Case (a): $\tau-\rho\geq\rho$. This allows for two sub-cases (i) $\mu\geq\rho \Leftrightarrow \tau-\rho\geq\tau-\mu\geq\rho$ and

(ii) $\mu < \rho \Leftrightarrow \tau - \mu > \tau - \rho \geq \rho$ (recall that $\tau \geq \rho + \mu$). In case (i),

$$\begin{aligned}\int_{\tau-\mu}^{\tau} \sigma_h(s,\tau)\,\sigma_f(s)\,ds &= \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot \frac{1}{\rho}[s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau - s)\cdot 1(s > \tau-\rho)]ds = \\ &= \int_{\tau-\mu}^{\tau} \frac{1}{\rho}[\rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau - s)\cdot 1(s > \tau-\rho)]ds = \\ &= (\mu - \rho) + \frac{1}{2}\rho = \mu - \frac{1}{2}\rho;\end{aligned}$$

whereas in case (ii),

$$\begin{aligned}\int_{\tau-\mu}^{\tau} \sigma_h(s,\tau)\,\sigma_f(s)\,ds &= \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot \frac{1}{\rho}[s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau - s)\cdot 1(s > \tau-\rho)]ds = \\ &= \int_{\tau-\mu}^{\tau} \frac{1}{\rho}[(\tau - s)\cdot 1(s > \tau-\rho)]ds = \int_{\tau-\mu}^{\tau} \frac{1}{\rho}(\tau-s)ds = \frac{1}{2}\frac{\mu^2}{\rho}.\end{aligned}$$

Furthermore,

$$\int_0^{\tau-\mu} \left(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{c}[s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau-s)\cdot 1(\tau > s > \tau-\rho)] - \\ {[s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\mu-\rho) + (\tau-\mu-s)\cdot 1(\tau-\mu > s > \tau-\mu-\rho)]}\end{array}\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{c}\rho \cdot 1(\tau-\mu-\rho \leq s \leq \tau-\rho) + (\tau-s)\cdot 1(\tau > s > \tau-\rho) \\ -(\tau-\mu-s)\cdot 1(\tau-\mu > s > \tau-\mu-\rho)\end{array}\right)^2 ds$$

The expression above simplifies:
(i) $\mu \geq \rho \Leftrightarrow \tau - \rho > \tau - \mu \geq \rho$

$$\int_0^{\tau-\mu} \left(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right)^2 ds = \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu-\rho}^{\tau-\mu} \left(\rho - (\tau-\mu-s)\right)^2 ds = \frac{1}{3}\rho.$$

In addition,

$$\int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\,ds = \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu}^{\tau} [s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau - s)\cdot 1(s > \tau-\rho)]^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2 \left(\int_{\tau-\mu}^{\tau-\rho} \rho^2 ds + \int_{\tau-\rho}^{\tau} (\tau-s)^2 ds\right) = (\mu-\rho) + \frac{1}{3}\rho = \mu - \frac{2}{3}\rho.$$

Thus,

$$\int_0^{\tau-\mu} \left(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right)^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s)\,ds = \mu - \frac{1}{3}\rho = \mu\left(1 - \frac{1}{3\pi^{\dagger}}\right).$$

(ii) $\mu > \rho \Leftrightarrow \tau - \mu > \tau - \rho \geq \rho$

$$\int_0^{\tau-\mu} \left(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right)^2 ds =$$

$$\left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{c} (\rho - (\tau-\mu-s)) \cdot 1(\tau-\mu-\rho \leq s \leq \tau-\rho) + (\tau - s) \cdot 1(\tau > s > \tau - \rho) \\ -(\tau-\mu-s) \cdot 1(\tau-\mu > s > \tau-\rho) \end{array} \right)^2 ds$$

$$\left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left[(\rho - (\tau-\mu-s)) \cdot 1(\tau-\mu-\rho \leq s \leq \tau-\rho) + \mu \cdot 1(\tau-\mu > s > \tau-\rho)\right]^2 ds$$

$$\left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu-\rho}^{\tau-\rho} [\rho - (\tau-\mu-s)]^2 ds + \left(\frac{\mu}{\rho}\right)^2 \int_{\tau-\rho}^{\tau-\mu} ds = -\frac{1}{3}\frac{\mu^2}{\rho^2}(2\mu - 3\rho) = \frac{\mu^2}{\rho} - \frac{2}{3}\frac{\mu^3}{\rho^2}$$

$$\int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\, ds = \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu}^{\tau} [s \cdot 1(s<\rho) + \rho \cdot 1(\rho \leq s \leq \tau-\rho) + (\tau-s) \cdot 1(s > \tau-\rho)]^2 ds =$$

$$= \left(\frac{1}{\rho}\right)^2 \int_{\tau-\mu}^{\tau} (\tau-s)^2 ds = \frac{1}{3}\frac{\mu^3}{\rho^2}$$

$$\int_0^{\tau-\mu} \left(\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right)^2 ds + \int_{\tau-\mu}^{\tau} \sigma_h^2(s)\, ds = \frac{\mu^2}{\rho} - \frac{2}{3}\frac{\mu^3}{\rho^2} + \frac{1}{3}\frac{\mu^3}{\rho^2} = \frac{\mu^2}{\rho} - \frac{1}{3}\frac{\mu^3}{\rho^2}$$

Case (b): $\tau - \rho < \rho$. Note that, since $\tau \geq \rho + \mu$, in this case the only possible subcase is $\mu < \rho$. Thus,

$$\int_{\tau-\mu}^{\tau} \sigma_f(s)\, \sigma_h(s,\tau)\, ds = \frac{1}{\rho} \int_{\tau-\mu}^{\tau} 1(s \geq \rho) \cdot \left[\begin{array}{c} s \cdot 1(s < \tau-\rho) + (\tau-\rho) \cdot 1(\tau-\rho \leq s \leq \rho) \\ +(\tau-s) \cdot 1(s > \rho) \end{array} \right] ds =$$

$$= \frac{1}{\rho} \int_{\tau-\mu}^{\tau} (\tau-s) \cdot 1(s>\rho) ds = \frac{1}{\rho} \int_{\tau-\mu}^{\tau} (\tau-s) ds = \frac{1}{2}\frac{\mu^2}{\rho}$$

Furthermore,

$$\int_0^{\tau-\mu} \left[\sigma_h(s,\tau) - \sigma_h(s,\tau-\mu)\right]^2 ds$$

$$= \left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{c} s \cdot 1(s < \tau-\rho) + (\tau-\rho) \cdot 1(\tau-\rho \leq s \leq \rho) + (\tau-s) \cdot 1(s > \rho) - \\ \left(\begin{array}{c} s \cdot 1(s < \tau-\mu-\rho) + (\tau-\mu-\rho) \cdot 1(\tau-\mu-\rho \leq s \leq \rho) \\ +(\tau-\mu-s) \cdot 1(s > \rho) \end{array} \right) \end{array} \right)^2 ds$$

$$= \left(\frac{1}{\rho}\right)^2 \int_0^{\tau-\mu} \left(\begin{array}{c} (s - (\tau-\mu-\rho)) \cdot 1(\tau-\mu-\rho < s < \tau-\rho) + \\ ((\tau-\rho) - (\tau-\mu-\rho)) \cdot 1(\tau-\rho \leq s \leq \rho) + (\tau - s - (\tau-\mu-s)) \cdot 1(s > \rho) \end{array} \right)^2 ds$$

$$= \left(\frac{1}{\rho}\right)^2 \left(\int_{\tau-\mu-\rho}^{\tau-\rho} (s - (\tau-\mu-\rho))^2 ds + \int_{\tau-\rho}^{\rho} \mu^2 ds + \int_{\rho}^{\tau-\mu} \mu^2 ds \right)$$

$$= -\frac{1}{\rho^2}\left(\mu^2(\mu - \tau + \rho) - \frac{1}{3}\mu^3 + \mu^2(\tau - 2\rho)\right) = \left(\frac{\mu}{\rho}\right)^2 \left(\frac{1}{3}\mu - (\mu - \rho)\right)$$

$$\int_{\tau-\mu}^{\tau} \sigma_h^2(s,\tau)\,ds = \frac{1}{\rho^2}\int_{\tau-\mu}^{\tau}(\tau-s)^2 ds = \frac{1}{3}\frac{\mu^3}{\rho^2}$$

$$\begin{aligned}\int_0^{\tau-\mu} \left(\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)\right)^2 ds + \int_{\tau-\mu}^{\tau}\sigma_h^2(s)\,ds &= \left(\frac{1}{\rho}\right)^2\left(\frac{1}{3}\mu^3-\mu^2(\mu-\rho)\right)+\frac{1}{3}\frac{\mu^3}{\rho^2} \\ &= -\frac{1}{3}\frac{\mu^2}{\rho^2}(\mu-3\rho)=\mu\pi^{\dagger}.\end{aligned}$$

■

Proof of Proposition 4. From Proposition 2 (in particular, eq. 38),

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mu^{-\frac{1}{2}}G^{-1}[I_\ell, FB]\left(\int_0^{\tau}\Omega(s,\tau)^{1/2}S^{1/2}d\mathcal{B}(s)-\int_0^{\tau-\mu}\Omega(s,\tau-\mu)^{1/2}S^{1/2}d\mathcal{B}(s)\right).$$

By similar arguments we can re-write as

$$m^{1/2}\widehat{\theta}_j \Rightarrow \mathcal{B}\left(\int_0^{\tau}\tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)'ds\right)-\mathcal{B}\left(\int_0^{\tau-\mu}\tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)'ds\right)$$

where $\tilde{\omega}(s,\tau)=\mu^{-\frac{1}{2}}G^{-1}[I_\ell,FB]\,\Omega(s,\tau)^{1/2}S^{1/2}$.

Furthermore,

$$\begin{aligned}\int_0^{\tau}\tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)'ds &= \mu^{-1}G^{-1}[I_\ell,FB]\int_0^{\tau}\Omega(s,\tau)^{1/2}S\Omega'(s,\tau)^{1/2}ds\,[I_\ell,FB]'G^{-1} = \quad (39)\\ & \mu^{-1}G^{-1}[I_\ell,FB]\begin{bmatrix}\int_0^{\tau}\sigma_f^2(s)\,dsS_{ff} & \int_0^{\tau}\sigma_f(s)\sigma_h(s,\tau)\,dsS_{fh}\\ \int_0^{\tau}\sigma_f(s)\sigma_h(s,\tau)\,dsS_{fh}' & \int_0^{\tau}\sigma_h^2(s,\tau)\,dsS_{hh}\end{bmatrix}\begin{bmatrix}I_\ell\\ B'F'\end{bmatrix}G^{-1}.\end{aligned}$$

In both recursive and rolling cases, $\int_0^{\tau}\sigma_f^2(s)\,ds=(\tau-\rho)$;

(i) Recursive case:

$$\begin{aligned}\int_0^{\tau}\sigma_f(s)\sigma_h(s,\tau)\,ds &= \int_0^{\tau}1(s\geq\rho)\cdot([\ln(\tau)-\ln(\rho)]\cdot 1(s<\rho)+[\ln(\tau)-\ln(s)]\cdot 1(s\geq\rho))ds\\ &= \int_\rho^{\tau}[\ln(\tau)-\ln(s)]\,ds = \int_\rho^{\tau}\ln(\tau)\,ds-\int_\rho^{\tau}\ln(s)\,ds =\\ &= \ln(\tau)(\tau-\rho)-(\ln(\tau)\tau-\tau)+(\ln(\rho)\rho-\rho)=\\ &= (\tau-\rho)-\rho\ln\left(\frac{\tau}{\rho}\right)=(\tau-\rho)\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right) \qquad (40)\end{aligned}$$

$$\int_0^\tau \sigma_h^2(s,\tau)\,ds = \int_0^\tau ([\ln(\tau)-\ln(\rho)]\cdot 1(s<\rho) + [\ln(\tau)-\ln(s)]\cdot 1(s\geq\rho))^2 ds \tag{41}$$
$$= \int_0^\rho [\ln(\tau)-\ln(\rho)]^2\,ds + \int_\rho^\tau [\ln(\tau)-\ln(s)]^2\,ds$$
$$= \rho\left(\ln^2(\tau) - 2\ln(\tau)\ln(\rho) + \ln^2(\rho)\right) + (\tau-\rho)\ln^2(\tau)$$
$$- 2\ln\tau\int_\rho^\tau \ln(s)\,ds + \int_\rho^\tau \ln^2(s)\,ds$$
$$= \rho\left(\ln^2(\tau) - 2\ln(\tau)\ln(\rho) + \ln^2(\rho)\right) + (\tau-\rho)\ln^2(\tau) -$$
$$- 2\ln(\tau)(\tau\ln(\tau) - \tau - \rho\ln(\rho) + \rho) +$$
$$+ (\tau\ln^2(\tau) - 2\tau\ln(\tau) + 2\tau) - (\rho\ln^2(\rho) - 2\rho\ln(\rho) + 2\rho)$$
$$= \rho\ln^2(\tau) - 2\rho\ln(\tau)\ln(\rho) + \rho\ln^2(\rho) + (\tau-\rho)\ln^2(\tau) -$$
$$- 2\tau\ln^2(\tau) + 2\tau\ln(\tau) + 2\rho\ln(\tau)\ln(\rho) - 2\rho\ln(\tau) +$$
$$+ \tau\ln^2(\tau) - 2\tau\ln(\tau) + 2\tau - \rho\ln^2(\rho) + 2\rho\ln(\rho) - 2\rho$$
$$= 2\tau - 2\rho - 2\rho\ln(\tau) + 2\rho\ln(\rho) = 2(\tau-\rho)\left(1 - \frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)$$

Thus,

$$\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds = \begin{bmatrix} \int_0^\tau \sigma_f^2(s)\,dsS_{ff} & \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS_{fh} \\ \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS'_{fh} & \int_0^\tau \sigma_h^2(s,\tau)\,dsS_{hh} \end{bmatrix}$$
$$= (\tau-\rho)\begin{bmatrix} S_{ff} & \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{fh} \\ \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S'_{fh} & 2\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{hh} \end{bmatrix}$$

and

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds$$
$$= \mu^{-1}G^{-1}[I_\ell, FB]\begin{bmatrix} \int_0^\tau \sigma_f^2(s)\,dsS_{ff} & \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS_{fh} \\ \int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,dsS'_{fh} & \int_0^\tau \sigma_h^2(s,\tau)\,dsS_{hh} \end{bmatrix}\begin{bmatrix} I_\ell \\ B'F' \end{bmatrix}G^{-1}$$
$$= G^{-1}\frac{(\tau-\rho)}{\mu}[I_\ell, FB]\begin{bmatrix} S_{ff} & \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{fh} \\ \left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S'_{fh} & 2\left(1-\frac{\rho}{\tau-\rho}\ln\left(\frac{\tau}{\rho}\right)\right)S_{hh} \end{bmatrix}\begin{bmatrix} I_\ell \\ B'F' \end{bmatrix}G^{-1}$$
$$\frac{(\tau-\rho)}{\mu}G^{-1}\left\{S_{ff} + \left(1-\rho\frac{\ln\frac{\tau}{\rho}}{\tau-\rho}\right)\left(S_{fh}B'F' + FBS'_{fh}\right) + 2FBS_{hh}B'F'\left(1-\rho\frac{\ln\frac{\tau}{\rho}}{\tau-\rho}\right)\right\}G^{-1};$$

(ii) Rolling Case:

Case (a): $\tau - \rho \geq \rho$

$$
\begin{aligned}
&\int_0^\tau \sigma_h(s,\tau)\,\sigma_f(s)\,ds \\
&= \int_0^\tau 1(s \geq \rho)\left(\frac{1}{\rho}\left[s \cdot 1(s<\rho) + \rho \cdot 1(\rho < s < \tau-\rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right]\right) ds \\
&= \int_\rho^\tau \frac{1}{\rho}\left[\rho \cdot 1(\rho < s < \tau-\rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right] ds \\
&= \frac{1}{\rho}\left(\int_\rho^{\tau-\rho} \rho ds + \int_{\tau-\rho}^\tau (\tau - s)\,ds\right) = \frac{1}{\rho}\left(\rho(\tau-2\rho) + \tau\rho - \frac{1}{2}\tau^2 + \frac{1}{2}(\tau-\rho)^2\right) \\
&= \tau - \frac{3}{2}\rho
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&\int_0^\tau \sigma_h^2(s,\tau)\,ds \\
&= \int_0^\tau \left(\frac{1}{\rho}\{s \cdot 1(s<\rho) + \rho \cdot 1(\rho < s < \tau-\rho) + (\tau - s)\cdot 1(s > \tau - \rho)\right)^2 ds \\
&= \left(\frac{1}{\rho}\right)^2 \left(\int_0^\tau s^2 \cdot 1(s<\rho)\,ds + \int_0^\tau \rho^2 \cdot 1(\rho < s < \tau-\rho)\,ds + \int_0^\tau (\tau-s)^2 \cdot 1(s > \tau-\rho)\,ds\right) \\
&= \left(\frac{1}{\rho}\right)^2 \left(\int_0^\rho s^2 ds + \rho^2 \int_\rho^{\tau-\rho} ds + \left(\frac{1}{\rho}\right)^2 \int_{\tau-\rho}^\tau (\tau-s)^2\,ds\right) \\
&= \left(\frac{1}{\rho}\right)^2 \frac{1}{3}\rho^3 + (\tau - 2\rho) + \left(\frac{1}{\rho}\right)^2 \left(\tau^2\rho + \frac{\tau^3 - (\tau-\rho)^3}{3} - 2\tau\frac{\tau^2 - (\tau-\rho)^2}{2}\right) \\
&= \tau - \frac{4}{3}\rho
\end{aligned}
\tag{43}
$$

Thus, $\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds = \begin{bmatrix} (\tau-\rho)S_{ff} & \left(\tau - \frac{3}{2}\rho\right)S_{fh} \\ \left(\tau - \frac{3}{2}\rho\right)S'_{fh} & \left(\tau - \frac{4}{3}\rho\right)S_{hh} \end{bmatrix}$, and

$$
\begin{aligned}
\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds &= \mu^{-1}G^{-1}[I_\ell, FB] \begin{bmatrix} (\tau-\rho)S_{ff} & \left(\tau - \frac{3}{2}\rho\right)S_{fh} \\ \left(\tau - \frac{3}{2}\rho\right)S'_{fh} & \left(\tau - \frac{4}{3}\rho\right)S_{hh} \end{bmatrix} \begin{bmatrix} I_\ell \\ B'F' \end{bmatrix} G^{-1} = \mu^{-1}G^{-1} \times \\
&\times \left\{S_{ff}(\tau-\rho) + \left(\tau - \frac{3}{2}\rho\right)\left(FBS'_{fh} + S_{fh}B'F'\right) + \left(\tau - \frac{4}{3}\rho\right) FBS_{hh}B'F'\right\} G^{-1};
\end{aligned}
$$

Case (b): $\tau - \rho < \rho$

$$\int_0^\tau \sigma_f(s)\,\sigma_h(s,\tau)\,ds$$
$$= \int_0^\tau \{1(s\geq\rho)\}\left\{\frac{s}{\rho}\cdot 1(s<\tau-\rho)+\frac{(\tau-\rho)}{\rho}\cdot 1(\tau-\rho<s<\rho)+\frac{(\tau-s)}{\rho}\cdot 1(s>\rho)\right\}ds =$$
$$= \int_0^\tau \frac{1}{\rho}(\tau-s)\cdot 1(s>\rho)\,ds = \frac{1}{\rho}\int_\rho^\tau (\tau-s)\,ds = \frac{1}{2\rho}(\tau-\rho)^2$$

$$\int_0^\tau \sigma_h^2(s)\,ds = \int_0^\tau \left(\frac{1}{\rho}\{s\cdot 1(s<\tau-\rho)+(\tau-\rho)\cdot 1(\tau-\rho<s<\rho)+(\tau-s)\cdot 1(s>\rho)\}\right)^2 ds$$
$$= \left(\frac{1}{\rho}\right)^2\left(\int_0^{\tau-\rho} s^2 ds + \int_{\tau-\rho}^{\rho}(\tau-\rho)^2 ds + \int_\rho^\tau (\tau-s)^2 ds\right)$$
$$= \left(\frac{1}{\rho}\right)^2\left(\frac{1}{3}(\tau-\rho)^3+(\tau-\rho)^2(2\rho-\tau)+\tau^2(\tau-\rho)+\frac{\tau^3-\rho^3}{3}-\tau^3+\tau\rho^2\right)$$
$$= \frac{1}{3\rho^2}(\rho-\tau)^2(4\rho-\tau)$$

Thus, $\int_0^\tau \Omega(s,\tau)^{1/2} S\Omega'(s,\tau)^{1/2}\,ds = \begin{bmatrix} (\tau-\rho)S_{ff} & \frac{1}{2\rho}(\tau-\rho)^2 S_{fh} \\ \frac{1}{2\rho}(\tau-\rho)^2 S'_{fh} & \frac{1}{3\rho^2}(\rho-\tau)^2(4\rho-\tau)S_{hh}\end{bmatrix}$,

and

$$\int_0^\tau \tilde{\omega}(s,\tau)\,\tilde{\omega}(s,\tau)'\,ds$$
$$= \mu^{-1}G^{-1}[I_\ell, FB]\begin{bmatrix} (\tau-\rho)S_{ff} & \frac{1}{2\rho}(\tau-\rho)^2 S_{fh} \\ \frac{1}{2\rho}(\tau-\rho)^2 S'_{fh} & \frac{(\rho-\tau)^2}{3\rho^2}(4\rho-\tau)S_{hh}\end{bmatrix}\begin{bmatrix} I_\ell \\ B'F'\end{bmatrix}G^{-1}$$
$$= \frac{(\tau-\rho)}{\mu}G^{-1}\left\{S_{ff}+\frac{(\tau-\rho)}{2\rho}\left(FBS'_{fh}+S_{fh}B'F'\right)-\frac{(\tau-\rho)(\tau-4\rho)}{3\rho^2}FBS_{hh}F'B'\right\}G^{-1}.$$

■

Proof of Theorem 5. The proof follows directly from Propositions 2 and 3. ■

Proof of Proposition 6. The result follows directly from Proposition 3 by imposing $F=0$. ■

Proof of Proposition 7. From Proposition 2,

$$m^{1/2}\widehat{\theta}_j \Rightarrow \int_0^\tau \widetilde{\omega}(s,\tau)^{1/2}\,d\mathcal{B}_\ell(s) - \int_0^{\tau-\mu}\widetilde{\omega}(s,\tau-\mu)^{1/2}\,d\mathcal{B}_\ell(s)$$
$$= \mu^{-\frac{1}{2}}G^{-1}[I_\ell, FB]\{\int_0^\tau \Omega(s,\tau)^{1/2}S^{1/2}d\mathcal{B}(s) - \int_0^{\tau-\mu}\Omega(s,\tau-\mu)^{1/2}S^{1/2}d\mathcal{B}(s)\}.$$

Note that

$$\int_0^{\tau} \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)' ds = \mu^{-1}G^{-1}\int_0^{\tau}[I_\ell, FB]\,\Omega(s,\tau)^{1/2} S\Omega(s,\tau)^{1/2}[I_\ell, FB]' dsG^{-1}$$

$$= \mu^{-1}G^{-1}\int_0^{\tau}[\sigma_f(s)^2 S_{ff} + [FBS_{hf} + S_{fh}B'F']\sigma_f(s)\sigma_h(s,\tau) + FBS_{hh}B'F'\sigma_h^2(s,\tau)^2]dsG^{-1}$$

$$= \mu^{-1}G^{-1}S_{ff}G^{-1}\int_0^{\tau}[\sigma_f(s)^2 + 2\sigma_f(s)\sigma_h(s,\tau) + \sigma_h^2(s,\tau)]ds, \tag{44}$$

and similarly, $\int_0^{\tau-\mu}\widetilde{\omega}(s,\tau-\mu)^{1/2}\widetilde{\omega}(s,\tau-\mu)' ds = \mu^{-1}G^{-1}S_{ff}G^{-1}\int_0^{\tau-\mu}[\sigma_f(s)^2 - 2\sigma_f(s)\sigma_h(s,\tau-\mu) + \sigma_h^2(s,\tau-\mu)^2]ds.$

By imposing condition (22) in Proposition 3, we have:

$$\int_0^{\tau}\omega(s,\tau)\omega(s,\tau)' ds = \mu^{-1}G^{-1}S_{ff}G^{-1}[\left(\int_{\tau-\mu}^{\tau}\sigma_f(s)^2 ds\right) - 2\left(\int_{\tau-\mu}^{\tau}\sigma_h(s,\tau)\sigma_f(s)ds\right) \tag{45}$$

$$+\left(\int_0^{\tau}\left[[\sigma_h(s,\tau)-\sigma_h(s,\tau-\mu)]^2\cdot 1(s\leq\tau-\mu) + \sigma_h(s,\tau)^2\cdot 1(\tau-\mu\leq s<\tau)\right]ds\right)].$$

Thus: (i) Recursive case. From the proof of Proposition 4 (equations 41 and 40) note that $\int_0^{\tau}\sigma_h^2(s,\tau)\,ds = 2\int_0^{\tau}\sigma_h(s,\tau)\sigma_f(s)\,ds$. Therefore, eq. (44) simplifies to $\int_0^{\tau}\tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)' ds = \mu^{-1}G^{-1}S_{ff}G^{-1}\int_0^{\tau}\sigma_f(s)^2 ds = \frac{(\tau-\rho)}{\mu}G^{-1}S_{ff}G^{-1}$. Furthermore, from eq. (45), we have: $\int_0^{\tau}\omega(s,\tau)\omega(s,\tau)' ds = \mu^{-1}G^{-1}\left[\mu S_{ff} - 2\mu\left(1-\widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right)S_{ff} + 2\mu\left(1-\widetilde{\pi}^{-1}\ln(1+\widetilde{\pi})\right)S_{ff}\right]G^{-1} = G^{-1}S_{ff}G^{-1}$.

(ii) Rolling case. Case (a): $\rho\leq\tau-\rho$. In this case, the distribution of $m^{1/2}\widehat{\theta}_j$ cannot be obtained using direct calculations based on eq. (44) because the latter simplifies to

$$\int_0^{\tau}\tilde{\omega}(s,\tau)\tilde{\omega}(s,\tau)' ds = \mu^{-1}\int_0^{\tau}[\sigma_f(s)^2 - 2\sigma_f(s)\sigma_h(s,\tau) + \sigma_h^2(s,\tau)^2]dsG^{-1}S_{ff}G^{-1} \tag{46}$$

$$= \frac{2}{3}\frac{\rho}{\mu}G^{-1}S_{ff}G^{-1},$$

and is independent of τ.

Case (b): $\rho>\tau-\rho$. In this case, given the values $\int_0^{\tau}\sigma_h(s,\tau)\sigma_v(s)\,ds = \frac{1}{2\rho}(\tau-\rho)^2$ and $\int_0^{\tau}\sigma_h^2(s,\tau)\,ds = \frac{1}{3\rho^2}(\rho-\tau)^2(4\rho-\tau)$ by proof of Proposition 4 (equations 42 and 43).

Therefore, eq. (44) simplifies to

$$\int_0^\tau \tilde{\omega}(s)\tilde{\omega}(s)'\,ds = \frac{1}{\mu}\int_0^\tau [\sigma_f^2(s) - 2\sigma_f(s)\sigma_h(s,\tau) + \sigma_h^2(s,\tau)]dsG^{-1}S_{ff}G^{-1}$$
$$= \frac{1}{\mu}\left[(\tau-\rho) - 2\frac{1}{2\rho}(\tau-\rho)^2 + \frac{1}{3\rho^2}(\rho-\tau)^2(4\rho-\tau)\right]G^{-1}S_{ff}G^{-1}$$
$$= \left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}. \tag{47}$$

■

Proof of Theorem 8. (a) From Propositions 2 and 7 we have

$$\begin{aligned}\mathcal{W}_{t,m} &= \widehat{\theta}_j V_{\theta,j}^{-1}\widehat{\theta}_j \\ &\Rightarrow \left[\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) - \mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right)\right]' V_\theta^{-1}\times \\ &\times\left[\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) - \mathcal{B}_\ell\left(\int_0^{\tau-\mu} \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right)\right]\end{aligned}$$

Under condition (22):

(i) Recursive case.

$V_{\theta,j} = m^{-1}G^{-1}S_{ff}G^{-1}$ and $\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) = (G^{-1}S_{ff}G^{-1})^{-1/2}\mu^{-1/2}\mathcal{B}_\ell(\tau-\rho)$.

Thus,

$\mathcal{W}_{t,m} \Rightarrow \mu^{-1}[\mathcal{B}_\ell(\tau-\rho) - \mathcal{B}_\ell(\tau-\mu-\rho)]'[\mathcal{B}_\ell(\tau-\rho) - \mathcal{B}_\ell(\tau-\mu-\rho)]$.

(ii) Rolling case. From Proposition 7, in particular eq. (47), we have that, for $\tau-\rho<\rho$,

$$\begin{aligned}&\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) \\ &= \mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)G^{-1}S_{ff}G^{-1}\right) \\ &= \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right).\end{aligned}$$

From eq. (46) we have that $\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) = \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)$. Thus,

$$\begin{aligned}&\mathcal{B}_\ell\left(\int_0^\tau \widetilde{\omega}(s,\tau)\widetilde{\omega}(s,\tau)'\,ds\right) \\ &= \left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\left[\begin{array}{c} 1(\mu+\rho\le\tau<2\rho)\cdot\mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right) \\ +1(2\rho\le\tau\le1)\cdot\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)\end{array}\right],\end{aligned}$$

$$\mathcal{B}_\ell\left(\int_0^{\tau-\mu}\widetilde{\omega}\left(s,\tau-\mu\right)\widetilde{\omega}\left(s,\tau-\mu\right)'ds\right)$$
$$=\left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\left[\begin{array}{c}1\left(\rho\leq\tau-\mu<2\rho\right)\cdot\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\\+1\left(2\rho\leq\tau-\mu\leq1-\mu\right)\cdot\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)\end{array}\right],$$

$$\mathcal{B}_\ell\left(\int_0^{\tau}\widetilde{\omega}\left(s,\tau\right)\widetilde{\omega}\left(s,\tau\right)'ds\right)-\mathcal{B}_\ell\left(\int_0^{\tau-\mu}\widetilde{\omega}\left(s,\tau-\mu\right)\widetilde{\omega}\left(s,\tau-\mu\right)'ds\right)$$
$$=\left(G^{-1}S_{ff}G^{-1}\right)^{1/2}\left[1\left(\mu+\rho\leq\tau<2\rho\right)\cdot\left\{\begin{array}{c}\mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)\\-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\end{array}\right\}+\right.$$
$$\left.\begin{array}{c}+1\left(2\rho<\tau\leq2\rho+\mu\right)\cdot\left\{\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\\+1\left(\tau>2\rho+\mu\right)\cdot0\end{array}\right]$$
$$\equiv\mathcal{B}_\ell\left(\int_0^{\tau}\omega\left(s,\tau\right)\omega\left(s,\tau\right)'ds\right).$$

From Proposition 7, $V_\theta=\left(G^{-1}S_{ff}G^{-1}\right)\left\{\left(\frac{2}{3\pi^\dagger}\right)\cdot1\left(\mu\geq\rho\right)+\left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)\cdot1\left(\mu<\rho\right)\right\}$. Thus,

$$\mathcal{W}_{t,m}\Rightarrow\mathcal{B}_\ell\left(\int_0^{\tau}\widetilde{\omega}\left(s,\tau\right)\widetilde{\omega}\left(s,\tau\right)'ds\right)\cdot V_\theta^{-1}\cdot\mathcal{B}_\ell\left(\int_0^{\tau}\omega\left(s,\tau\right)\omega\left(s,\tau\right)'ds\right)=$$
$$=\left[1\left(\mu+\rho\leq\tau<2\rho\right)\left\{\mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\right.$$
$$\left.+1\left(2\rho<\tau\leq2\rho+\mu\right)\cdot\left\{\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\right]'\times$$
$$\times\left\{\left(\frac{2}{3\pi^\dagger}\right)\cdot1\left(\mu\geq\rho\right)+\left(1-\frac{1}{3}\left(\pi^\dagger\right)^2\right)\cdot1\left(\mu<\rho\right)\right\}^{-1}\times$$
$$\times\left[1\left(\mu+\rho\leq\tau<2\rho\right)\left\{\mathcal{B}_\ell\left(\left(\frac{\tau-\rho}{\mu}\right)\left(1-\frac{(\tau-\rho)^2}{3\rho^2}\right)\right)-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\right.$$
$$\left.+1\left(2\rho<\tau\leq2\rho+\mu\right)\left\{\mathcal{B}_\ell\left(\frac{2}{3}\frac{\rho}{\mu}\right)-\mathcal{B}_\ell\left(\left(\frac{\tau-\mu-\rho}{\mu}\right)\left(1-\frac{(\tau-\mu-\rho)^2}{3\rho^2}\right)\right)\right\}\right]'.$$

(b) Follows directly from Propositions 6 and 7. ■

Tables

Table 1a. Critical Values for the Fluctuation Optimality Test

Recursive Case

Panel A. 10% Significance Level

	μ								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	10.2669	8.7578	7.5765	6.8180	5.9651	5.3264	4.7263	4.0770	3.0670
2	21.0526	17.6387	15.4595	13.6645	12.3778	11.0838	9.8099	8.1885	6.3733
3	31.6098	26.5056	23.1435	20.4020	18.2414	16.4521	14.3639	12.1936	9.4068
4	42.2477	35.3095	31.0490	28.1807	24.8024	21.8613	19.4414	16.4647	12.9179
5	51.4326	43.3741	37.9558	34.0329	30.1304	26.8493	23.6270	20.4345	15.8604
6	62.1959	52.8653	46.8010	41.3255	37.0648	33.2631	29.7693	25.7976	20.1981
7	72.5923	61.1458	53.6086	48.3716	42.1891	37.6945	33.5416	28.3144	22.4318
8	83.0521	69.8155	60.1221	53.3352	48.3534	42.9999	37.7007	33.7535	26.4047
9	93.3700	79.1396	70.1581	62.3072	56.0609	50.2067	44.5958	38.5961	29.5163
10	102.8387	87.9789	77.0303	69.0955	62.9926	55.9850	47.9871	40.8464	33.0343

Panel B. 1% Significance Level

	μ								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	11.6856	10.2590	9.0347	8.3078	7.5188	6.7878	6.1378	5.3392	4.2895
2	24.1870	20.5629	18.4802	16.6605	14.9386	135726	12.3433	11.0913	8.8522
3	35.7436	31.3579	27.4429	24.6209	22.2881	20.4842	18.4831	16.2724	13.1799
4	48.4109	41.1313	37.1256	34.1003	31.0601	27.5384	24.5786	22.1307	17.6315
5	58.7816	51.0609	44.9732	40.7340	37.7266	34.3607	30.4638	26.8477	22.0854
6	71.1544	61.8007	56.3129	51.4453	46.1987	42.5460	38.7247	33.8356	27.8409
7	82.3462	71.4629	64.6099	57.2321	52.4756	47.1016	43.2170	38.0804	30.8315
8	95.5454	83.3650	74.4333	66.5070	60.7184	54.4951	49.0898	43.4037	35.4405
9	106.9382	93.7497	83.4796	75.8295	69.7516	62.8953	58.3720	51.6320	41.2317
10	118.7537	102.0123	93.9141	84.5024	78.1770	69.8137	61.9619	54.9538	44.4878

Panel C. 5% Significance Level

ℓ	μ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	14.8490	13.5353	12.7226	12.3497	11.6315	10.5737	9.2474	8.1820	7.2755
2	30.6697	57.2850	25.0862	23.2379	21.7915	20.1452	18.7319	16.8365	13.9806
3	46.1154	40.2553	37.4200	34.8092	31.8805	30.3981	28.4565	26.4705	22.5937
4	61.3731	55.2706	50.4924	47.5501	44.1246	40.8818	39.4697	34.5798	28.6159
5	75.2285	68.3475	64.1851	57.1331	54.4350	51.9411	47.2560	43.7015	36.6641
6	88.7717	80.6740	75.2264	70.9772	67.7189	63.1786	60.5449	53.2299	44.7424
7	104.0166	95.1507	86.7878	80.1468	76.4849	70.8361	64.1169	60.3049	52.7625
8	125.2995	110.3452	102.4552	95.8943	85.2440	79.5294	72.7809	65.8315	58.2705
9	138.8842	127.1147	116.4493	106.1689	101.8481	95.8864	87.8756	82.9540	717847
10	156.9100	133.9111	125.3871	117.0503	110.6774	100.6489	88.4913	81.9793	69.7523

Note. The table reports $\kappa_{\alpha,\ell}$, the critical values at 10%, 5%, and 1% significance levels respectively, for $max_{j\in\{R+m,\ldots,T\}}\ \mathcal{W}_{j,m}$ for the recursive scheme under Condition in equation (22) and for the case when parameter estimation error is irrelevant as in equation (24). The table reports critical values for various $\mu = [m/T]$ and number of restrictions, ℓ.

Table 1b. Critical Values for the Fluctuation Optimality Test

Rolling Case

Panel A. 10% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	6.4740	5.5899	7.7008	7.2863
2	9.1746	8.5913	10.7956	10.6739
3	11.5663	10.8638	12.9363	12.3039
4	13.2258	12.6272	14.4234	13.9984
5	14.9127	14.6108	17.0396	16.5904
6	16.4976	15.6346	19.0793	18.3600
7	18.7578	17.7502	20.5095	19.8632
8	19.9329	19.2351	22.7822	22.0157
9	21.7496	21.2777	23.9299	23.9710
10	23.1398	22.2947	25.2720	24.8407

Panel B. 5% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	7.7122	6.9621	8.8102	8.5989
2	10.5702	10.0698	12.4778	12.1265
3	13.2956	12.3069	14.5513	13.9501
4	14.8771	14.2805	16.6307	15.6392
5	16.8451	16.6441	19.0969	18.6127
6	18.5144	17.5945	20.9080	20.1921
7	21.0426	19.7563	22.6405	21.9120
8	22.9293	21.4715	25.1263	24.2192
9	24.2818	23.4890	26.4981	26.1489
10	25.7621	24.6734	26.8059	26.7959

Panel C. 1% Significance Level

ℓ	$\rho = 0.5; \mu = 0.25$	$\rho = 0.5; \mu = 0.3$	$\rho = 0.3; \mu = 0.25$	$\rho = 0.3; \mu = 0.3$
1	11.0943	10.3372	11.7440	11.0335
2	13.7842	14.1051	15.6558	15.7895
3	17.2460	16.4848	18.3441	18.0914
4	18.3709	18.0079	21.0888	21.4351
5	21.6826	20.6014	22.3863	22.5591
6	22.7820	22.9287	25.9931	25.1753
7	24.9451	23.4760	26.5961	25.3780
8	27.9311	26.4139	28.7270	28.1257
9	30.1262	28.8866	32.1298	30.3945
10	31.0603	30.4121	31.7719	32.4430

Note. The table reports $\kappa_{\alpha,\ell}$, the critical values at 10%, 5%, and 1% significance levels respectively, for $max_{j\in\{R+m,\ldots,T\}}\ \mathcal{W}_{j,m}$ for the rolling scheme under Condition in equation (22). The table reports critical values for various $\mu = [m/T]$, ρ= $[R/T]$ and number of restrictions, ℓ.

Table 1c. Critical Values for the Fluctuation Rationality Test

Survey and Model-Free Forecasts

Panel A. 10% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	10.5066	9.0503	8.0245	7.1035	6.3957	5.6112	5.1113	4.6141	3.9748
2	21.2392	18.0544	15.8290	13.9122	13.0720	11.1526	10.4549	9.0570	7.8723
3	31.4497	26.8866	23.7832	21.4577	19.6097	17.4180	15.3225	13.5010	11.4381
4	43.5150	36.9028	32.8187	28.4075	25.1774	23.3645	20.5785	17.6700	15.5384
5	52.4148	45.7998	39.6896	35.7848	32.0200	28.4850	26.2204	23.1738	19.1090
6	62.6771	54.3749	47.4711	42.4503	38.4920	34.9394	30.4063	27.9807	23.8787
7	74.8406	62.3659	56.2449	49.0721	44.4213	39.6189	36.4280	33.0852	26.9654
8	84.5728	72.8813	63.2267	56.8973	51.5069	45.7856	41.3975	36.7853	31.2008
9	94.9541	81.4986	72.0148	64.7598	57.1844	51.4226	46.4180	41.2775	33.9880
10	104.5994	90.8578	80.1290	69.7875	63.1391	58.7175	51.6605	45.565**0**	38.6993

Panel B. 5% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	12.0846	10.5969	9.6550	8.7536	7.7583	6.961	6.4958	6.1299	5.3700
2	23.9304	21.0152	18.8106	16.9024	16.4506	14.5181	13.2906	11.9523	10.6583
3	35.8110	31.4406	28.0387	25.7908	24.5830	21.9864	19.7824	17.7214	15.1007
4	49.4366	43.1530	39.3683	34.2290	31.3434	29.3066	26.0227	22.9933	20.9018
5	59.4929	54.2582	46.9204	43.5296	40.2826	36.1054	33.2983	30.5946	25.6162
6	71.5833	63.6046	56.9141	51.3008	47.6269	43.5316	38.8591	36.5334	31.9232
7	85.5798	73.1198	67.0436	60.3735	54.9621	51.0055	46.4178	42.5572	37.3060
8	96.5994	82.8340	75.8613	69.3683	64.7512	57.4847	54.7416	48.6285	40.4862
9	107.9342	95.6691	85.9190	79.9041	70.2681	64.2407	60.5165	53.4894	46.1068
10	120.5426	107.9044	95.7044	84.9419	78.7060	71.8195	65.8906	59.5306	51.3579

Panel C. 1% Significance Level

	$\widetilde{\mu}$								
ℓ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	15.2034	14.1709	12.8862	12.5024	11.3655	10.2615	9.4840	9.4669	8.8229
2	30.8516	28.2962	25.9860	24.0338	23.3590	21.7058	20.1484	18.3749	16.6201
3	45.5649	41.8301	38.5925	35.8044	34.0350	32.5569	30.2612	27.1766	23.8648
4	62.9754	54.9380	55.6239	48.0043	45.2153	44.5599	41.1181	35.1465	31.2726
5	78.8954	69.0344	63.8089	59.8863	56.6435	53.3724	50.3065	48.5804	41.4091
6	91.9499	84.6385	76.9111	73.6821	67.9723	63.2696	59.3231	55.8028	49.0648
7	112.8762	96.4070	91.6098	86.1555	77.8255	75.7540	69.5349	63.2412	59.2658
8	125.4023	109.8004	105.4471	96.4564	98.2139	84.6103	84.8680	77.9564	67.3690
9	140.4446	126.4322	119.2996	109.7265	101.7085	92.3127	91.4050	79.0176	70.5083
10	153.1288	145.7988	132.9398	121.6310	113.1604	107.4479	98.5991	92.6189	84.2430

Note. The table reports $\kappa_{\alpha,\ell}$, the critical values at 10%, 5%, and 1% significance levels respectively, for $max_{\widetilde{j}\in\{m,\dots,P\}}\ \mathcal{W}_{\widetilde{j},m}$ for the case when parameter estimation error is irrelevant as in Corollary 9. The table reports critical values for various $\widetilde{\mu}= [m/P]$ and number of restrictions, ℓ.

Table 2a: Size. Recursive Case

	Traditional Test		Fluctuation Test	
R/P	100	200	100	200
Panel A. Mean Prediction Error				
25	0.0486	0.0496	0.0500	0.0648
50	0.0454	0.0494	0.0522	0.0630
100	0.0572	0.0482	0.0582	0.0678
200	0.0468	0.0520	0.0580	0.0712
300	0.0534	0.0454	0.0526	0.0624
400	0.0508	0.0538	0.0546	0.0632
Panel B. Efficiency Test				
25	0.0498	0.0544	0.0686	0.0832
50	0.0542	0.0478	0.0644	0.0874
100	0.0546	0.0518	0.0726	0.0842
200	0.0512	0.0502	0.0670	0.0812
300	0.0570	0.0546	0.0658	0.0820
400	0.0566	0.0484	0.0694	0.0800

Table 2b: Size. Rolling Case

Panel A. Mean Prediction Error				
25	0.1130	0.1550	0.0940	0.1440
50	0.0690	0.0780	0.0540	0.0610
100	0.0670	0.0620	0.0580	0.0610
200	0.0630	0.0550	0.0560	0.0710
300	0.0540	0.0530	0.0740	0.0560
400	0.0510	0.0450	0.0400	0.0460
Panel B. Efficiency Test				
25	0.9240	0.9990	0.6680	0.9020
50	0.2260	0.7210	0.1610	0.3400
100	0.0630	0.1290	0.0850	0.1290
200	0.0450	0.0600	0.0840	0.1250
300	0.0600	0.0360	0.0770	0.1100
400	0.0470	0.0510	0.0710	0.0980

Note. Tables 2a and 2b report empirical rejection frequencies of the test statistics $\max_{j\in\{R+m,\ldots,T\}} \mathcal{W}_{j,m}$ (column labeled "Fluctuation Test") and the traditional test statistics (column labeled "Traditional Test") under the recursive and rolling estimation schemes respectively (see DGP in Section 3). The first column provides the R values; the columns under the header give the P values. Nominal size is 0.05 and m = 50.

Table 3: Power Analysis

Panel A. DGP 1 - Mean Forecast Error						
	A. Non-Stationary		B. Non-Stationary		C. Stationary	
b	I. Traditional	II. Fluctuation	I. Traditional	II. Fluctuation	I. Traditional	II. Fluctuation
0	0.0482	0.0448	0.0482	0.0448	0.0482	0.0448
0.1	0.0486	0.0504	0.0924	0.0856	0.0822	0.0680
0.2	0.0460	0.0676	0.2040	0.2058	0.1570	0.1080
0.3	0.0432	0.0848	0.3882	0.4286	0.2704	0.1732
0.4	0.0350	0.1014	0.6000	0.6684	0.4138	0.2526
0.5	0.0312	0.1358	0.7722	0.8650	0.5494	0.3318
0.6	0.0222	0.1456	0.9022	0.9672	0.6900	0.4094
0.7	0.0214	0.1740	0.9554	0.9928	0.7640	0.4596
0.8	0.0138	0.1870	0.9878	0.9992	0.8454	0.4998
0.9	0.0086	0.1826	0.9964	0.9998	0.8828	0.5208
1.0	0.0044	0.1886	0.9990	1.0000	0.9078	0.5104

Panel B. DGP 2 - Efficiency						
	A. Non-Stationary		B. Non-Stationary		C. Stationary	
b	I. Traditional	II. Fluctuation	I. Traditional	II. Fluctuation	I. Traditional	II. Fluctuation
0	0.0540	0.0490	0.0540	0.0490	0.0540	0.0490
0.50	0.0680	0.0998	0.0854	0.0710	0.2020	0.1598
1.00	0.0362	0.2152	0.4048	0.2992	0.7878	0.6100
1.50	0.0110	0.4110	0.7372	0.7108	0.9908	0.9190
2.00	0.0016	0.6438	0.8390	0.9148	1.0000	0.9912
2.50	0	0.8278	0.8442	0.9832	1.0000	0.9992
3.00	0	0.9260	0.7842	0.9950	1.0000	0.9996
3.50	0	0.9782	0.6770	0.9964	1.0000	1.0000
4.00	0	0.9922	0.4972	0.9972	1.0000	1.0000
4.50	0	0.9980	0.3240	0.9986	1.0000	1.0000
5.00	0	1.0000	0.1812	0.9982	1.0000	1.0000

Note. The table reports empirical rejection frequencies of the test statistics $max_{j\in\{R+m,\ldots,T\}}W_{j,m}$ (column labeled "Fluctuation") and the traditional test statistics (column labeled "Traditional") under the recursive estimation scheme. Non-Stationary cases A and B refer to DGP A and DGP B, Stationary case C refers to DGP C in Section 3, respectively. Nominal size is 0.05. R = 300, P = 100, m = 50.

Table 4. MSFE Comparisons

Horizon	Greenbook	BCEI	SPF
Sample Start Date:	1968:IV	1980:I	1968:IV
0	1.06	0.97	1.32
1	1.76	1.29	2.28
2	2.33	1.56	3.02
3	2.47	1.92	3.67
4	2.71	2.31	4.30
5	2.70	2.61	- -

Note. MSFE is calculated as $(\pi_{t+h} - \hat{\pi}_{t+h,t})^2$ for various forecast horizons h.

Table 5. Inflation Forecast Rationality Tests

Horizon	N. Obs.	Fluctuation	Traditional
		Greenbook	
0	149	39.64*	0.15
1	149	46.89*	0.65
2	143	49.41*	0.33
3	134	41.59*	0.02
4	109	37.89*	0.01
5	74	91.96*	3.33
		BCEI	
0	100	43.74*	11.95*
1	100	51.97*	15.69*
2	100	74.22*	22.38*
3	98	135.79*	44.67*
4	74	167.51*	64.54*
		SPF	
0	149	45.11*	0.13
1	149	66.36*	0.16
2	148	77.84*	0.07
3	145	158.77*	0.46

Note. The table reports the Traditional and Fluctuation Wald test statistics, $\mathcal{W}_P$ and $\max_{\tilde{j}\in\{m,\ldots,P\}} \mathcal{W}_{\tilde{j},m}$, respectively. The traditional tests (column labeled "Traditional") are based on the indicated number of observations. The Fluctuation test (column labeled "Fluctuation") results are based on $m = 60$. The significance of the test statistics at the 5% significance level is indicated by asterisks.

Table 6. Fed's Information Advantage Over Private Sector's Forecasts

Horizon	N. Obs.	Fluctuation	Traditional
		BCEI	
0	100	49.81*	14.11*
1	100	93.02*	34.87*
2	100	60.00*	20.93*
3	98	28.10*	11.05*
		SPF	
0	149	39.71*	36.96*
1	149	47.68*	18.78*
2	142	38.46*	19.29*
3	134	51.07*	34.22*

Note. The table reports the Traditional and Fluctuation Wald test statistics (W_P and $\max_{\tilde{j}\in\{m,...,P\}} \mathcal{W}_{\tilde{j},m}$, respectively) for their respective null hypotheses. The traditional tests are based on the indicated number of observations. The Fluctuation test results are based on $m = 60$. The significance of the test statistics at the 5% significance level is indicated by asterisks.

Figures

Figure 1: One-quarter-ahead U.S. Inflation Forecasts and Realized Values

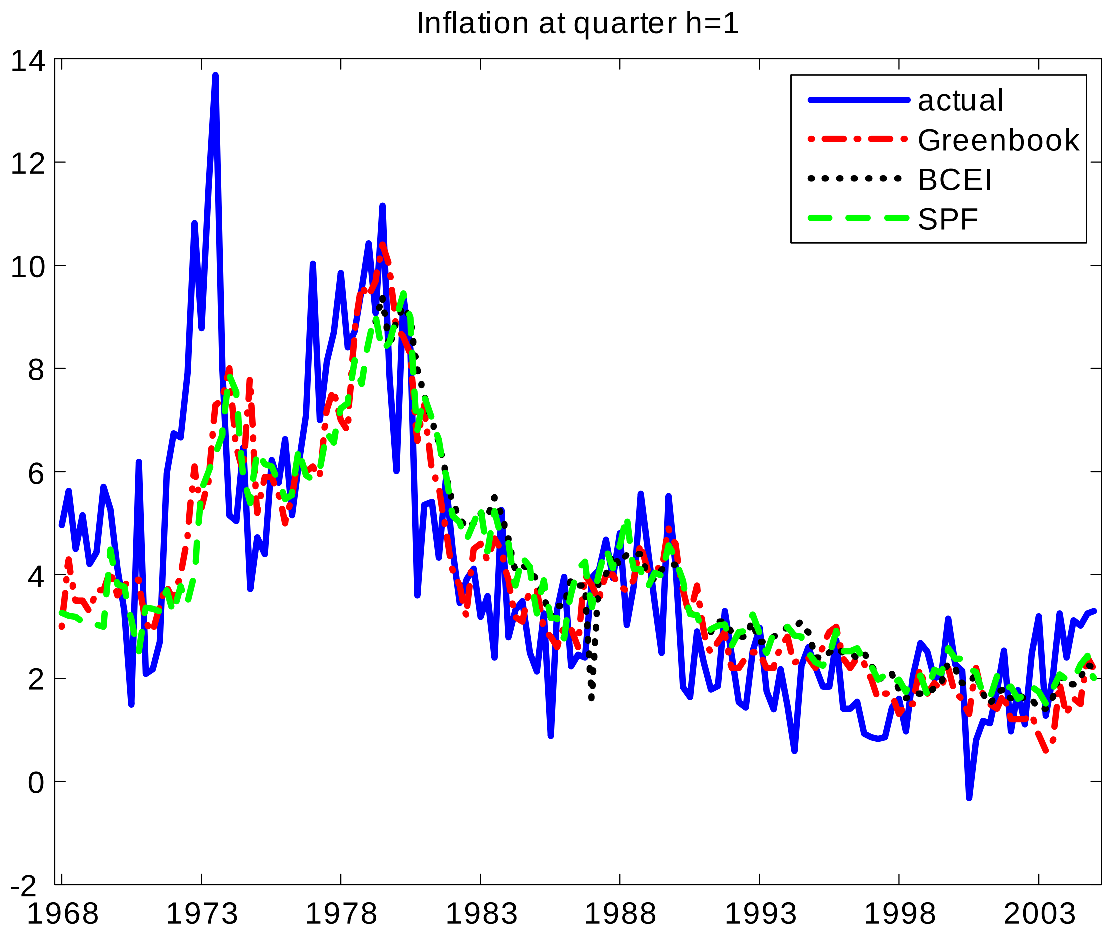
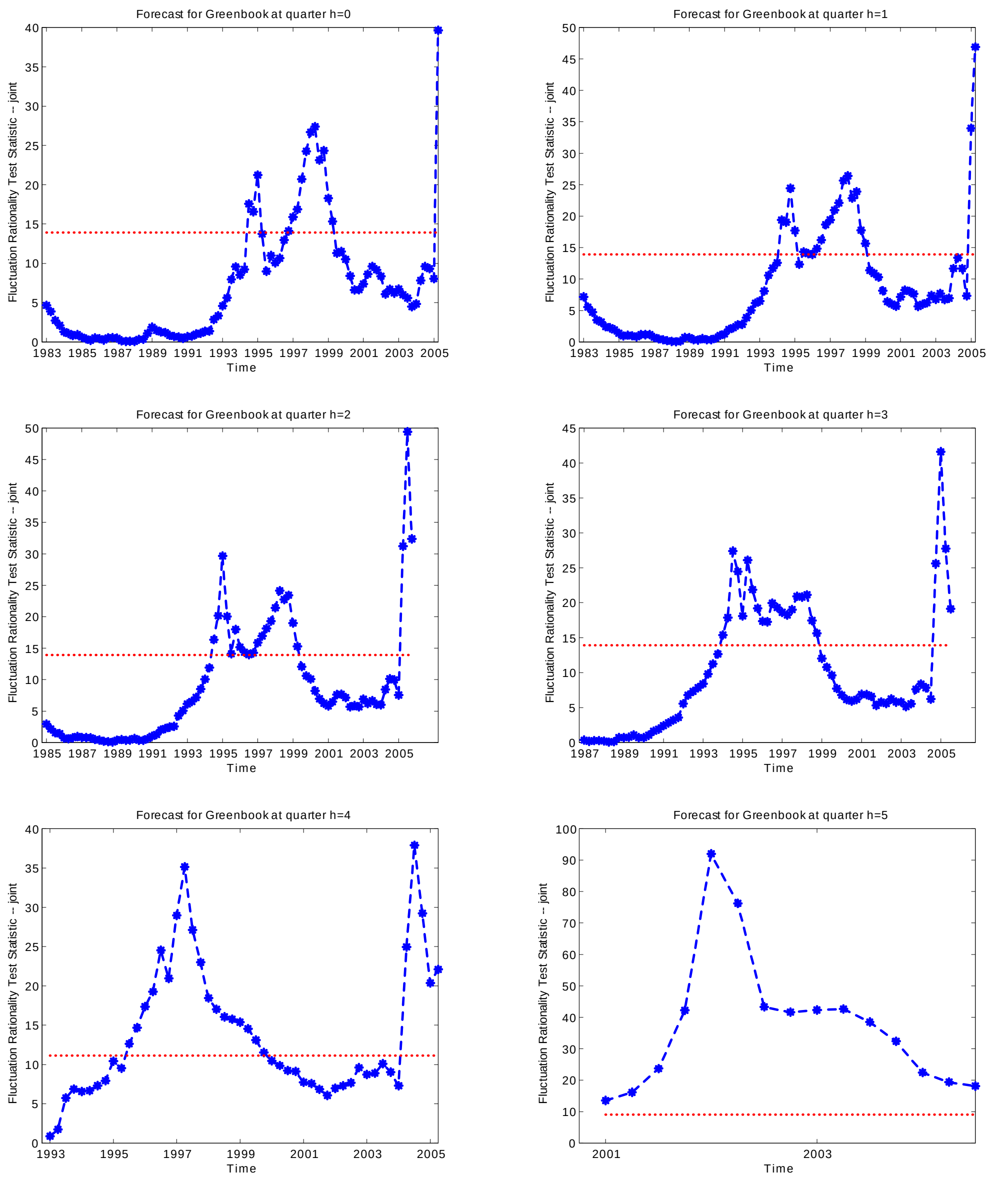


Figure 2: Inflation Forecasts

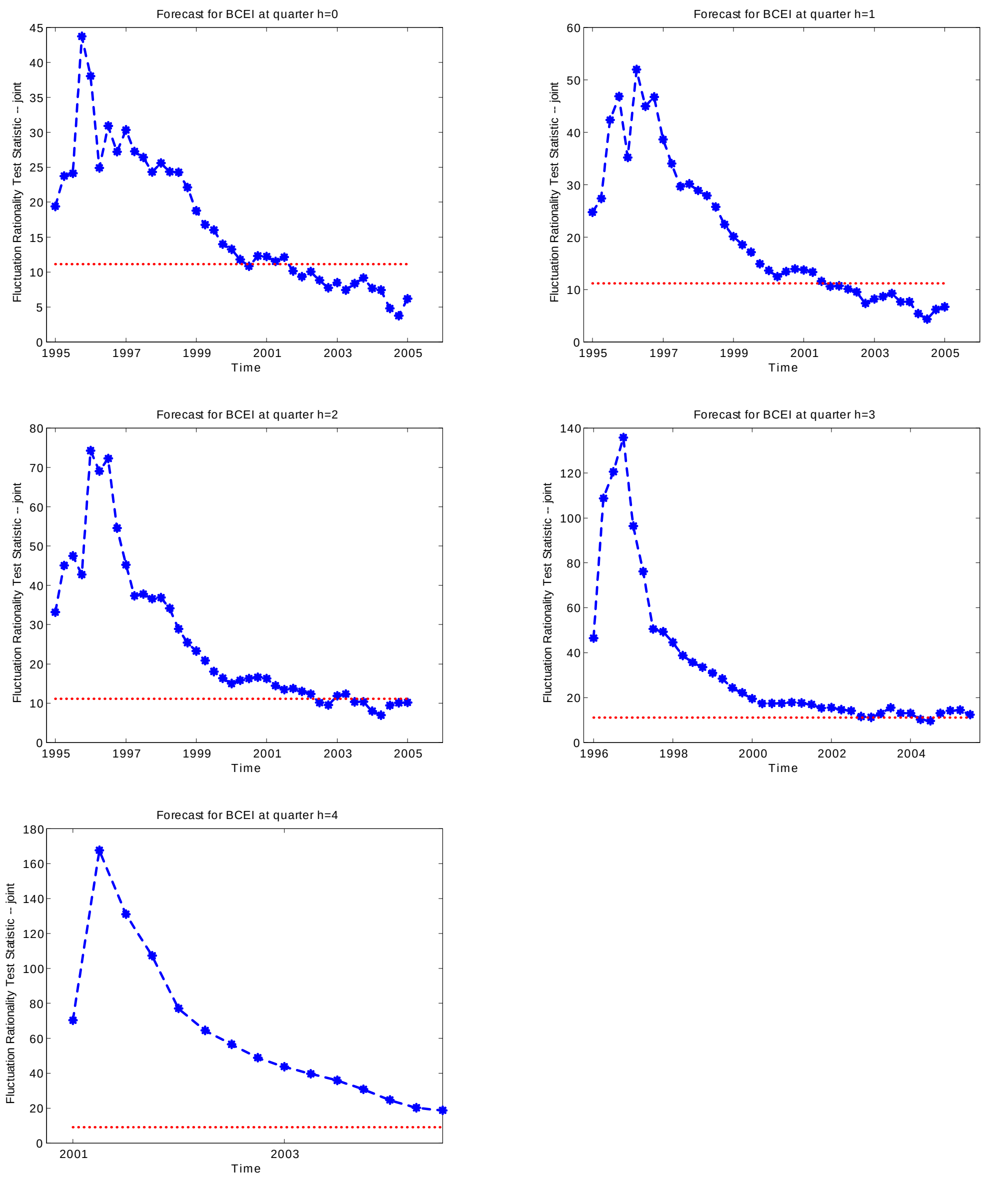
Note. The figure plots Greenbook, SPF, and BCEI forecasts of inflation for various forecast horizons h in conjunction with the realized values of inflation for the corresponding horizon. If a forecast for a specific horizon by the corresponding agency does not exist, it is depicted as a missing value.

Figure 3: Fluctuation Optimality Test for Greenbook Forecasts



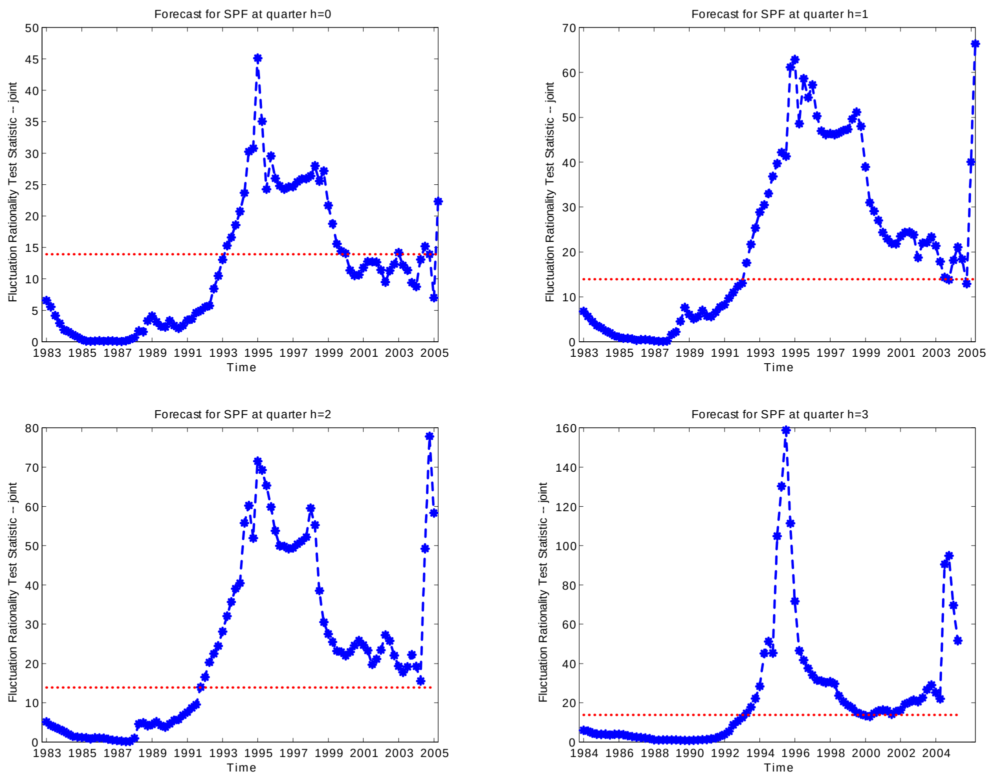
Note. The figure reports the time path of the test statistics $W_{j,m}$ for the null hypothesis of forecast rationality under a recursive estimation scheme. $m = 60$ and the dotted line ("...") corresponds to the critical value at 5% significance level. If the test statistic is above the dotted line, we reject the null hypothesis of rationality at any point in time. The dates in the horizontal axis suggest a particular break-date.

Figure 4: Fluctuation Optimality Test for BCEI Forecasts



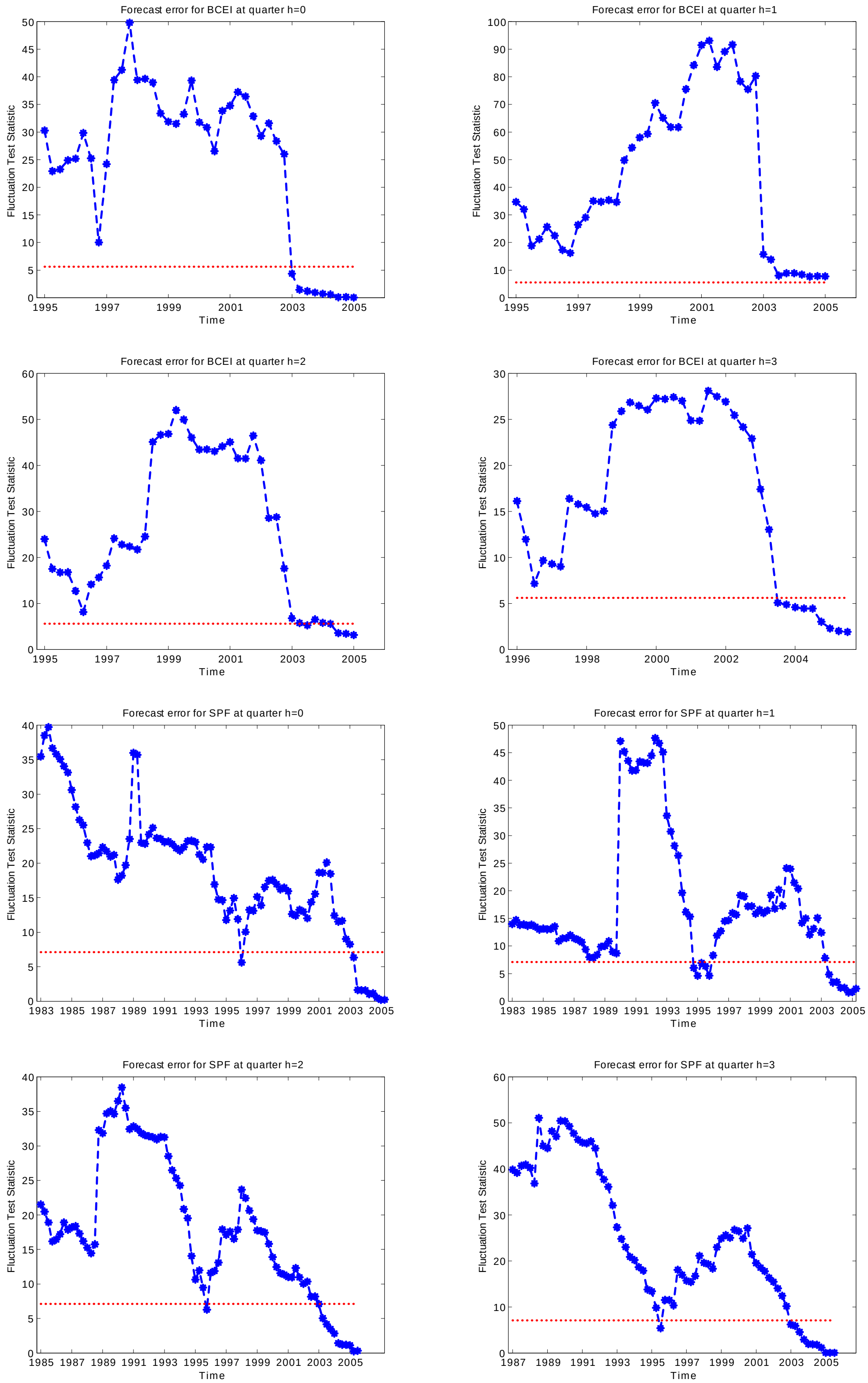
Note. The figure reports the time path of the test statistics $W_{j,m}$ for the null hypothesis of forecast rationality under a recursive estimation scheme. $m = 60$ and the dotted line ("...") corresponds to the critical value at 5% significance level. If the test statistic is above the dotted line, we reject the null hypothesis of rationality at any point in time. The dates in the horizontal axis suggest a particular break-date.

Figure 5: Fluctuation Optimality Test for SPF Forecasts



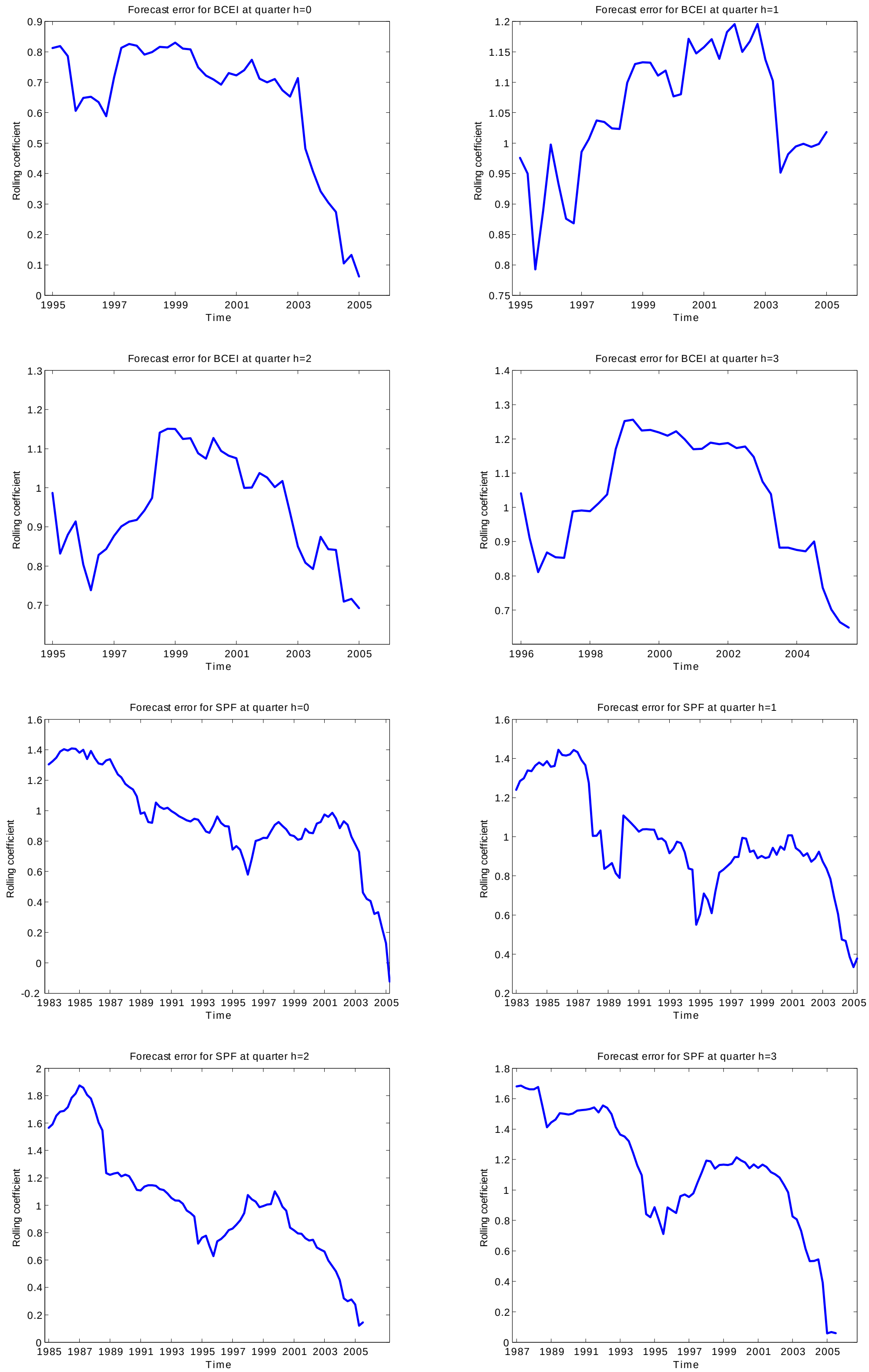
Note. The figure reports the time path of the test statistics $W_{j,m}$ for the null hypothesis of forecast rationality under a recursive estimation scheme. $m = 60$ and the dotted line ("...") corresponds to the critical value at 5% significance level. If the test statistic is above the dotted line, we reject the null hypothesis of rationality at any point in time. The dates in the horizontal axis suggest a particular break-date.

Figure 6: Fed's Informational Advantage



Note. The figure reports the test statistics $W_{j,m}$for the null hypothesis $\beta_g = 0$ over time. $m = 60$.

Figure 7: Fed's Informational Coefficients



Note. The figure shows the rolling estimate of β_g as in equation (9) based on $m = 60$.